Geometry

5. Enter Bernhard Riemann

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In the previous article the author examined curves and surfaces. One might hope to continue by analogy in many dimensions. The concept of working in many dimensions is so bewildering (yet today so matter-of-course) that it needed the genius of Bernhard Riemann to show us exactly how it can be done. In just one lecture on the foundations of geometry he completely changed our way of thinking. Later geometers were to spend entire lifetimes trying to finish what Riemann had begun. Some even see the genesis of *General Relativity* in his lecture.

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A Manifold is a "Curvey-Curvey" Thing

Manifold: An object that displays extendibility in many dimensions, i.e. a multidimensional object.

How does one think about a higher dimensional object? This is the first question that Riemann addresses in his great lecture "On the hypotheses that lie at the foundations of geometry"[1]. While there are a number of objects that we can examine physically in dimensions up to and including three it is difficult to think of dimensions beyond *except abstractly* —in one's mind. Riemann suggests the following approach as one possible method. We can think of a curve as that which is traced out by a moving (or changing) point, a surface as that traced out by a changing curve, and so on. A manifold of dimension n is traced out by a changing manifold of dimension $n-1$. As we study such manifolds and learn their properties we will 'see' them more clearly.[2]

Analytically these n dimensions can be thought of as repre-

[1] This lecture was made particularly difficult to understand because Riemann had to lecture to a general (non-mathematical) audience and so he had to avoid too many symbols. This unfortunate circumstance continues to this day!

[2] This is a common approach in mathematics — we take a plausible definition and then work out all the properties. This gives us a feel for what the definition really entails.

sented[3] by functions $x_1, \ldots, x_n$ on the manifold called the 'coordinate' functions. Specifying the values of each of these n functions simultaneously, uniquely specifies a point within the manifold. Thus other functions on the manifold can be thought of as functions $f(x_1, \ldots, x_n)$ of n variables. In fact functions of many variables had been studied before Riemann—it was his *geometric* interpretation that gave these earlier results more meaning and scope, and thus led to a spurt in the study of such functions. One point that needs to be emphasized, is that there is no obvious choice of coordinates—indeed there could be a number of such choices as we see even in the case of dimensions 2 and 3. We will try to examine properties that do not depend on any particular choice of coordinates.

[3] For more knowledgeable readers — we are following the convention of Riemann's lecture in that the study is only 'local' at the moment. Moreover I beg forgiveness of such readers for entirely neglecting analytical difficulties like the type of functions involved — smooth, analytic etc.

The above definition gives us the property called *extension* by Riemann; it says that around any point our manifold extends in n independent directions. But he points out that this is not enough to specify the geometry of the manifold. The additional *metric* property is one that allows us to measure distances. In a radical departure from earlier work (except perhaps that of Gauss) he specifies distance by assigning magnitudes to 'line elements', which we have called velocity vectors. In other words, if one could measure the speed $s(t)$ at every instant t as we travel along a curve, then we could compute the distance we have travelled as $\int s(t)\,dt$. The distance between two points is then the *infimum* (or greatest lower bound) of the distances travelled along various curves between two points. From Gauss' theory of surfaces the length of a line element $dx = (dx_1, \ldots, dx_n)$ is specified by a form[4] $(\sum g_{ij}(x)\, dx_i\, dx_j)^{1/2}$. This means that the length of a curve $(x_1(t), \ldots, x_n(t))$ from t_0 to t_1 is given by the integral

[4] Riemann actually does consider more general forms briefly but we will skip them here for the sake of brevity.

$$l = \int_{t_0}^{t_1} \sqrt{\sum_{i,j=1}^{n} g_{ij}(x(t))\, \dot{x}_i(t)\, \dot{x}_j(t)}\; dt$$

The Calculus of Variations

Euler (and later Lagrange) developed the principal ideas for the *Calculus of Variations. If* $L(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a function and we wish to extremise

$$\int_{t_0}^{t_1} L(x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t))\, dt$$

along all paths $x(t) = (x_1(t), \ldots, x_n(t))$ that go from $p = x(t_0)$ to $q = x(t_1)$; then this method tells us that we need to solve the equations

$$\frac{\partial L}{\partial x_k}\Big|_{x=x(t), y=\dot{x}(t)} - \sum_{i=1}^{n} \frac{\partial^2 L}{\partial x_i \partial y_k}\Big|_{x=x(t), y=\dot{x}(t)} \dot{x}_i(t)$$

$$= \sum_{i=1}^{n} \frac{\partial^2 L}{\partial y_i \partial y_k}\Big|_{x=x(t), y=\dot{x}(t)} \ddot{x}_i(t)$$

In our case we want $L = (\sum_{i,j=1}^{n} g_{ij}(x)\dot{x}_i \dot{x}_j)^{1/2}$. This is not likely to have a nice form since the same curve may appear in many guises (parametrisations). It turns out that a better way is to extremise the 'energy'

$$E = \int_{t_0}^{t_1} \sum_{i,j=1}^{n} g_{ij}(x(t))\, \dot{x}_i(t)\, \dot{x}_j(t)\, dt$$

for which $L = (\sum_{i,j=1}^{n} g_{ij}(x)\dot{x}_i \dot{x}_j)$. The geodesic equations then become

$$\sum_{ij=1}^{n} \left(\frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_i} \right) \dot{x}_i \dot{x}_j = \sum_{i=1}^{n} g_{ik} \ddot{x}_i$$

The paths of shortest length or geodesics are 'straight lines' for our geometry; and one advantage over the method of Gauss is that this definition does not depend on any ambient Euclidean space.

Now Euler had given a way to find the extremal values of such integrals (for the application to Fermat's least time principle; see box above). Applying his methods one deduces an equation for a path of extremal (minimum or perhaps maximum!) length (see box on the calculus of variations). Now we make the further 'natural' assumption (which was later overturned in the space-time context by Minkowski and Einstein) that at each point the form $\sum g_{ij}\, a_i\, a_j$ is positive for any collection of numbers $(a_1, \ldots, a_n)$. Moreover, the given form can easily be adapted so that $g_{ij} = g_{ji}$; so we assume this. One can then show that the extremal paths are indeed of shortest length (at least locally[5]). Thus the paths of shortest length or geodesics are 'straight lines' for our geometry; and one advantage over the methods of Gauss is that this definition does not depend on any ambient Euclidean space.

[5] See previous footnote 3.

The next question is what quantities need to be determined to determine the geometry uniquely. The functions g_{ij} are clearly sufficient but don't seem to be necessary, since they change depending on the system of coordinates chosen. In a different system of coordinates $(y_1, \ldots, y_n)$, one computes easily (exercise) that the g_{ij} transform into

$$g'_{ij} = \sum g_{kl} \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j}$$

To take this into account Riemann argues heuristically as follows: The metric is given by the $n(n+1)/2$ functions g_{ij} (not n^2 since the symmetries $g_{ij} = g_{ji}$ have to be taken into account). A change of coordinates is given by n functions (the new coordinate functions). The difference is $n(n-1)/2$ functions which ought to be the number of functions required to determine the geometry. He then proceeds to construct $n(n-1)/2$ functions as candidates. These are the principal curvatures whose definition is outlined below[6].

[6] *A word of warning:* the calculations that follow may seem complicated to the uninitiated reader, even one armed with a paper and pencil. The details can be found in the books listed at the end of the article.

There is a distinguished choice of coordinates around any chosen point o. For each point near 0 let its coordinates be given by the velocity (i.e. direction and speed) with which one must start at o to reach the point in unit time (*Exercise*: What are the coordinates

The Geodesic Normal Coordinates

We write the Taylor expansions of the functions

$$g_{ij}(x) = g^o_{ij} + l_{ij} + q_{ij} + \textbf{higher order terms in the } x_i\text{'s}$$

where l_{ij} (respectively q_{ij}) is a linear (respectively quadratic) function of the x_i 's. Moreover, by choosing an orthonormal system at the origin o we can assume that g^o_{ij} is 1 *if* $i = j$ and 0 if $i \neq j$. Now we substitute the given geodesics $(ta_1, \ldots, ta_n)$ into the equations of the geodesic derived in the previous box.

$$\sum_{i,j=1}^{n} \left(\frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_i} \right) a_i a_j = 0$$

Equating the coefficients of terms of various orders in t we obtain

$$\sum_{i,j=1}^{n} \left(\frac{\partial l_{ij}}{\partial x_k} - \frac{\partial l_{ik}}{\partial x_j} - \frac{\partial l_{jk}}{\partial x_i} \right) x_i x_j = 0$$

$$\sum_{i,j=1}^{n} \left(\frac{\partial q_{ij}}{\partial x_k} - \frac{\partial q_{ik}}{\partial x_j} - \frac{\partial q_{jk}}{\partial x_i} \right) x_i x_j = 0$$

Now some simple manipulations show that $l_{ij} = 0$. Some more complicated manipulations show that $c_{ij,kl}$ do indeed have the required form of equation (1)

The notion of principal curvature discussed here is *different* from that introduced for surfaces; however the name 'curvature' for these quantities can be justified.

for o?). In other words, the coordinates are so chosen that the paths $(ta_1, \ldots, ta_n)$ are geodesics starting at o for every choice of numbers $(a_1, \ldots, a_n)$ (and for small enough values of t). We substitute this in the equations for the geodesics and perform a simple calculation (see box on geodesic normal coordinates) to obtain the Taylor expansion of the metric in these special coordinates (called *geodesic normal coordinates*),

$$\sum_{i,j=1}^{n} g_{ij}\ dx_i\, dx_j = \sum_{i=1}^{n} dx_i^2 + \sum_{i,j,k,l=1}^{n} c_{ij,kl}\ x_k x_l\ dx_i\, dx_j$$

$$+ \text{ higher order terms in the } x_i\text{'s}$$

We are in a small region of Euclidean space if and only if curvature is zero.

Quadratic Forms

Let $Q(x_1, \ldots, x_n)$ be a quadratic form. We show by induction that there is an orthonormal linear change of coordinates $x_i = \sum_j m_{ij} y_j$ so that

$$Q(x_1, \ldots, x_n) = \sum \lambda_j y_j^2$$

Let S denote the collection of all points $(x_1, \ldots, x_n)$ such that $\sum x_i^2 = 1$. Let $(a_1, \ldots, a_n)$ be a point on S so that $Q(a_1, \ldots, a_n)$ is maximum among all points on S. We can easily show that for any point $(x_1, \ldots, x_n)$ on S such that $\sum x_i a_i = 0$ we have

$$Q(sa_1 + tx_1, \ldots, sa_n + tx_n) = s^2 Q(a_1, \ldots, a_n) + t^2 Q(x_1, \ldots, x_n)$$

Choose an orthonormal change of coordinates so that $(a_1, \ldots, a_n)$ becomes $(1, \ldots, 0)$. The expression we then obtain is

$$Q(x_1, \ldots, x_n) = \lambda_1 y_1^2 + Q'(y_2, \ldots, y_n)$$

where $\lambda_1 = Q(a_1, \ldots, a_n)$ and Q' depends on a smaller number of variables. An induction argument completes the proof.

Moreover, the $c_{ij,kl}$ are numbers such that

$$\sum_{i,j,k,l=1}^{n} c_{ij,kl}\, a_i a_j b_k b_l = \sum_{\substack{1 \le i < j \le n \\ 1 \le k < l \le n}} d_{ij,kl}\, (a_i b_j - a_j b_i)(a_k b_l - a_l b_k) \tag{1}$$

for some numbers $d_{ij,kl}$. The right hand side can be thought of as a quadratic form in the variables $A_{ij} = (a_i b_j - a_j b_i)$ where $1 \le i < j \le n$. This quadratic form can be put into diagonal form by an orthonormal linear substitution $A_{ij} = \sum m_{ij,kl} B_{kl}$ (see box on quadratic forms).

The 'diagonal entries' k_{ij} are then called the *principal curvatures* at o (the minus sign is for historical reasons). This notion of principal curvatures is *different* from that introduced for surfaces in the previous article; however, the name 'curvature' for these quantities is justified as follows. Consider a pair of vectors

$(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ and the surface described parametrically as the collection of all points $(sa_1 + tb_1, \ldots, sa_n + tb_n)$. The Gaussian curvature of this surface at o can be computed and shown to be

$$\frac{-1}{4} \Big(\sum_{\substack{1 \le i < j \le n \\ 1 \le k < l \le n}} d_{ij,kl} \; (a_i b_j - a_j b_i) \, (a_k b_l - a_l b_k) \Big)$$

Since curvature being exactly zero is precisely the condition for being in Euclidean space, this lays the question of whether space is Euclidean or not, firmly at the door of the experimental sciences.

For manifolds of dimension 2 (i.e. surfaces), this is precisely the Gaussian curvature introduced in the previous article. Now note that there are $n(n-1)/2$ variables A_{ij} and thus there are $n(n-1)/2$ principal curvatures. Thus we have obtained the required $n(n-1)/2$ functions on the manifold; unfortunately the count of Riemann doesn't quite work beyond this point.[7] However, it is true that we are in a small region of Euclidean space if and only if the curvature is zero. It now appears that these depend on a special choice of coordinates, but since this choice was determined by the metric, it turns out that these quantities have an interpretation independent of the choice of coordinates (this was demonstrated by Riemann and later simplified through the work of Christoffel, Cartan and Koszul). Thus we have found a collection of quantities that depends only on the manifold and not on a chosen coordinate system. Moreover, these quantities generalise the notion of curvature and as a by-product we also have a proof of Gauss' *Theorema Egregium.*

[7] The margin here is too small for a proof!

Riemann then addressed himself to the question of what this means for space, i.e. the world around us. First of all, this curvature could be computed by making measurements. Though measurements (in his time) had not shown up any curvature, he says this only demonstrates that the curvature is extremely small. Since curvature being exactly zero is precisely the condition for being in Euclidean space, this lays the question of whether space is Euclidean or not, firmly at the door of the experimental sciences.

The remaining problem is that of the global structure of space. Riemann pointed out that the property of unboundedness[8] is the property he called 'extension'. This does not contradict the possibility of the universe being finite. In fact, he points out that if the curvatures of space are everywhere greater than a positive constant, then the universe must close upon itself like a sphere. Thus the global structure of our geometry is directly related to the quantities computed locally, i.e. curvatures.

[8] In the words of an old lady responding to a lecture by Eddington, "if the umbrella has a boundary I could just go to the edge of the universe and poke out my umbrella!"

Summary

Gauss, who was extremely sparing in his praise, came out of Riemann's lecture transfixed with amazement and full of compliments.

It is not at all surprising that Gauss, who was extremely sparing in his praise (he only noted to Bolyai, Lobachevsky, Jacobi, Abel and others that their work was good—since it was in conformity with his own calculations!), came out of Riemann's lecture transfixed with amazement and full of compliments. Most of the details of calculations in this lecture were only available in later papers of Riemann; these appeared late since Riemann was a man with many interests. Other papers by Riemann that were to make a lasting impression were the ones on the theory of functions of a complex variable and the paper on the *Prime Number Theorem* (which led to the famous Riemann hypothesis of number theory). In any case Riemann transformed geometry completely and this explains why the entry of Bernhard Riemann into geometry must always be noted with thunderous applause!

Suggested Reading

J W Milnor. Lectures on Morse Theory, Annals of Math. Studies. No. 55, Princeton University Press, New Jersey, 1963.
Book which deals with many topics covered here rather nicely.

M Spivak. Comprehensive Intoduction to Differential Geometry. vol.II, Publish or Perish, Berkeley, 1970.
The author depended upon the translation of Riemann's lecture given in the above book.

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