Chaos Modelling with Computers

Unpredictable Behaviour of Deterministic Systems

Balakrishnan Ramasamy and T S K V Iyer

Chaos is a type of complicated behaviour found in non-linear dynamical systems. Computers are playing an important role in the growth of this science.

Chaos is one of the major scientific discoveries of our times. In fact many scientists rank it along with relativity and quantum mechanics as one of the three major scientific revolutions of this century. Chaos is a science of everyday things: it has been implicated in areas ranging from heart failure, meteorology, economic modelling, population biology to chemical reactions, neural networks, fluid turbulence and more speculatively even manic-depressive behaviour. It also seems to occur everywhere — in rising columns of cigarette smoke, in fluttering flags, in dripping faucets, in traffic jams and so on. Computers have played a major role in the discovery and subsequent developments in this field. The computer is to chaos what cloud chambers and particle accelerators are to particle-physics. Numbers and functions are chaos' mesons and quarks. In this article we provide an introduction to chaos and the role that computers play in this field.

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Chaos and Dynamical Systems

The laws of science aim at relating cause and effect and thereby making predictions possible. For example, based on the laws of gravitation, eclipses can be predicted thousands of years in advance. But there are other natural phenomena that are not predictable though they obey the same laws of physics. The weather, the flow of a mountain stream, the roll of a dice are examples of such phenomena. It was believed until recently that precise predictability can in principle be achieved, by gathering and processing sufficient amount of information.

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Such a viewpoint has been altered by a striking discovery: simple deterministic systems can generate random like behaviour. The randomness is fundamental; gathering more information will not make it go away. Randomness generated in this way has come to be called *chaos*.

The discovery of chaos has created a new paradigm in scientific thinking. On the one hand it places fundamental limits on the ability to make predictions. On the other hand, the determinism inherent in chaos implies that many random phenomena are more predictable than had been thought. Chaos allows us to find order amidst disorder. The result is a revolution affecting many different branches of science.

Chaos has put an end to the Laplacian fantasy of mechanistic determinism. Laplace once boasted that "The present state of the system of nature is evidently a consequence of what it was in the preceding moment, and if we conceive of an intelligence which at a given instant comprehends all the relations of the entities of this universe, it could state the respective positions, motions, and general affects of all these entities at any time in the past or future." But according to chaos, even if we were able to write down the equations governing every particle in the universe, if we knew the state of the system at any point of time *only* approximately, we cannot predict the behaviour of the system in the long term because the small errors inherent in measurements amplify very fast, thus making prediction impossible. This property of *sensitive dependence on initial conditions* is one of the characteristics of chaos.

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Chaos emerges from the larger context of the theory of dynamical systems. A *dynamical system* is one that allows us to predict the future given the past. A dynamical system is made up of two parts: the notion of a phase (the essential information about a system) and a dynamic (a rule that describes how the state evolves with time). The evolution of the system can be visualized in a phase

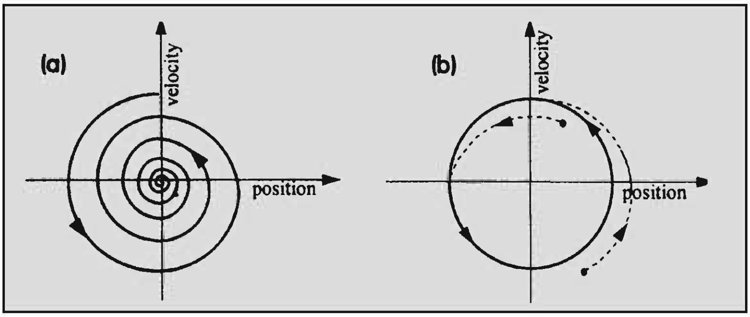


Figure 1 (left) Phase space portrait for a simple damped pendulum has a fixed point attractor; (right) phase space portrait for a pendulum clock shows a limit cycle. Any arbitrary initial condition (represented by the dots) is attracted to the limit cycle.

space, an abstract construct whose coordinates are the components of the state.

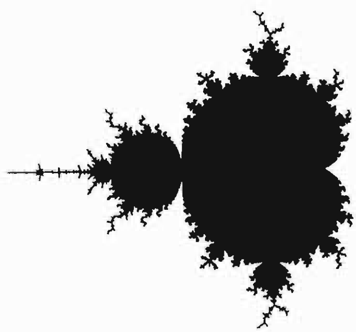
The simple pendulum is a good example of a dynamical system. The position and velocity are all that are needed to determine the motion of a pendulum. The phase is thus a point in a plane, whose coordinates are position and velocity. Newton's laws provide the dynamic (or rule), that describes how the phase evolves. As the pendulum swings the phase moves along an orbit in a plane. For an undamped pendulum the orbit is a loop whereas for a damped pendulum the orbit spirals to a point called the *fixed point* as the pendulum comes to rest.

The example of a pendulum also introduces us to the concept of an *attractor*. An attractor is a finite region in phase space, to which the system settles down in the long run. For example the fixed point is an attractor for a simple damped pendulum (*Figure 1a*). It can be thought of as attracting all the nearby points to itself. Another type of attractor could be found in the behaviour of a pendulum clock. Since the energy lost due to friction is balanced by the energy input to the system, the pendulum executes periodic motion. The phase space portrait of a pendulum clock is a cycle. Irrespective of how the pendulum is set swinging, it approaches the same cycle in the long-term limit . Such attractors are called *limit cycles* (*Figure 1b*). Systems whose attractors are classical[1] such as fixed points and limit cycles, have the property that small measurement errors remain bounded and long term behaviour is predictable.

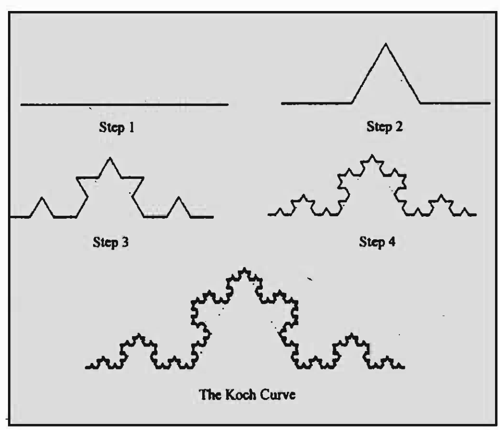
[1] Most systems settle down in the long run to a fixed point or a limit cycle. These classical attractors remain bounded in spite of small variations in initial conditions.

Fractals : A Geometry of Nature

"Clouds are not spheres; mountains are not cones; coastlines are not circles and bark is not smooth, nor does lightning travel in a straight line", Benoit B Mandelbrot, the discoverer of fractals is fond of saying. Traditional Euclidean geometry is insufficient in describing natural objects such as clouds, mountains, coastlines, etc. To explain such natural shapes Mandelbrot invented fractal geometry.Fractals have the property of self-similarity, i.e. parts of the fractal object resemble the whole across all scales of magnification. In other words fractals display symmetry across scales. Fractals provide the mathematics necessary to describe the phase space portrait of chaotic systems. In general, fractal objects have a fractional dimension unlike traditional Euclidean shapes whose dimension is an integer. However, there are exceptions. The well-known Mandelbrot set has a fractal boundary with a fractal dimension equal to 2, the dimensions of an Euclidean plane. Interestingly this important fact was established only recently, in 1991, by Mitsuhiro Shishikura. Another fractal object which has a fractal dimension of 2 is the so called 'Skewed Web'. This is a three dimensional object which is an analogue of the famous Sierpinski's Gasket. In this context it may also be mentioned that the network of blood vessels is not only a fractal but is said to have a fractal dimension equal to 2. (Refer articles by Ian Stewart given in Suggested Reading).



The Koch curve is a good example of a fractal. The curve is constructed as follows. A line segment is taken and is divided into three parts. The middle one-third of the line segment is replaced by two line segments as shown in Step 2 of the figure. The above operation is now performed on the four line segments (Step 3). This process is carried on ad-infinitum. At the end of the process we get a fractal object called the Koch curve. The Koch curve has a dimension of 1.261... This is because the Koch curve is neither a line (dimension 1) nor a plane (dimension 2), but something in-between. It can be observed that the Koch curve is self-similar.



Generating the Koch curve starting from a line segment

The Discovery Of Chaos

Edward Lorenz, a meteorologist at MIT first discovered chaos in the early 60's. Lorenz wrote a system of equations that modelled the earth's weather and simulated its behaviour on his Royal MacBee computer. His computer spewed out a series of numbers, which represented various weather parameters of his model. One day wanting to examine a sequence at greater length, he took a short cut. Instead of starting the whole run again, he started mid-way through, by typing in the numbers from an earlier printout. When he examined the results, he discovered that his new weather patterns were diverging very rapidly from the patterns of the last run. Lorenz found out that the problem lay in the numbers he had typed. In the computer's memory, six decimal places were stored whereas only three appeared on the printout. An error of one part in a thousand had changed his weather patterns drastically. Lorenz called his discovery "the butterfly effect" - the notion that a butterfly flapping its wings in Bombay will set off a tornado in Japan a week later. Technically, the butterfly effect is called sensitive dependence on initial conditions, which is one of the hallmarks of chaos.

The advent of chaos introduces us to a new type of attractor - a *strange attractor* or a chaotic attractor. Geometrically a strange attractor is a *fractal* (see box on fractals), i.e. it reveals more detail as it is increasingly magnified. In strange attractors, arbitrarily nearby orbits diverge exponentially fast and so stay together for only a short time. Strange attractors (and hence chaos) are found in certain non-linear dynamical systems. Non-linear systems, unlike their linear counterparts, do not have closed form solutions. Hence one has to numerically simulate the behaviour of the non-linear system. Therefore, it is not surprising that the growth in the field of chaos has occurred hand-in-hand with the widespread availability of computing power.

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The original discovery of chaos, by meteorologist, E N Lorenz, (see box on discovery of chaos) in the following set of differential equations, is a good example of the role that computers play in chaos.

$$\begin{aligned} dx/dt &= 10y - 10x \\ dy/dt &= -xz + 28x - y \\ dz/dt &= xy - (8/3)z \end{aligned} \qquad (1)$$

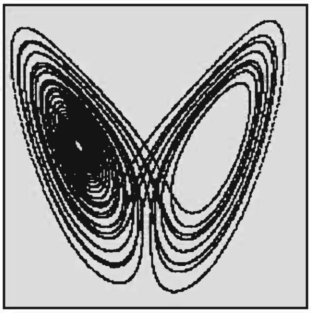

***Figure 2** Trajectory of the Lorenz attractor in the x-z axis, obtained by plotting equations (1).*

The trajectory of this system of equations projected on the x-z plane, is called the *Lorenz attractor, (Figure 2)*. The books by Gleick and Gulick discuss this attractor in greater detail.

Simplicity and Universality in Transition to Chaos

Chaos occurs even in the most deceptively simple systems. For example, it was observed that the following equation, called the *logistic equation* (see Robert May's article in *Nature*) captured the essence of Lorenz's system of equations.

$$x_{n+1} = \mu . x_n (1 - x_n) \qquad (2)$$
where $0 < x_n < 1$ and $1 < \mu < 4$.

The above equation is also a dynamical system, but the evolution of the system is in discrete time instead of continuous time. An initial value for x_n is chosen and is called x_0. Equation (2) then gives us the value of x_1. This simple calculation is repeated endlessly, feeding the output of one calculation as input for the next. (In computer parlance this process is called iteration.) In analysing the equation we are interested in finding out the behaviour of x_n as $n \to \infty$. In actual practice, we observe the behaviour of x_n after a few thousand iterations (to allow the transients to die down).

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When the parameter μ is less than three, for any initial condition (i.e. the value of x_0) between 1 and 3, the value of x_n converges to a fixed point or steady state (*Figure* 3a). As the value of the parameter μ is increased beyond three, the limit value of x_n oscillates between two values. For example, for a μ value of 3.2, the value of x_n keeps oscillating between the values 0.7994 and 0.5130. The system is said to have a period of two. The time-evolution of a period two system is depicted in the diagram in *Figure* 3b. At the parameter value of 3, the system is said to have undergone a *period-doubling bifurcation,* because its behaviour changes from steady state to that of period two.

As the parameter value is increased further, the system undergoes

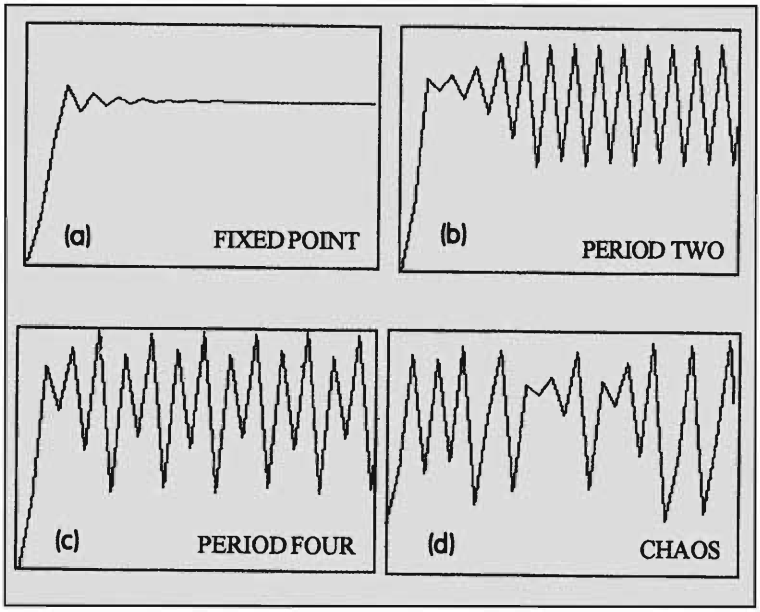


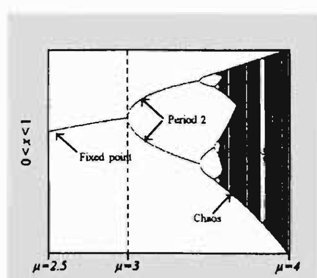
***Figure 3** Behaviour of $x_{n+1} = \mu\ x_n\ (1-x_n)$ for various μ values. The graph shows values of x_n (along the y axis) plotted against increasing values of x_n in the x axis. (a) when μ ranges between 1 and 3. (b) when μ ranges between 3 and 3.44 (approx.) (c) when μ ranges between 3.44 (approx.) and 3.54 (approx.) (d) graph for chaotic regime.*

successive period doublings. Period two gives way to period four, which gives way to period eight and so on. The behaviour of the system as the parameter μ is increased is shown in *Figure 4*, generated by the computer. Along the x-axis, the μ value increases from 2.5 to 4. The y-axis represents the long-term behaviour of the system — i.e. the value that x_n settles down to finally. It can be seen that the bifurcations occur faster and faster and suddenly break off. Beyond 3.57, the periodicity of the system gives way to chaos and long term values of x_n do not settle down at all. In the midst of this complexity, stable cycles of periods such as 3 or 7, suddenly return (called *periodic windows*), only to break off once again into chaos. In fact there is an interesting theorem, which states that in any one-dimensional system, if for some parameter value, the system has a period of three, then the same system (for some other parameters) will also display periods of all other values, as well as completely chaotic behaviour.

Chaos occurs even in the most deceptively simple systems.

The period-doubling cascade is one of the standard routes to chaos. It occurs in many non-linear systems that depict chaotic behaviour. For example, in the following dynamical system, period-doubling cascades occur before the system

***Figure 4** Bifurcation diagram of the equation $x_{n+1} = \mu . x_n\ (1 - x_n)$.*



becomes chaotic.

$$x_{n+1} = \mu \sin(\pi x_n) \quad (3)$$
$$\text{where } 0 < x_n < 1 \text{ and } 0 < \mu < 1.$$

Is there any similarity between systems that take the period-doubling route to chaos? The principle of universality, discovered by Feigenbaum, states that irrespective of the nature of the system, if it takes the period-doubling route to chaos, then the parameter spacings (the range of parameter values for which the period is the same) occur in geometric progression. Feigenbaum computed the ratio of convergence and the constant named after him, is one of the fundamental physical constants. Later, Feigenbaum's results were reformulated in a rigorous mathematical framework but the initial insights were provided by numerical simulations.

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Feigenbaum's principle of universality can be stated rigorously as follows. Let μ_0 be the parameter value for which the first period doubling occurs (in other words μ_0 is the parameter value at which period 1 gives way to period 2). Let μ_1, μ_2, μ_3 be the succeeding period doublings. Now we define F_k as,

$$F_k = [\mu_k - \mu_{k-1}] / [\mu_{k+1} - \mu_k] \quad \text{for } k = 2, 3, 4\ldots .$$

The sequence F_2, F_3, F_4 ... converges to a number F_∞ called *Feigenbaum constant.* F_∞ is the same for all systems taking the period doubling route to chaos.

In the next section, we take two dissimilar systems and compute the geometric convergence ratio of the parameter spacings and thereby verify the principle of universality.

Computer Assisted Verification of Universality

An algorithm for computing the parameter value at which a period doubling occurs is presented. The algorithm can be better understood from the accompanying flow chart *(Figure 5)*. The para-meter values at which the period-doublings occur can then be used to compute the parameter spacings and the spacing ratios.

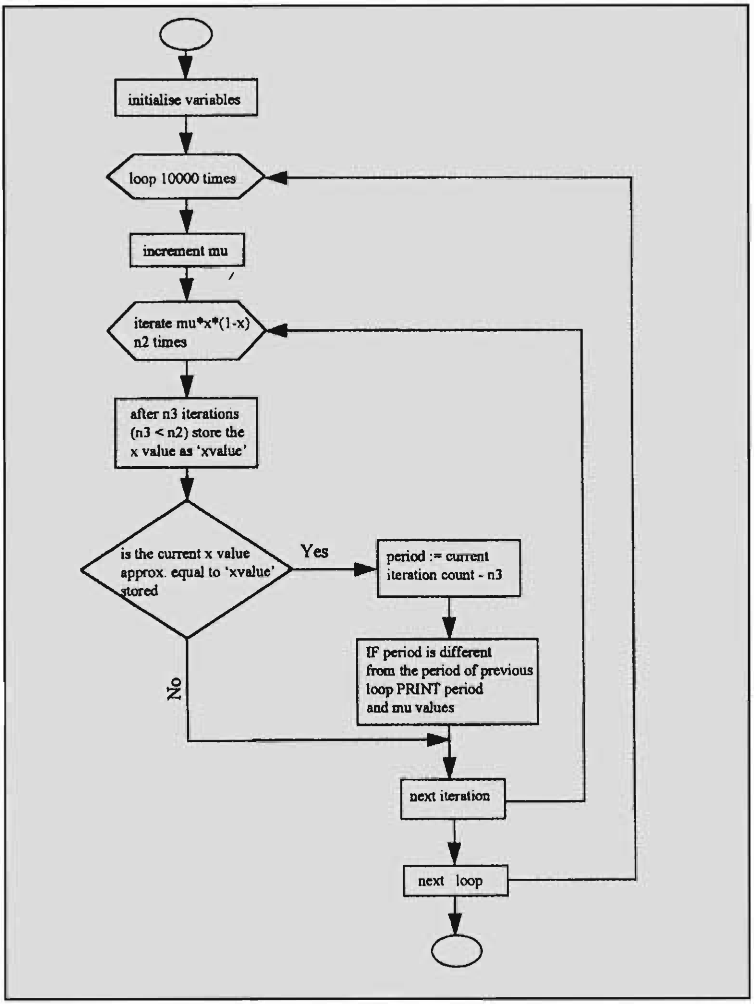

Figure 5 Flowchart for computing the parameter values for which period doubling occurs.

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The idea of the algorithm is to increment the parameter values in an outer loop and to iterate the equation in the inner loop. ABS is a function that returns the absolute value of the input value. RAND is a procedure that returns a random number between 0 and 1. The first few iterations (an arbitrary choice of 8000 iterations in the algorithm) are used to allow the transients to die down. At a fixed *count* value (8000 in the algorithm and n3 in the flow chart) the value of x is stored in the *x value* variable. In the subsequent iterations, we monitor if the value stored in *x value* arrives again. The *count* value for which *x value* arrives again less the *count* value at which *x value* was stored (8000 in this case) gives us the *period* of the system. If the current period (given by *period*)

```
/* Variables used :
       pcount, count          : loop counter variables
       prevperiod, period     : stores the period values
       r                      : stores the parameter values
       x, xvalue              : used to store the actual iteration values
       flag                   : boolean that determines when to exit out of the WHILE loop. */

  prevperiod, pcount, count, period : Integer;
  mu, x, xvalue : float;
  flag : boolean;
  prevperiod := 0;
  FOR pcount := 0 TO 10000          /* outer loop for incrementing parameter value */
       mu := 3.43 + 0.15/10000 * pcount;      /* increment the parameter value */
       count := 1; x := RAND; flag := TRUE;  /* initialise variables before inner loop */
       WHILE    count < 15000  AND  flag = TRUE  DO
                count := count + 1;
                x := mu * x * (1-x);          /* perform iteration of the system*/.
                IF count = 8000 THEN
                     xvalue := x;             /* store value of x at 8000th iteration */
                END IF
                IF count > 8000 THEN    /* monitor and see when xvalue  returns */
                     IF ABS(xvalue - x) < 0.00001 AND  x > 0 THEN
                          period := count - 8000;
                     ENDIF
                     IF period <> prevperiod THEN
                          PRINT period, mu;
                          prevPeriod := period;
                     END IF
                     flag := FALSE;
                END IF
       END WHILE
  NEXT pcount
```

The initial discovery of the universal constant by Feigenbaum was done using a pocket calculator.

is different from the previous period (stored in *prevperiod*), the period value (*period*) and the parameter value (r) is printed. The parameter spacing and Feigenbaum constant are then calculated manually.

The algorithm was applied to two dissimilar systems $x_{n+1} = \mu .\, x_n\,(1-x_n)$ and $x_{n+1} = \mu .\sin(\pi .\, x_n)$. The results obtained are in *Tables 1* and *2*.

The numerical studies indicate that for non-linear, one-dimensional systems, which take the period doubling route to chaos, the ratio of the spacing of the parameter values could be universal, in the sense that, for a wide class of systems, it is independent of the details of the equations. The above experiment provides a good metaphor for the role that computers have played in the field of chaos. In fact, the initial discovery of the universal

Table 1. Analysis of the logistic equation $x_{n+1} = \mu . x_n\ (1-x_n)$

Period#	starting μ value	parameter spacing	Feigenbaum constant
1	1.0000000000	2.0000000000	
2	3.0000000000	0.4494897427	4.44948
4	3.4494897427	0.0946228040	4.75033
8	3.5441125467	0.0203330686	4.65364
16	3.5644456153	0.0043638742	4.65940
32	3.5688094895	0.0009362636	4.66064
64	3.5697457531		

Table 2. Analysis of the sine function $x_{n+1} = \mu \sin(\pi x_n)$

Period#	starting μ value	parameter spacing	Feigenbaum constant
1	0.3000000000	0.4198154390	
2	0.7198154390	0.1134053575	3.70190
4	0.8332207965	0.0253723920	4.46963
8	0.8585931885	0.0054861570	4.62480
16	0.8640793455	0.0011784030	4.65558
32	0.8652577485	0.0002528400	4.66066
64	0.8655105885		

constant by Feigenbaum was done using a pocket calculator, using a procedure similar to the one used above.

Conclusion

The discovery of chaos has far reaching implications in many branches of science. Chaos has provided us with a new way of looking at nature. It has helped us to find order in places where we earlier found only disorder. In conclusion, two points are worth reiterating. First, chaos can occur in deceptively simple dynamical systems. Second, the computer is an essential and indispensable tool in studying chaotic dynamics.

Suggested Reading

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