Geometry

4. Curves vs Surfaces

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In the first three parts the author covered what can roughly be called 'classical' geometry. These are the aspects of geometry that all science and engineering graduates have at some time studied. Unfortunately the 'modern' material you will now encounter, which is a beautiful combination of ideas from algebra and analysis, has not entirely made it to the common curriculum. So arm yourselves with a paper and pencil[0] and begin your excursion.

Is a Curve Curved?

Curve: A one-dimensional locus of points[1].

Now that we have accepted coordinate geometry as the basis of our study, we need to re-examine earlier notions like lines and planes in this context. In some sense, the more natural notion is that of a curve rather than a line. Huygens, Leibnitz and Newton (independently) formulated the notion of curvature of a curve. (This was developed by Serret-Frenet into a multiplicity of invariants for curves in higher dimensions. We will concentrate on the curvature defined by Huygens et al). A line then becomes a curve of curvature zero.

It is always confusing to look at the dictionary meaning of a word when it is also a mathematical term. The word 'line' in English corresponds to the mathematical term 'curve', whereas the English word 'curve' is used in the sense of a line which is *not* straight. A 'straight line' (in English) is what one would call a 'line' in geometry. As we shall see it is not easy to distinguish a curve from a straight line intrinsically—thus we use the mathematical term curve to denote a one dimensional locus which *may or may not be* curved!

[0] The readers are encouraged to solve or at least attempt all exercises given since (to quote a famous mathematician) 'mathematics is not banana eating.'

[1] This is a fancy way of saying 'that which is traced by the tip of a moving finger'—which having writ moves on.

Since we will only be dealing with non-singular[2] loci we use the analytic definition that near each point $p_0 = (x_0, y_0, z_0)$ of the curve it is given in parametric form as $p(t) = (x(t), y(t), z(t))$ where these coordinates are given by regular functions of t near 0. Then the non-singularity of the locus translates to the non-vanishing of the velocity vector $(\dot{x}(t), \dot{y}(t), \dot{z}(t))$.

[2] Mathematicians are prone to describing things which are easy to study by nice names like 'regular' while those which are difficult are given epithets like 'singular'.

In order to understand how such a locus is curved, we travel along it at constant speed and check to see whether we experience any acceleration. This is in keeping with Newton's law that a body moves along a (straight) line at constant speed unless it is subjected to acceleration; thus Newton's law will turn out to be a geometric *definition* of a line. In symbols, let $p(t) = (x(t), y(t), z(t))$ be a parametrisation of the curve so that we have speed equal to a constant, i.e.

$$(\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2) = \text{constant independent of } t.$$

(Exercise: Those who know their calculus should be able to show that such a parametrisation always exists). The acceleration is then a vector perpendicular to the velocity

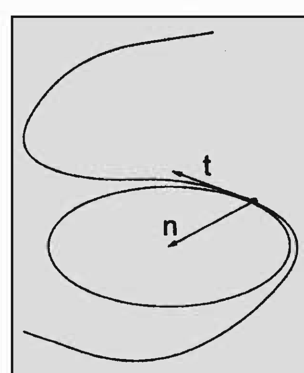
$$\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t) + \dot{z}(t)\ddot{z}(t) = 0.$$

(Exercise: Why does this equation hold?) The magnitude $k(t) = (\ddot{x}(t)^2 + \ddot{y}(t)^2 + \ddot{z}(t)^2)^{1/2}$ of the acceleration is called the curvature of the curve at the point $p(t)$. If one works this out for the circle

$$(x(t), y(t), z(t)) = (r\cos(t), r\sin(t), 0)$$

then we obtain $k(t) = 1/r$ (Exercise: Check this). This corroborates our intuition that a circle with a large radius is nearly a straight line, i.e. has curvature close to zero.

***Figure 1** The osculating circle. (t: tangent vector; n: normal vector.)*



A more geometric approach is as follows. Just as the tangent line at a point is given by the linear equation that approximates the curve upto terms of order 2 or more, the *osculating circle* is the circle that approximates the curve upto terms of order 3 or more (see *Figure 1*). The curvature of the curve at the point is defined to be the inverse of the radius of the osculating circle (here we allow a circle to *become* a line when its radius is infinite). It is not hard to show that the analytic definition coincides with the geometric one. This points the way to the higher order invariants of Serret-Frenet which are obtained by examining curves of higher degree which approximate the given one to even higher orders.

One problem is that this curvature is not *intrinsic*. An object that travels along the curve *without* interacting with the surroundings[3] will not observe the acceleration since the latter is perpendicular to the curve. One way of seeing this is to think of a signal moving in a TV cable or optical telephone fibre cable—the coiling or straightening of the cable has no effect on the signal. It is reasonably clear from this viewpoint that there is *no* intrinsic notion of curvature for a curve; curvature for a curve is determined by the manner in which the curve is embedded in (sits in) space—the beauty of a curve is truly in the eye of the beholder! To confront intrinsic curvature (and beauty) one must study higher dimensional loci.

[3] Like the arrow of Arjun that travelled to its target looking neither left nor right.

Curvature is Superficial!

> **Surface:** A two-dimensional locus of points or a one-dimensional locus of curves.

Analogous to the case of curves, the analytic definition of a surface is given by a vector valued function $p(u, v) = (x(u, v), y(u, v), z(u, v))$ such that the two vectors

$$\frac{\partial p(u,v)}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \text{ and } \frac{\partial p(u,v)}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

are linearly independent (i.e. one is not a multiple of the other). The study of curvature of such loci was initiated by Euler and Meusnier and carried to its "remarkable" conclusion (Theorema Egregium) by Gauss.

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We have already introduced the notion of curvature for any curve on the surface. We can expect that the curvature of a surface would be determined by understanding the curvature of the curves on it. In particular, as Euler did, one may consider a curve on the surface which appears 'straight' on the surface, i.e. so that the acceleration experienced while travelling along this curve is perpendicular to the surface. These curves describe the distance minimising paths between points on the surface (at least in a small region). One may think about a person walking along a 'straight line' on the surface of the earth and travelling at constant speed; no acceleration is required (assuming that there is no friction) but of course there is an unnoticed acceleration due to gravity which holds the person to the surface of the earth. Because of this description such curves are called *geodesics*. Clearly the curvature of geodesics would be directly related to the curvature of the surface; in fact Meusnier showed that the curvature of other curves on the surface can be easily determined once we know the curvature of geodesics.

If we have two surfaces and an identification between the two so that distances are preserved, then the curvatures must also be the same. This is why there can never be a perfect map of the surface of the earth or a perfect astronomical chart.

Each geodesic emanating from a point p is determined by its initial velocity, which is a linear combination of the basic tangent vectors,

$$\mathbf{t} = a\frac{\partial p}{\partial u} + b\frac{\partial p}{\partial v} = a\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) + b\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

Fix a vector $\mathbf{n}$ of unit length that is perpendicular to all these vectors. The acceleration experienced on a geodesic along $\mathbf{t}$ is then $k(a,b)\mathbf{n}$ for some scalar function $k(a,b)$ of the two parameters determining this vector. The result proved by Euler was that $k(a,b) = La^2 + 2Mab + Nb^2$. Normalising $k(a,b)$ by the square of the length of the vector $\mathbf{t}$ we obtain a function of the direction alone. Let k_1 be the maximum normalised value of $k(a,b)$ so obtained, say along $\mathbf{t}_1$. Let $\mathbf{t}_2$ be orthogonal to $\mathbf{t}_1$ so that $\mathbf{t}_1 \times \mathbf{t}_2 = \mathbf{n}$ and let k_2 be the normalised value of $k(a,b)$ in this direction; it can be shown that this is the minimum normalised value. The numbers k_1 and k_2 were called the principal curvatures by Euler.

Gauss (who was perhaps inspired by astronomy or by the cartographic surveys he was carrying out for the ruler of Germany) gave a new interpretation to Euler's theory. First consider the length of the vector **t** considered as a function $g(a,b)$ of the two parameters. This has the form $g(a,b) = E(u,v)\,a^2 + 2\,F(u,v)\,ab + G(u,v)b^2$ and determines distances along paths on the surface—hence it is intrinsic. The second fundamental form is $k(a,b)$ introduced above. The curvature $K = k_1\,k_2$ (now called Gaussian curvature in his honour) is then the ratio of the discriminants of these two forms

$$K = \frac{LN - M^2}{EG - F^2}$$

Even an ant crawling along a surface (or a cartographer in the days before aerial travel) can determine the curvature. Hence the beauty of a surface is skin deep and yet is naturally associated with it!

The key results are the following:

- The Gaussian curvature can be expressed in terms of E, F, G and their partial derivatives alone. Hence it is an *intrinsic* invariant.
- The integral of the Gaussian curvature on a triangle whose sides are geodesics is [(the sum of the angles of the triangle}) – π] x (the area of the triangle).

  Hence again it is *intrinsic*.

Exercise: What are k_1 and k_2 for a cylinder of radius r? As a more challenging exercise the reader is invited to check that the points A, B, C on the cycle tube pictured in *Figure 2* exhibit positive, zero and negative Gaussian curvature respectively.

What is new and remarkable here is that the (Gaussian) curvature is an *intrinsic* invariant. In other words, if we have two surfaces and an identification between the two so that distances are preserved, then the curvatures must also be the same. This is why there can never be a perfect map of the surface of the earth or a perfect astronomical chart—a curved surface cannot be identified with a flat one in a manner that preserves distances. The second result of Gauss allows one to compute the average value of curvature in a small region of the surface. Thus even an ant

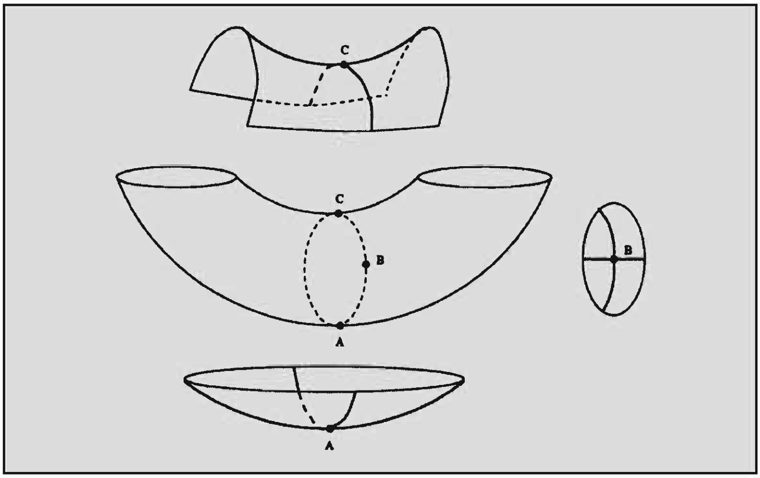

Figure 2 Curvature on a piece of cycle tube. (The thick lines denote the geodesics of extremal curvature.)

crawling along a surface (or a cartographer in the days before aerial travel) can determine the curvature. Hence the beauty of a surface is skin deep and yet is naturally associated with it!

Summary

While curvature is a property which seems to be emanating out of curves, these are in fact not intrinsically curved—the curvature of a curve is entirely the result of how the curve lies in space. However, the intrinsic curvature of a surface (the Gaussian curvature) is a quantity that can be measured without reference to the 'outside'. This curvature relates to the sum of angles of a triangle which is closely related to the parallel postulate.

In spite of all the positive features of Gaussian curvature it has one major drawback which is that its definition is not intrinsic since it depends on the extrinsic curvatures of curves contained in the surface. However, since it is an intrinsic invariant one should expect an intrinsic definition. Further, it is not clear what the correct analogue in higher dimensions should be (in particular in the all important dimension 3). Historically, it was philosophically unacceptable to compute invariants for space by postulating a higher dimensional universe in which it is contained. It was Riemann under the insistence of Gauss who produced the definitive answer which we shall study next time.

Suggested Reading

A very good general introduction to Differential Geometry can be found in the following books:

N J Hicks. Notes on Differential Geometry. Van Nostrand. 1965.

M Spivak. Differential Geometry. Vol. II. Publish or Perish, Berkeley. USA. 1970.

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