On the Infinitude of the Prime Numbers

Euler's Proof

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Euclid's elegant proof that there are infinitely many prime numbers is well known. Euler proved the same result, in fact a stronger one, by *analytical* methods. This article gives an exposition of Euler's proof introducing the necessary concepts along the way.

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Introduction

In this article, we present Euler's very beautiful proof that there are infinitely many prime numbers. In an earlier era, Euclid had proved this result in a simple yet elegant manner. His idea is easy to describe. Denoting the prime numbers by $p_1, p_2, p_3, \ldots$, so that $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$, he supposes that there are n primes in all, the largest being p_n. He then considers the number N where

$$N = p_1 p_2 p_3 \cdots p_n + 1,$$

and asks what the prime factors of N could be. It is clear that N is indivisible by each of the primes $p_1, p_2, p_3, \ldots, p_n$ (indeed, $N \equiv 1 \pmod{p_i}$ for each i, $1 \le i \le n$). Since every integer greater than 1 has a prime factorization, this forces into existence prime numbers other than the p_i. Thus there can be no largest prime number, and so the number of primes is infinite.

The underlying idea of Euler's proof is very different from that of Euclid's proof. In essence, he proves that the *sum of the reciprocals of the primes is infinite*; that is,

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots = \infty.$$

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In technical language, the series $\Sigma_i 1/p_i$ *diverges*. Obviously, this cannot possibly happen if there are only finitely many prime numbers. The infinitude of the primes thus follows as a corollary. Note that Euler's result is stronger than Euclid's.

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Convergence and Divergence

A few words are necessary to explain the concepts of convergence and divergence of infinite series. A series $a_1 + a_2 + a_3 + \cdots$ is said to *converge* if the sequence of partial sums,

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \cdots,$$

approaches some limiting value, say L; we write, in this case, $\sum_1^\infty a_i = L$. If, instead, the sequence of partial sums grows without any bound, we say that the series *diverges*, and we write, in short, $\sum_1^\infty a_i = \infty$.

A statement of the form $\Sigma a_i = \infty$ is to be regarded as merely a short form for the statement that the sums a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, ..., do not possess any limit. It is important to note that ∞ is *not* to be regarded as a number! We shall however frequently use phrases of the type '$x = \infty$' (for various quantities x) during the course of this article. The meaning should be clear from the context.

Examples:

- The series $1/1 + 1/2 + 1/4 + \cdots + 1/2^n + \cdots$ converges (the sum is 2, as is easily shown).
- The series $1/1 + 1/3 + 1/9 + \cdots + 1/3^n + \cdots$ converges (the sum in this case is 3/2).
- The series $1+1+1+\cdots$ diverges (rather trivially).
- The series $1-1+1-1+1-1+1-\cdots$ also fails to converge, because the partial sums assume the values $1, 0, 1, 0, 1, 0, \ldots$, and this sequence clearly does not possess a limit.
- A more interesting example: $1 - 1/2 + 1/3 - 1/4 + \ldots$; a careful analysis shows that it too is convergent, the limiting sum being ln 2 (the natural logarithm of 2)

Divergence of the Harmonic Series $\Sigma\, 1/i$

In order to prove Euler's result, namely, the divergence of $\Sigma\, 1/p_i$, we need to establish various subsidiary results. Along the way we shall meet other examples of divergent series. To start with, we present the proof of the statement that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots = \infty.$$

This rather non-obvious result is usually referred to as *the divergence of the harmonic series.* The proof given below is due to the Frenchman Nicolo Oresme and it dates to about 1350. We note the following sequence of equalities and inequalities:

$$\frac{1}{1} = \frac{1}{1},$$

$$\frac{1}{2} = \frac{1}{2},$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2},$$

$$\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16} = \frac{1}{2},$$

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and so on. This shows that it is possible to group consecutive sets of terms of the series $1/1 + 1/2 + 1/3 + \cdots$ in such a manner that each group has a sum exceeding 1/2. Since the number of such groups is infinite, it follows that the sum of the whole series is itself infinite. (Note the crisp and decisive nature of the proof!)

Based on this proof, we make a more precise statement. Let $S(n)$ denote the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

e.g., $S(3) = 11/6$. Generalizing from the reasoning just used, we find that

$$S(2^n) > 1 + \frac{n}{2}. \tag{3.1}$$

(Please fill in the details of the proof on your own.) This means that by choosing n to be large enough, the value of $S(2^n)$ can be made to exceed any given bound. For instance, if we wanted the sum to exceed 100, then (3.1) assures us that a mere 2^{198} terms would suffice! This suggests the extreme slowness of growth of $S(n)$ with n. Nevertheless it does grow without bound; loosely stated, $S(\infty) = \infty$.

The result obtained above, (3.1), can also be written in the form,

$$S(n) > 1 + \frac{1}{2}\log_2 n.$$

Exercise: Write out a proof of the above inequality.

A much more accurate statement can be made, but it involves calculus. We consider the curve Ω whose equation is $y = 1/x$, $x > 0$. The area of the region enclosed by Ω, the x-axis and the ordinates $x = 1$ and $x = n$ is equal to $\int_1^n \frac{1}{x}\,dx$, which simplifies to $\ln n$. Now let the region be divided into $(n-1)$ strips of unit width by the lines $x = 1, x = 2, x = 3, \ldots, x = n$ (see *Figure 1*).

The growth of $S(n)$ with n is extremely slow. For the sum to exceed 100 we would require 2^{198} terms!

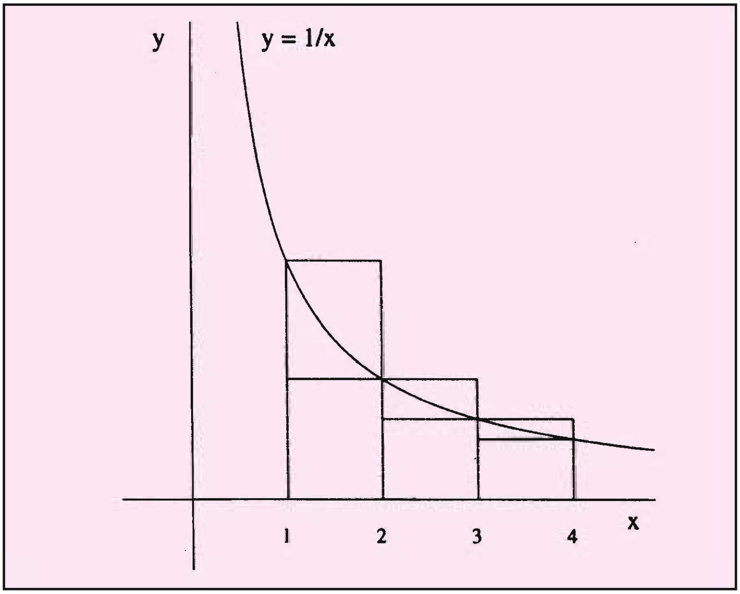

***Figure 1** The figure shows how to bound ln n by observing that ln n is the area enclosed by the curve y = 1/x, the x-axis and the ordinates x = 1 and x = n.*

Consider the region enclosed by Ω, the x-axis, and the lines $x = i - 1, x = i$. The area of this region lies between $1/i$ and $1/(i-1)$, because it can be enclosed between two rectangles of dimensions $1 \times 1/i$ and $1 \times 1/(i-1)$, respectively. (A quick examination of the graph will show why this is true.) By letting i take the values $2, 3, 4, \ldots, n$, and adding the inequalities thus obtained, we find that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1}. \quad (3.2)$$

Relation (3.2) implies that

$$\ln n + \frac{1}{n} < \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n + 1 \quad (3.3)$$

and this means that we have an estimate for $S(n)$ (namely, $\ln n + 0.5$) that differs from the actual value by no more than 0.5. A still deeper analysis shows that for large values of n, an excellent approximation for $S(n)$ is $\ln n + 0.577$, but we shall not prove this result here. It is instructive, however, to check the accuracy of this estimate. Write $f(n)$ for $\ln n + 0.577$. We now find the following:

In general, when mathematicians find that a series $\sum a_i$ diverges, they are also curious to know how *fast* it diverges.

$n =$	10	100	1000	10000	100000
$S(n) =$	2.92897	5.18738	7.48547	9.78761	12.0902
$f(n) =$	2.87959	5.18217	7.48476	9.78734	12.0899

The closeness of the values of $f(n)$ and $S(n)$ for large values of n is striking. (The constant 0.577 is related to what is known as the Euler-Mascheroni constant.)

In general, when mathematicians find that a series Σa_i diverges, they are also curious to know how *fast* it diverges. That is, they wish to find a function, say $f(n)$, such that the ratio $(\sum_1^n a_i) / f(n)$ tends to 1 as $n \to \infty$. For the harmonic series $\Sigma 1/i$, we see that one such function is given by $f(n) = \ln n$. This is usually expressed by saying that the harmonic series diverges like the logarithmic function. We note in passing that this is a very slow rate of divergence, because ln n diverges more slowly than n^{ε} for any $\varepsilon > 0$, *no matter how small* ε *is*, in the sense that $\ln n / n^{\varepsilon} \to 0$ as $n \to \infty$ for every $\varepsilon > 0$. Obviously the function ln ln n diverges still more slowly.

Exercise: Prove that if $a > 1$, then the series

$$\frac{1}{1^a} + \frac{1}{2^a} + \frac{1}{3^a} + \cdots$$

converges. (The conclusion holds no matter how close a is to 1, but it does not hold for $a = 1$ or $a < 1$, a curious state of affairs!) Further, use the methods of integral calculus (and the fact that for $a \neq 1$, the integral of $1/x^a$ is $x^{(1-a)}/(1-a)$ to show that the sum of the series lies between $1/(a-1)$ and $a/(a-1)$.

The fact that the sum $1/1 + 1/2^2 + 1/3^2 + \cdots$ is finite can be shown in another manner that is both elegant and elementary. We start with the inequalities, $2^2 > 1 \times 2$, $3^2 > 2 \times 3$, $4^2 > 3 \times 4$, $\cdots$, and deduce from these that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots$$

The sum on the right side can be written in the form,

$$1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots, \qquad (3.4)$$

which (after a whole feast of cancellations) simplifies to $1 + 1/1$, that is, to 2. (This is sometimes described by stating that the series 'telescopes' to 2.) Therefore the sum $1 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots$ is less than 2. We now call upon a theorem of analysis which states that if the partial sums of any series form an increasing sequence and are at the same time bounded, that is, they do not exceed some fixed number, then they possess a limit. We conclude, therefore, that the series $\Sigma\, 1/i^2$ does possess a finite sum which lies between 1 and 2.

The fact that the sum $1/1^2 + 1/2^2 + 1/3^2 + \ldots$ is finite can be shown in a manner that is both elegant and elementary.

The divergence of the harmonic series was independently proved by Johann Bernoulli in 1689 in a completely different manner. His proof is worthy of deep study, as it shows the counter-intuitive nature of infinity.

Bernoulli starts by assuming that the series $1/2 + 1/3 + 1/4 + \cdots$ (note that he starts with 1/2 rather 1/1) does have a finite sum, which he calls S. He now proceeds to derive a contradiction in the following manner. He rewrites each term occurring in S thus:

$$\frac{1}{3} = \frac{2}{6} = \frac{1}{6} + \frac{1}{6}, \quad \frac{1}{4} = \frac{3}{12} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12}, \cdots,$$

and more generally,

$$\frac{1}{n} = \frac{n-1}{n(n-1)} = \frac{1}{n(n-1)} + \frac{1}{n(n-1)} + \cdots + \frac{1}{n(n-1)},$$

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with $(n-1)$ fractions on the right side. Next he writes the resulting fractions in an array as shown below:

1/2	1/6	1/12	1/20	1/30	1/42	1/56	...
	1/6	1/12	1/20	1/30	1/42	1/56	...
		1/12	1/20	1/30	1/42	1/56	...
			1/20	1/30	1/42	1/56	...
				1/30	1/42	1/56	...
					1/42	1/56	...
						1/56	...

Note that the column sums are just the fractions $1/2, 1/3, 1/4, 1/5, \ldots$; thus S is the sum of all the fractions occurring in the array. Bernoulli now sums the rows using the telescoping technique used above (see equation (3.4)). Assigning symbols to the row sums as shown below,

$$A = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \cdots,$$

$$B = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \cdots,$$

$$C = \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \cdots,$$

$$D = \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \cdots,$$

he finds that:

$$A = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$$
$$= 1,$$

$$B = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots$$
$$= \frac{1}{2},$$

$$C = \frac{1}{3}, \qquad \text{(arguing likewise)},$$

$$D = \frac{1}{4},$$

Bernoulli's proof is worthy of deep study, as it shows the counter-intuitive nature of infinity.

and so on. Thus the sum S, which we had written in the form $A + B + C + D + \cdots$, turns out to be equal to

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \quad .$$

Now this looks disappointing -- just as things were beginning to look promising! We seem to have just recovered the original series after a series of very complicated steps. But in fact something significant has happened: *an extra* '1' *has entered the series.* At the start we had defined S to be $1/2 + 1/3 + 1/4 + \cdots$; **now we find that** S **equals** $1 + 1/2 + 1/3 + 1/4 + \cdots$. **This means that** $S = S + 1$. **However, no finite number can satisfy such an equation. Conclusion:** $S = \infty$!

There are many other proofs of this beautiful result, but I shall leave you with the pleasant task of coming up with them on your own. Along the way you could set yourself the task of proving that each of the following sums diverge:

- $1/1 + 1/3 + 1/5 + 1/7 + 1/9 + \cdots$;
- $1/1 + 1/11 + 1/21 + 1/31 + 1/41 + \cdots$;
- $1/a + 1/b + 1/c + 1/d + \cdots$, where $a, b, c, d, \ldots$, are the successive terms of any increasing arithmetic progression of positive real numbers.

Elementary Results

The fundamental theorem of arithmetic states that every positive integer greater than one can be expressed in precisely one way as a product of prime numbers.

The next result that we shall need is the so-called fundamental theorem of arithmetic: *every positive integer greater than 1 can be expressed in precisely one way as a product of prime numbers.* We shall not prove this very basic theorem of number theory. For a proof, please refer to any of the well-known texts on number theory, e.g., the text by Hardy and Wright, or the one by Niven and Zuckermann.

We shall also need the following rather elementary results: (i) if k is any integer greater than 1, then

$$\frac{1}{1 - 1/k} = 1 + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \frac{1}{k^4} + \cdots,$$

which follows by summing the geometric series on the right side, and (ii) if a_i, b_j are any quantities, then

$$\left(\sum_i a_i\right)\left(\sum_j b_j\right) = \sum_{i,j} a_i b_j,$$

where, in the sum on the right, each pair of indices (i, j) occurs *precisely once.*

Now consider the following two equalities, which are obtained from (4.1) using the values $k = 2, k = 3$:

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots,$$

$$\frac{1}{1-\frac{1}{3}} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots.$$

We multiply together the corresponding sides of these two equations. On the left side we obtain $2 \times 3/2 = 3$. On the right side we obtain the product

$$(1 + 1/2 + 1/2^2 + 1/2^3 + \cdots) \times (1 + 1/3 + 1/3^2 + 1/3^3 + \cdots)$$

Expanding the product, we obtain:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

$$+ \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots + \frac{1}{18} + \frac{1}{36} + \frac{1}{72} + \ldots,$$

that is, we obtain the sum of the reciprocals of all the positive integers that have only 2 and 3 among their prime factors. The fundamental theorem of arithmetic assures us that each such integer occurs *precisely once* in the sum on the right side. Thus we obtain a nice corollary: if A denotes the set of integers of the form $2^a 3^b$, where a and b are non-negative integers, then

$$\sum_{z \in A} \frac{1}{z} = 3.$$

We obtain a nice corollary: if A denotes the set of integers of the form $2^a 3^b$, where a and b are non-negative integers, then $\sum_{z \in A} 1/z = 3$.

If we multiply the left side of this relation by $(1 + 1/5 + 1/5^2 + 1/5^3 + \cdots)$ and the right side by $3/(1-1/5)$, we obtain the following result:

$$\sum_{z \in B} \frac{1}{z} = \frac{3}{1-\frac{1}{5}} = \frac{15}{4},$$

where B denotes the set of integers of the form $2^a\,3^b\,5^c$, where a, b and c denote non-negative integers.

Continuing this line of argument, we see that infinitely many such statements can be made, for example:

- If C denotes the set of positive integers of the form $2^a\,3^b\,5^c\,7^d$, where a, b, c and d are non-negative integers, we then have $\Sigma_{z \in C}\, 1/z = (15/4)\,(7/6) = 35/8$.
- If D denotes the set of positive integers of the form $2^a\,3^b\,5^c\,7^d\,11^e$, then $\Sigma_{z \in D}\, 1/z = (35/8)\,(11/10) = 77/16$.

Infinitude of the Primes

Euler was capable of stunning reasoning; some of the steps in his proofs are so daring that they would leave today's mathematicians gasping for breath.

Suppose now that there are only finitely many primes, say $p_1, p_2, p_3, \ldots, p_n$, where $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$. We consider the product

$$\frac{1}{1-1/2}\;\frac{1}{1-1/3}\;\frac{1}{1-1/5}\;\cdots\;\frac{1}{1-1/p_n}$$

This is obviously a finite number, being the product of finitely many non-zero fractions. Now this product also equals

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)\times\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right)\times$$
$$\left(1+\frac{1}{5}+\frac{1}{5^2}+\cdots\right)\times\;\cdots\;\times\left(1+\frac{1}{p_n}+\frac{1}{p_n^2}+\cdots\right)$$

When we expand out this product, we find, by continuing the line of argument developed above, that we obtain *the sum of the reciprocals of all the positive integers*. To see why, we need to use the fundamental theorem of arithmetic and the assumption that 2, 3, 5, …, p_n are *all* the primes that exist; these two statements together imply that every positive integer can be expressed *uniquely* as a product of non-negative powers of the n primes 2, 3, 5, … p_n. From this it follows that the expression on the right side is precisely the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots,$$

written in some permuted order. But by the Oresme-Bernoulli theorem, the latter sum is infinite! So we have a contradiction: the finite number

$$\frac{1}{1 - 1/2}\ \frac{1}{1 - 1/3}\ \frac{1}{1 - 1/5}\ \cdots\ \frac{1}{1 - 1/p_n}$$

has been shown to be infinite -- an absurdity! The only way out of this contradiction is to drop the assumption that there are only finitely many prime numbers. Thus we reach the desired objective, namely, that of proving that there are infinitely many prime numbers.

It is not for nothing that Euler is at times referred to as analysis incarnate!

Note that, as a bonus, there are several formulas that drop out of this analysis, more or less as corollaries. For instance, we find that

$$\frac{1}{1 - 1/2^2}\ \frac{1}{1 - 1/3^2}\ \frac{1}{1 - 1/5^2}\ \cdots = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

that is, the infinite product and the infinite sum both converge to the same (finite) value. By a stunning piece of reasoning, including a few daring leaps that would leave today's mathematicians gasping for breath, Euler showed that both sides of the above equation are equal to $\pi^2/6$. Likewise, we find that

$$\frac{1}{1 - 1/2^4}\ \frac{1}{1 - 1/3^4}\ \frac{1}{1 - 1/5^4}\ \cdots = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots,$$

and this time both sides converge to $\pi^4/90$. Euler proved all this and much much more; it is not for nothing that he is at times referred to as *analysis incarnate!*

The Divergence of $\Sigma\, 1/p$

As mentioned earlier, Euler showed in addition that the sum

$$\sum_{i \geq 1} \frac{1}{p_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$$

is itself infinite. We are now in a position to obtain this beautiful result. For any positive integer $n \geq 2$, let P_n denote the set of prime numbers less than or equal to n. We start by showing that

$$\prod_{p \in P_n} \frac{1}{1 - 1/p} > \sum_{j=1}^{n} \frac{1}{j}. \tag{6.1}$$

Our strategy will be a familiar one. We write down the following inequality for each $p \in P_n$, which follows from equation (4.1):

$$\frac{1}{1 - 1/p} > 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}.$$

The '>' sign holds because we have left out all the positive terms that follow the term $1/p^n$. Multiplying together the corresponding sides of all these inequalities ($p \in P_n$), we obtain:

$$\prod_{p \in P_n} \frac{1}{1 - 1/p} > \prod_{p \in P_n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)$$

When we expand out the product on the right side, we obtain a sum of the form $\Sigma_{j \in A}\, 1/j$ for some set of positive integers A. This set certainly includes all the integers from 1 to n because the set P_n contains all the prime numbers between 1 and n. Inequality (6.1) thus follows immediately.

Next, we already know (see equation (3.3)) that

$$\sum_{j=1}^{n} \frac{1}{j} > \ln n + \frac{1}{n} > \ln n\,. \tag{6.2}$$

Leonhard Euler

Since universities were not the major research centres in his days, Leonhard Euler (1707-1783) spent most of his life with the Berlin and Petersburg Academies. Pious, but not dogmatic, Euler conducted prayers for his large household, and created mathematics with a baby on his lap and children playing all around. Euler withheld his own work on calculus of variations so that young Lagrange (1736-1813) could publish it first, and showed similar generosity on many other occasions. Utterly free of false pride, Euler always explained how he was led to his results saying that "the path I followed will perhaps be of some help". And, indeed, generations of mathematicians followed Laplace's advice: "Read Euler, he is our master in all!".

Combining (6.1) and (6.2), we obtain the following inequality:

$$\prod_{p \in P_n} \frac{1}{1 - 1/p} > \ln n.$$

Taking logarithms on both sides, this translates into the statement,

$$\sum_{p \in P_n} \ln \left(\frac{1}{1 - 1/p} \right) > \ln \ln n. \tag{6.3}$$

Our task is nearly over. It only remains to relate the sum $\Sigma_{p \in P_n} 1/p$ with the sum on the left side of (6.3). We accomplish this by showing that the inequality

$$\frac{7x}{5} > \ln \frac{1}{1-x} \tag{6.4}$$

holds for $0 < x \leq 1/2$.

To see why (6.4) is true, draw the graph of the curve Γ whose equation is $y = \ln(1/(1-x))$, over the domain $-\infty < x < 1$, (see *Figure 2*). Note that Γ passes through the origin and is convex over

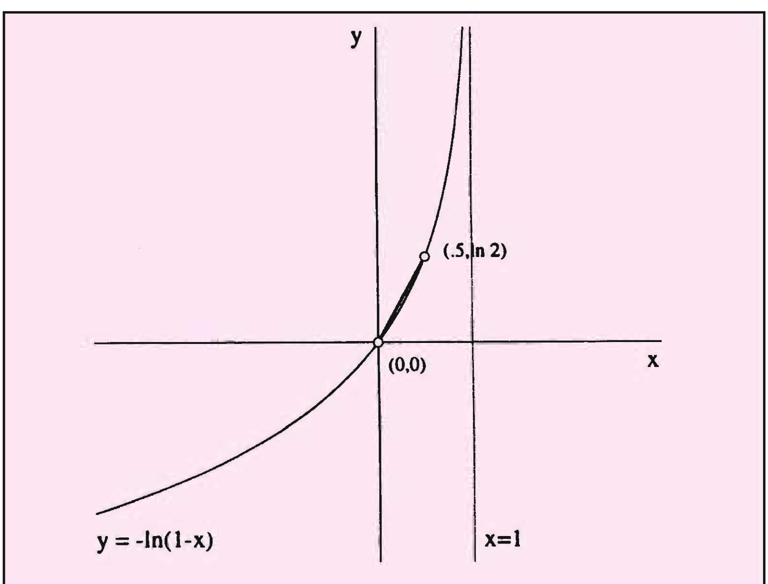

***Figure 2** The graph shows that for $0 \leq x \leq 1/2$, we have $(2 \ln 2)x \geq \ln(1/(1-x))$.*

Euler's Prodigious Output

It has been estimated that Euler's 886 works would fill 80 large books. Dictating or writing on his slate, Euler kept up his unparalleled output all through his life. Though totally blind for the last 17 years of his life, he promised to supply the Petersburg Academy with papers until 20 years after his death; one came out 79 years after he died! The most prolific mathematician in history died while playing with his grandchildren and drinking tea. (*All boxed notes on Euler taken from the IBM poster* Men of Modern Mathematics 1966.)

its entire extent. (Proof: Write $f(x) = -\ln(1-x)$; then $f'(x) = 1/(1-x)$ and $f''(x) = 1/(1-x)^2 > 0$ for all $x < 1$.)

The convexity of Γ implies that the chord joining the points $A(0,0)$ and $B(1/2, \ln 2)$ lies completely *above* the curve. The equation of AB is $y = (2\ln 2)\,x$, so over the range $0 \le x \le 1/2$ we have the inequality:

$$(2\ \ln\ 2)\,x \ge \ln\left(\frac{1}{1-x}\right).$$

Since $\ln 2 \approx 0.69315 < 0.7 = 7/10$, (6.4) follows.

Inequality (6.4) implies that

$$x > \frac{5}{7}\ln\left(\frac{1}{1-x}\right)$$

for $x = 1/2, x = 1/3, x = 1/5, \ldots$. Therefore, by addition,

$$\sum_{p \in P_n} \frac{1}{p} > \frac{5}{7}\left(\sum_{p \in P_n} \ln \frac{1}{1 - 1/p}\right). \qquad (6.5)$$

Combining (6.3) and (6.5), we deduce that

$$\sum_{p \in P_n} \frac{1}{p} > \frac{5}{7} \ln\ \ln\ n\,.$$

As $n \to \infty$, the right side diverges to infinity, therefore so does the left side, so we reach our desired objective, that of showing the divergence of $\Sigma_i\, 1/p_i$.

An Alternate Proof

Here is an alternate proof of the claim that $\Sigma_i\, 1/p_i$ diverges. The proof has been written in an 'old-fashioned' style and purists will protest. Nevertheless, we shall present the proof and let readers

The Genius of Euler

Euler's work on the zeta function, partitions and divisor sums remind us that he founded analytic number theory, while his creation of the theory of residues of powers and his proof of Fermat theorems are permanent contributions to elementary number theory. Various Euler equations establish his claims to mechanics, calculus of variations and hydrodynamics (where he gave the Lagrangian form as well as the Eulerian). The systematic theory of continued fractions is his, as is a major method in divergent series — justified, a century later, by analytic continuation. Analytic trigonometry, quadric surfaces, theory of investment and annuities, and linear differential equations with constant coefficients are among the many elementary subjects whose present form is chiefly due to Euler.

judge for themselves. Let S denote the sum $\Sigma_i 1/p_i$. We shall make use of the following result:

$$e^x \geq 1 + x \quad \text{for all real values of } x,$$

with equality holding precisely when $x = 0$. The graphs of e^x and $1+x$ show why this is true; the former graph is convex over its entire extent (examine the second derivative of e^x to see why), while the latter, a line, is tangent to the former at the point (0,1), and lies entirely below it everywhere else. Substituting the values $x = 1/2, x = 1/3, x = 1/5, \ldots$, successively into this inequality, we find that

$$e^{1/2} > 1 + \frac{1}{2},\ e^{1/3} > 1 + \frac{1}{3},\ e^{1/5} > 1 + \frac{1}{5},\ \cdots .$$

Multiplying together the corresponding sides of these inequalities, we obtain:

$$e^S > \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\cdots .$$

The infinite product on the right side yields the following series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \cdots .$$

This series is the sum of the reciprocals of all the positive integers whose prime factors are all distinct; equivalently, the positive integers that have no squared factors. These numbers are sometimes referred to as the *quadratfrei* or *square-free* numbers. Let Q denote this sum. We shall show that this series itself diverges, in other words, that $Q = \infty$. This will immediately imply that $S = \infty$ (for $e^S > Q$), and Euler's result will then follow.

We consider the product

$$Q \times \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

This product, when expanded out, gives the following series:

Johann Bernoulli

Johann Bernoulli (1667-1748), the younger brother of Jakob Bernoulli (1654-1705) took it upon himself to spread Leibniz's calculus across the European continent. Johann's proof of the divergence of the harmonic series first appeared (1689) in Jakob's treatise and with uncharacteristic fraternal affection, Jakob even prefaced the argument with an acknowledgement of his brother's priority.

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots,$$

that is, we obtain the harmonic series. To see why, note that every positive integer n can be *uniquely* written as a product of a square-free number and a square; for example, $1000 = 10 \times 10^2$, $2000 = 5 \times 20^2$, $1728 = 3 \times 24^2$, and so on. Now when we multiply

$$\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\frac{1}{11}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\cdots\right)$$

with

$$\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)$$

we find, by virtue of the remark just made, that the reciprocal of each positive integer n occurs *precisely once* in the expanded product. This explains why the product is just the harmonic series. Now recall that the sum

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

is finite (indeed, we have shown that it is less than 2). It follows that

$$Q \times (\text{some finite number}) = \infty .$$

Therefore $Q = \infty$, and Euler's result ($\Sigma_i 1/p_i = \infty$) follows. QED!

Readers who are unhappy with this style of presentation, in which ∞ is treated as an ordinary real number, will find it an interesting (but routine) exercise to rewrite the proof to accord with more exacting standards of rigour and precision.

Conclusion

A much deeper – but also more difficult – analysis shows that the sum $1/p_1 + 1/p_2 + 1/p_3 + \ldots + 1/p_n$ is approximately equal to

Suggested Reading

G H Hardy, E M Wright. An Introduction to the Theory of Number. 4th ed., Oxford, Clarendon Press, 1960.

Ivan Niven, Herbert S Zuckermann. An Introduction to the Theory ofNumbers. Wiley Eastern Ltd., 1989.

Tom Apostol. An Introduction to Analytic Number Theory. Narosa Publishing House. 1979.

XVI. *Summa ſeriei infinitæ harmonicè progreſſionalium*, $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ *&c. eſt infinita.*

Id primus deprehendit Frater: inventa namque per præced. ſumma ſeriei $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}$, &c. viſurus porrò, quid emergeret ex iſta ſerie, $\frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \frac{5}{30}$, &c. ſi reſolveretur methodo Prop. XIV. collegit propoſitionis veritatem ex abſurditate manifeſta, quæ ſequeretur, ſi ſumma ſeriei harmonicæ finita ſtatueretur. Animadvertit enim,

Seriem A, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$, &c. ∞ (fractionibus ſingulis in alias, quarum numeratores ſunt 1, 2, 3, 4, &c. transmutatis) ſeriei B, $\frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \frac{5}{30} + \frac{6}{42}$, &c. ∞ C + D + E + F, &c.

C. $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42}$, &c. ∞ per præc. $\frac{1}{1}$
D. .. $+ \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42}$, &c. ∞ C − $\frac{1}{2}$ ∞ $\frac{1}{2}$
E. ... $+ \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42}$, &c. ∞ D − $\frac{1}{6}$ ∞ $\frac{1}{3}$
F. ..... $+ \frac{1}{20} + \frac{1}{30} + \frac{1}{42}$, &c. ∞ E − $\frac{1}{12}$ ∞ $\frac{1}{4}$
&c. ∞ &c.
} ∞ G; unde ſequitur, ſeriem G ∞ A, totum parti, ſi ſumma finita eſſet.

Ego

Johann's divergence proof, from Jakob's **Tractatus de seriebus infinitis**, *republished in 1713. (From page 197 of* **Journey through Genius** *by William Dunham).*

ln ln n. This is usually stated in the following form: as n tends to ∞, the fraction

$$\frac{1/p_1 + 1/p_2 + 1/p_3 + \cdots + 1/p_n}{\ln \ln n}$$

tends to 1. This is indeed a striking result, reminiscent of the earlier result that $1/1 + 1/2 + 1/3 + \cdots + 1/n$ is approximately equal to $\ln n$. It shows the staggeringly slow rate of divergence of the sum of the reciprocals of the primes. The harmonic series $\Sigma_i 1/i$ diverges slowly enough – to achieve a sum of over 100, for instance, we would need to add more than 10^{43} terms, so it is certainly not a job that one can leave to finish off over a weekend. (Do you see where the number 10^{43} comes from?) On the other hand, to achieve a sum of over 100 with the series $\Sigma_i 1/p_i$, we need to add something like $10^{10^{43}}$ terms!! This number is so stupendously large that it is a hopeless task to make any visual image of it. Certainly there is no magnitude even remotely comparable to it in the whole of the known universe.

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