Geometry

3. Towards a Geometry of Space and Time

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In the first two articles of this series the author described Euclidean geometry, coordinate geometry, trigonometry and measure theory. In this article he introduces non-Euclidean geometry and discusses tangents to curves and surfaces. These seemingly different notions will be brought together in the future when he discusses differential geometry.

Working at Parallel Purposes

Parallel: A pair of lines in a plane is said to be parallel if they do not meet.

Mathematicians were at war with one another because Euclid's axioms for geometry were not entirely acceptable to all. Archimedes, Pasch and others introduced further axioms as they thought that Euclid had missed a few, while other mathematicians were bothered by the non-elementary nature of the parallel axiom. They wondered if it could be proved on the basis of the other axioms.

One form of the Parallel Axiom of Euclidean Geometry is:

There is exactly one *line that is parallel to a given line and passes through a given point not on it.*

By constructing a pair of right angles it is not hard to show (using the remaining axioms) that there is at least *one* parallel line as required by the above axiom (exercise). So we are tantalisingly close and only need to show that the line is unique. Many mathematicians devoted much of their careers to this problem. By the method of *reductio ad absurdum* (reduction to absurdity)[1] they

[1]Captured by the following verse or worse:
If the theorem isn't true
Then the sky isn't blue
Which is so absurd
That the truth you must have heard

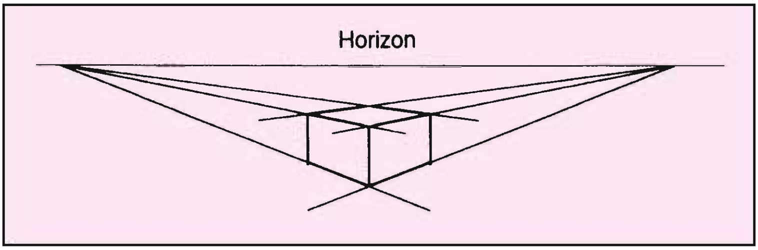

***Figure 1** A matter of perspective.*

attempted to derive (from the supposedly wrong hypothesis that there is another parallel) a number of 'results' that would seem absurd. The most successful in this was Saccheri. Many of his 'results' actually became theorems in non-Euclidean geometry — results which he thought were wrong!

Indeed, Lobachevsky and Bolyai showed that there is a perfectly valid geometry in which the parallel axiom is replaced by the following:

There are at least two lines that are parallel to a given line and pass through a given point not on it.

Subsequently, Poincaré, Klein, Beltrami and others refined non-Euclidean geometry. It was shown that Euclidean and non-Euclidean geometry are equi-consistent — one is as consistent as the other.

From a different perspective artists had all along pointed out that parallel lines *do meet* at the horizon *(Figure 1)*. In fact all pairs of coplanar lines meet and parallel lines are singled out by the fact that their point of meeting is at the horizon. The horizon is itself another line (called the line 'at infinity'). A perfectly consistent geometry can thus be formed with the axiom:

There are no parallel lines. Any pair of coplanar lines meet.

Euclidean and non-Euclidean geometry are equi-consistent — one is as consistent as the other.

Of course, the axioms of separation need to be modified somewhat to accommodate the fact that every line becomes 'circular'. This geometry is the *projective geometry* of Poncelet which reappeared as the *elliptic geometry* of Riemann.

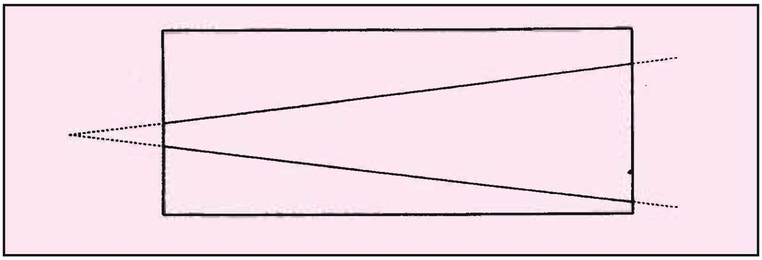

Figure 2 Geometry on a sheet of paper.

The Local Axioms

Incidence. Each pair of distinct points determines a unique line and so on.

Dimension Any two planes that meet have at least two points in common. There are four non-coplanar points.

Separation. Each point on a line divides the line into two rays; each line on a plane divides the plane into two half planes and so on.

LUB property. Given a sequence of points A_n and a point B so that A_{n+1} is between A_n and B, i.e. B is an upper bound for the A_n's, there is a point C which is a least upper bound.

The existence of so many parallel choices for the parallel axiom appears to spell trouble since it calls into question the introduction of co-ordinate geometry. (We recall from the first article of this series that the introduction of co-ordinates depended on the parallel axiom.) However, it was shown by Klein, Beltrami and others that coordinates are a natural consequence of the axioms of 'local space'. (Note that there can be no talk of parallels since even lines which might meet 'far away' do not meet in the given region; see *Figure 2*). More precisely we restrict ourselves to the axioms of incidence, dimension, separation and the least upper bound property (LUB) — axioms that appear to be satisfied by a small region of space surrounding us (see box). It turns out that there is a natural way to embed such a geometry in co-ordinate geometry (so that the lines embed as lines, planes as planes and so on). Thus we can safely return to the study of co-ordinate geometry secure with the knowledge that all non-Euclidean phenomena can be found there.

A question that perhaps still nags practical-minded people is whether the geometry of the space around us is Euclidean or not. The first person to try to test this was Gauss but the scale he chose was not large enough to show the non-Euclidean nature of space. In this century as a test of Einstein's theory of gravitation the 'curved' nature of space was finally shown.

Time to Take Off on a Tangent

Tangent line. A line with the *maximal order of contact* with the given curve at the given point among all lines through this point.

Since time immemorial — or at least since time became measurable, people have wanted to know how fast or slow things change or move. A crude way of doing this is to measure the change that takes place in a specified amount of time or to use a stop watch to measure how long it takes to achieve a specified amount of displacement. In the interests of accuracy one must use smaller and smaller time periods. To follow this logic to its limit one must use a zero time period but that is apparently absurd. The word *limit* in the previous sentence gives a clue to the correct approach — and this is how 'instantaneous velocity' or derivative was finally given analytical meaning. However, this analytical definition of derivative in terms of limits really came much later with the work of Cauchy and Weierstrass. Newton and Liebnitz argued on the basis of other concepts.

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One was the use of 'infinitesimals' — or infinitely small entities. This was vehemently argued against at the time[2] but is in fact quite a straight-forward algebraic method which can be made perfectly valid when the position is an algebraic (or more generally, analytic) function of time. As an example, to find the instantaneous velocity of an object occupying the position $(t+t^2, t^3)$ at time t we substitute $(t+h)$ in place of t,

$$((t+h) + (t+h)^2, (t+h)^3) = ((t+t^2)+(1+2t)h, (t^3) + (3t^2)h)$$

when h is an 'infinitesimal' such that h^2 is zero but h is itself not zero[3]. Now the coefficient of h, that is $(1+2t,\ 3t^2)$ gives the instantaneous velocity 'vector'.

[2] It is not accepted by many teachers today; so don't use this method in your exam papers!

[3] Those who find this difficult to digest can think of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ which has the same property.

The second method was based on what is now called the Liebnitz rule of derivation. There are in fact three rules

$$D(f+g) = Df + Dg\ ;\ D(\alpha f) = \alpha\, Df\ ;\ D(fg) = (Df)g + f(Dg)$$

where f and g are functions and α a constant. Combined with the 'choice' $Dt = 1$ we easily see that $D(t + t^2, t^3) = (1+2t, 3t^2)$. This is another algebraic method and one that has been used very fruitfully in commutative algebra in the recent past.

Newton was principally a mathematician whatever some other

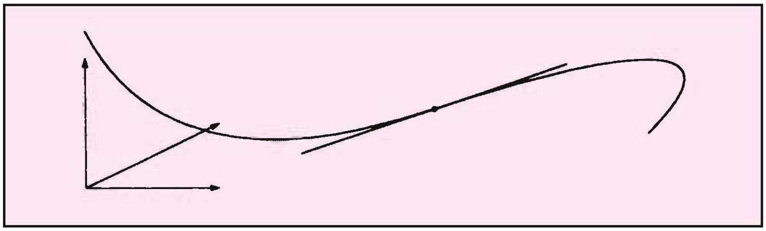

Figure 3 Tangent to a curve.

people may tell you! At least he was a geometer since he gave the geometric definition of instantaneous velocity by means of the tangent. First, one considers the trajectory as a curve with time plotted as one of the axes; in the example above we have the parametric curve $(x(t), y(t), t) = (t+t^2, t^3, t)$. The slope of the tangent line (as defined above) with respect to the time axis gives the derivative (*Figure 3*).

Thus the study of nebulous physical quantities like time (is it justified to treat it as one of the axes — i.e. a real variable?) and dubious analytic constructs like limits (at least until the work of Cauchy and Weierstrass) is replaced by the clear[4] geometrical notion of tangents.

[4] Clarity is clearly in the eye of the beholder!

Another way to look at tangents is by means of the 'apparent locus'. This is how the curve will appear at a specified point, to a person constrained to move along the given curve.

This becomes clearer if we consider higher dimensional tangents. Since an ant is constrained to move along the surface of the earth, which we may assume to be a sphere, at any position it imagines the earth to be planar — this plane being the tangent plane to the sphere at that point[5] (*Figure 4*). The locus of all tangent lines then makes up the 'apparent horizon'; this is how the horizon appears to an ant constrained to move along the given locus. These apparently simple notions play an important role in geometry since they allow us to study tangency even for more complicated situations. (Exercise for the adventurous: What is the apparent horizon of the locus $x^2+yz-x^3=0$ at the origin?)

[5] As opposed to the usual notion of bird's eye view the apparent locus can perhaps be called *the ant's eye view*!

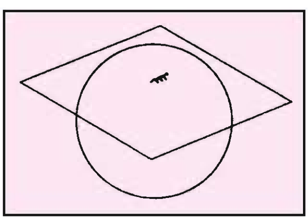

Figure 4 Ant's eye view.

Differential calculus thus becomes the most important tool with which one can study the more general loci that co-ordinate geometry allows us to introduce.

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Summary

A non-Euclidean geometry is one where the notion of parallel line is changed to include a multiplicity of parallels. Even such a geometry can be given co-ordinates. Thus co-ordinate geometry wins the day. Differential calculus is the most useful tool in the study of co-ordinate geometry. This explains why most undergraduate studies in mathematics begin with *calculus and analytic geometry*.

We will now take a break for rumination (or to chew gum). When we return we shall see how Gauss and Riemann put together the above tools so that today even an ant can decide whether space is curved.

Suggested reading

H S M Coxeter. Non-Euclidean Geometry. University of Toronto Press. 1961.
A good introduction to projective geometry and other non-Euclidean geometries.

D Hilbert. Foundations of Geometry. Open Court Publishers, La Salle, Illinois, USA. 1971.
A more advanced treatment can be found in this book.

D Hilbert and S Cohn-Vossen. Geometry and the Imagination . Chelsea, NY, USA. 1952.
A difficult but juicy book.

R Courant and H Robbins. What is Mathematics? Oxford University Press. 1941.
A must-read book for a look at real mathematics.

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This series on Geometry will resume in May 1996.

The manifold genius of Helmholtz... The *Dictionary of Scientific Biography* lists Hermann Helmholtz's eminence in the following fields: energetics, physiological acoustics, physiological optics, epistemology, hydrodynamics and electrodynamics — without mentioning his work in non-Euclidean geometry.