# Geometry

## 2. A Circle of Ideas

*Kapil H Paranjape*

After spending about a decade at the School of Mathematics, TIFR, Bombay, Kapil H Paranjape is currently with Indian Statistical Institute, Bangalore.

**In this article the author addresses the origins of trigonometry and the idea of limits. The concept of 'limit' is crucial in the development of integral calculus, the subject which deals with the measurements of lengths, areas and volumes of general figures. What emerges is that the roots of trigonometry and integral calculus are already implicit in the early geometrical studies.**

In the previous article we looked at the origins of synthetic and analytic geometry. More practical minded people, the builders and navigators, were studying two other aspects of geometry—trigonometry and integral calculus. These are actually algebraic and analytic studies with the initial input coming in a left handed (right-brained[1]) way.

[1] It is the right half of the brain that is supposed to do spatial thinking and control the left hand. This perhaps gives one explanation why so many left handers are good tennis players.

### Trigonometry

**Circle:** The locus of all points in a plane which are equidistant from a given point.

Quite independent of the rigours of Euclid's synthetic geometry, the geometry of the circle was extensively studied by travellers and builders. The astronomers and navigators of ages past could take one look at the stars and figure out the time, date and their location[2]. This circle of ideas revolves around studying the points on a circle. There are, broadly speaking, two classes of methods for fixing a point on the circle. (Exercise: Quickly figure out these two before reading the next paragraph.)

[2] Nowadays we rush to the quartz watch instead—perhaps because pollution has made it impossible to see the stars.

One class is the 'constructible' one, of which an example is as follows: Choose a point $P$ on the circle and a line $l$ not containing

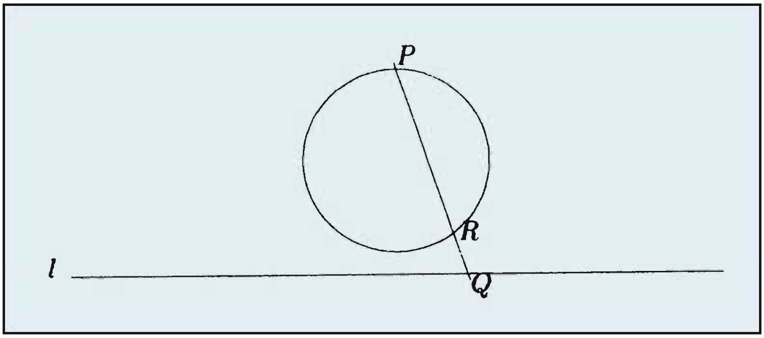



***Figure 1** Rational parametrisation of the circle.*

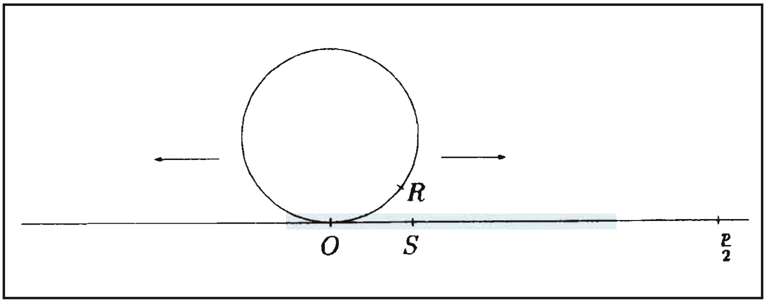



***Figure 2** Unrolling the circle.*

$P$. Each point $Q$ of the line $l$ *uniquely* fixes a point of the circle. This is the 'other' point $R$ of intersection of the line $PQ$ with the circle (see *Figure 1*). The other class involves unrolling the circle (see *Figure 2*); this can be done by *angular* parametrisation (Exercise: why is this called angular?). In this method each point $R$ of the circle is represented by a collection of points of the form $S \pm n.p$ on the line where $S$ is some point on the line, $p$ is the perimeter of the circle and $n$ is a natural number. (Here $a \pm n.p$ denotes the translation of the point $a$ towards the left/right by the distance $n.p$.)

Quite independent of the rigours of Euclid's synthetic geometry, the geometry of the circle was extensively studied by travellers and builders. The astronomers and navigators of ages past could take one look at the stars and figure out the time, date and location.

The advantage of the first method is both practical and philosophical. From the practical perspective, only that which can be constructed is useful; from the philosophical perspective the unrolling of the circle is an operation which is outside the axiomatic setup (some work has to be done to prove this!). The advantage of the second method is that it is intuitive and aesthetic; it fully captures the symmetry of the circle which is broken by the first method. So how do we reconcile the two methods?

To fix things we choose a point $P$ of the circle and let the line $l$ be the tangent line to the circle at the anti-podal point $O$ (if $d$ denotes the line joining $P$ to the centre of the circle then $l$ is the line through $O$ which is perpendicular to $d$; see *Figure 3*). We choose coordinates on the line so that $O$ is the origin. Then $Q$ in the first construction represents a number $t$. The unrolling procedure assigns to each point of the circle other than $P$, a point $S$ on $l$ with coordinate $\theta$ between $-p/2$ and $p/2$. The combination then gives us an assignment $\theta(t)$ for every number $t$ thought of as a point on the line $l$; those who prefer the second method could also think of the assignment $t(\theta)$. In order to realise the rotational symmetry in the constructive approach we must be able to find $t_3 = t(\theta(t_1) + \theta(t_2))$; in other words we must be able to express the combination of two rotations (denoted by $\theta(t_1)$ and $\theta(t_2)$) in constructible coordinates. The construction is sketched in *Figure 3* and the formula is

$$t_3 = [4(t_1 + t_2)] / [4 - t_1 t_2].$$

It is an easy exercise in coordinate geometry to verify this (braver people may also attempt a proof using Euclidean geometry alone).

The above formula summarises the *essence* of trigonometry. To put things in a more familiar setting we note that

$$t(\theta) = 2\tan(\theta/2).$$

The above formula then becomes the familiar addition law for tan. We note that the coordinates of the point $(x,y) = (\sin(\theta), -\cos(\theta))$ are expressed in terms of $t = t(\theta)$ as

$$(x,y) = (4t / [t^2 + 4], [t^2 - 4] / [t^2 + 4]).$$

**Construction of Regular Polygons**

While the circle is an endless source of geometrical ideas we now describe another historically important problem—the construction of regular polygons. Let $\theta_n$ denote the angle subtended by the side of a regular $n$-sided polygon at its centre. One can then show using the addition laws above that $t_n = t(\theta_n)$ satisfies an equation of degree $n-1$. Some number theory can be brought in to show that this equation can be solved by a succession of square roots (of sums of squares) if and only if $n = 2^m q$ where $q$ is a product of distinct Fermat *primes*. This gives Gauss' famous theorem about the constructibility of polygons. In particular, this shows that the 17-sided regular polygon is constructible, a fact realised by Gauss at the age of 19!

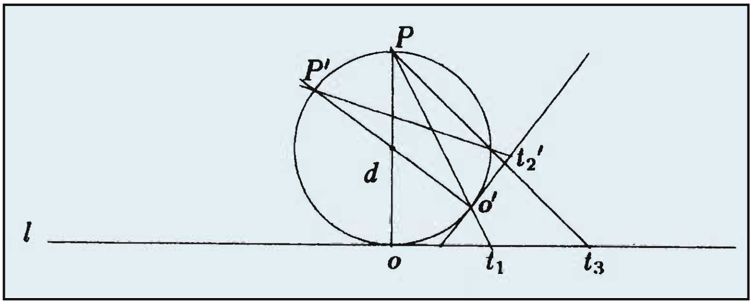



***Figure 3 The addition law.***

Conversely, given $(x,y)$ such that $x^2 + y^2 = 1$ we have

$$t = 2x/(1-y) = 2(1+y)/x$$

In other words we can work interchangeably between the $t$ coordinate and the $(x,y)$ coordinates for the circle. (Exercise: derive the addition law for sin and cos using the above formulae.)

In different parts of the world mathematicians came upon different versions of the above formulae (and gave us a seemingly endless series of school problems on trigonometry). A complete exposition was given in the works of Indian and Arab mathematicians, but it was left to de Moivre and Euler to put the finishing touches as we shall see later.

In different parts of the world mathematicians came upon different versions of the trigonometric formulae. A complete exposition was given in the works of Indian and Arab mathematicians, but it was left to de Moivre and Euler to put the finishing touches.

## This is the Limit

**Limit:** (of a sequence) A point such that the points of the sequence eventually approach it to within any previously specified distance.

Some of the Greek mathematicians were quite confused! For example, let us take an empty cup and put it under a tap. Assume that it is half full in a minute. It is then 3/4-th full in another half minute and 7/8-th full a quarter minute after that and so on. Will the cup ever be full? In other words, can the sequence {1/2, 3/4, 7/8, 15/16,...} be said to *become* 1 in some sense? The problem here is clearly an abstract mathematical one—most thirsty people would grab the cup after some time!

This enigma was resolved by that famous Greek mathematician Archimedes who introduced the axiom: *Given a succession of points, $A_n$ for $n = 1,2,3,...$ and a point B so that $A_{n+1}$ is between $A_n$ and B for all n. There is a point C between $A_n$ and B for every n so that if D also has the same property then D is between C and B (see Figure 4). (In other words C is the limit of the sequence $A_n$)*

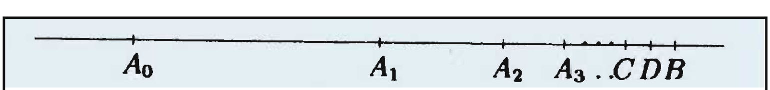
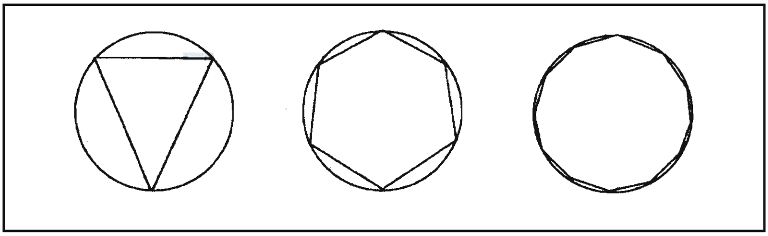



***Figure 4** The axiom introduced by Archimedes.*

*Figure 5 Filling out a circle.*



It has one of the basic properties of axioms in that it conforms to our picture of the world around us. However, it was unacceptable to a number of Greek mathematicians since it asserted the existence of a point without giving a construction. We use it since it is quite essential to much of modern mathematics and its applications.

Not every figure is measurable and non-measurable figures can behave in strange ways. A ball can (in principle) be cut into three non-measurable pieces which when put together in a different way give a ball of twice the size!

With this axiom in hand we can try to find areas of figures and volumes of solids, of varied shapes. The fundamental idea is that of *approximation*. We find a sequence of objects which we can measure exactly and which successively give better and better approximations to the value we want (*Figure 5*). But this method begs the question—how do we know that these values approximate the value we want?

A number of different approaches to measurement were tried out till Lebesgue finally resolved the question at the beginning of this century. He answered the question—what are *all* the figures that we can call *measurable*? We first look at all the exactly measurable (constructively measurable) figures like polygons, within the given figure $F$ (see *Figure 5*). Let $i(F)$ denote the smallest number greater than all of these measures. (The existence of such a number is guaranteed by the axiom given above). Similarly, let $o(F)$ denote the largest number less than the measure of all exactly measurable figures that enclose the given figure (see *Figure 6*). If $i(F) = o(F)$, then we say $F$ is measurable. Lebesgue then showed how this gives rise to a consistent theory of measurement. (Exercise: By this method try to find a good approximation to the area of a circle. Show that it is good say up to the 2nd decimal place.)

A word of warning—not every figure is measurable and non-measurable figures can behave in strange ways. A ball can (in principle) be cut into finitely many non-measurable pieces which

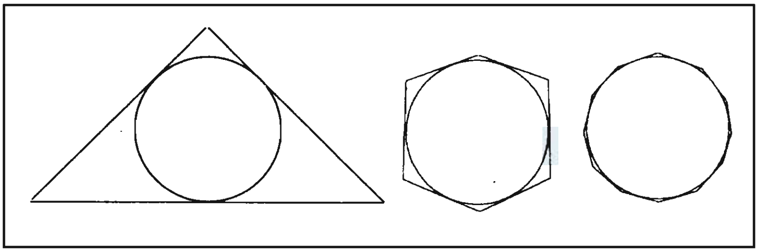


*Figure 6 Enclosing a circle.*

when put together in a different way give a ball of twice the size! This is the so called 'Banach-Tarski paradox'. However, it is not easy to sketch a non-measurable figure! It would seem that a figure that can be sketched is measurable. In fact there is a device called the *pantograph* which will give the area enclosed by a closed curve if we trace the curve. (Exercise: Re-invent this device!). So I cannot show you a picture of a non-measurable figure.

This study of areas and volumes is today called measure theory and is a fundamental branch of analysis and probability theory. A slightly different and more geometric approach led to integral calculus as we shall see in a later article.

The study of areas and volumes is today called measure theory and is a fundamental branch of analysis and probability theory.

## Summary

Trigonometry is summarised by the single formula that gives the addition law. All other identities in trigonometry follow from either this one (which summarizes the relation between the constructible co-ordinates and the intuitive one on the circle) or the relation between various different constructible coordinates on the circle. The main reason for the unending sequence of school exercises is to develop algebraic skills.

Measurement of lengths, areas and volumes is not too difficult for the well-known types of figures (polygons, prisms, pyramids, etc.). We can use these figures to "fill up" more complicated figures to measure them. To do this we need to introduce numbers that are limits of other numbers, in other words *real* numbers.

Let us plunge ahead! Is the parallel axiom justified? Are coordinates still valid without it? How can we perform experiments to find out if our (Euclidean) intuition is correct? Await the exciting next instalment.

*Address for correspondence*
Kapil H Paranjape,
Indian Statistical Institute,
8th Mile, Mysore Road,
Bangalore 560 059, India.