Geometry

1. The Beginnings

Kapil H Paranjape

After spending about a decade at the School of Mathematics, TIFR, Bombay, Kapil H Paranjape is currently with Indian Statistical Institute, Bangalore.

Mathematics is as much an art as a science. Thus to understand why we study the problems we do today we must examine the history of the subject. In this series of articles we will try to examine how the geometric concepts that are in use today evolved. (A note of warning: the 'history' here is more a personal view than a historian's.) As in art, understanding is enhanced by doing. Readers are encouraged to attempt the exercises scattered in the text.

The Origin (s)

Origin: the starting point of a flow *or* the centre of a coordinate system.

We are often told that geometry (=geo+metry) arose out of the attempt to measure land area. But this view ignores the development of geometry for navigation by travellers who used stars, for the design of buildings, in art and painting and so on. In fact geometry and geometrical thinking is one of the fundamental activities of the brain — the other being algebraic thinking (these two modes are sometimes called the spatial and verbal functions of the brain).

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The first comprehensive treatment of geometry which we can call *mathematical* from a modern perspective is that of Euclid. From a few basic concepts (point, line, angle etc.) and few basic statements (the five axioms) he wished to deduce *all* the known (geometrical) phenomena using some logical principles (which he called common notions). In other words he was constructing a 'theory of everything'. At the same time he was aware that our 'imperfect' world did not quite meet all the requirements — the truly geometric world was the Platonic universe of the heavens; a

Table 1. Hilbert's Axioms for Euclidean Geometry

1) Incidence. Each pair of distinct points determines a unique line and so on.
2) Separation. Each point on a line divides the line into two rays; each line divides the plane into two half planes and so on.
3) Congruence. Along any ray one can mark a segment congruent to a given one; given any ray and a half plane adjacent to it, for any angle we can find a congruent angle lying in the half plane based on the given ray. The "side-angle-side" postulate for congruence of triangles.
4) Archimedean property. Given any pair of segments some multiple of the first segment is longer than the second one.
5) Parallel postulate. Given a point and a line not containing it there is a unique line through this point parallel to the given line.

The study of real numbers required a whole new geometric insight due to Weierstrass, Dedekind, Cantor and others. In Euclid's time only Eudoxus and Archimedes came close to this insight.

modern perspective would be that he had a 'model' for the universe.

An important mathematical feature of Euclid's theory is that rules of deduction are very strict; nothing — not even so-called common sense or intuition — can be taken for granted[1]. However, Euclid too fell into some traps set by common sense. One of the most (logically) circular parts of his theory is his use of the circle! A number of corrected or alternate approaches to Euclidean geometry exist today but none has the all encompassing breadth of his 'build–the–whole–thing–up–from–nothing' approach. The closest is Hilbert's approach (see the box for a quick summary).

It can never be over-emphasised that common sense has a love-hate relationship with science.

Nowadays the real number system that comes up in Euclidean measurement is often constructed algebraically (via the decimal numbers) and then *imposed* on geometry via the Ruler Placement Postulate. However, the idea (embodied in the Ruler Placement Postulate) that a line is the set of its points would have been totally unacceptable to the Greek mathematicians. In fact one of Euclid's attempts was to give a geometric construction of the number system. Each number *r* is *represented* by a pair of line segments (the

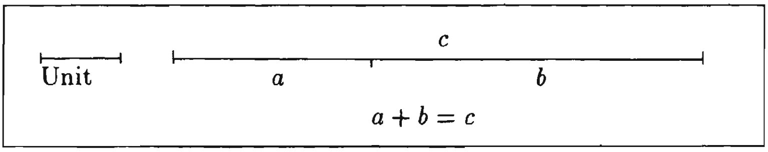

Figure 1 Adding numbers.

first gives the 'unit' of measurement and the second gives r when measured in those units)[2]. Some simple constructions (see the *Figures 1,2,3*) show how we can add, multiply, divide and take square roots of numbers represented as above. (Exercise: Justify these constructions using Hilbert's axioms.)

[2] The fact that some line segments give rise to irrational numbers like $\sqrt{2}$ disturbed the common sense of Euclid and his contemporaries but was *not* in any way an inconsistency in the theory.

However, not every number of geometrical interest arises by successive application of the above constructions to the unit length. Two important unsolved problems of Euclid's time were (1) 'unrolling' the circle (2) 'doubling' the cube. In fact, as we now know from the theory of measure, 'most' real numbers *cannot* be constructed by means of straightedge (ruler) and compass; in Greek mathematics the only permissible entities were those constructed in this way. The study of real numbers required a whole new geometric insight due to Weierstrass, Dedekind, Cantor and others. In Euclid's time only Eudoxus and Archimedes came close to this insight[3].

[3] A relatively 'unknown Indian' by the name of Madhavacharya appears to have also come quite close.

Co-ordinating the Plane/Brain[4]

[4] Unfortunately we mathematicians seem to fall into the 'gum-chewer' category; we can't chew gum (verbal) and walk in a straight line (spatial) at the same time!

Co-ordinates: A pair (triple) of numbers uniquely identifying a point on the plane (in space).

By the time mathematics had wound its way via the Indian and Arabic traditions, the algebraic and arithmetic aspects had seen

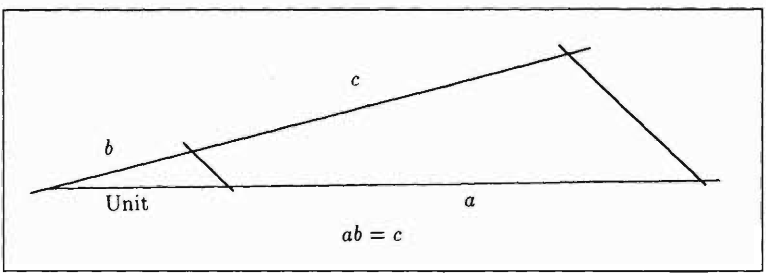

Figure 2 Multiplying and dividing numbers.

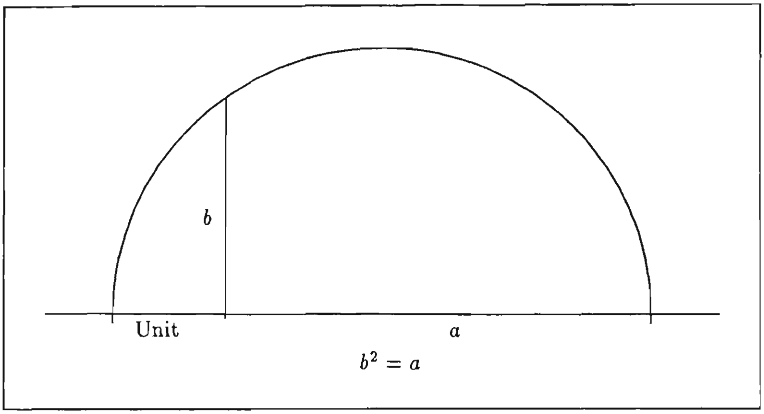

Figure 3 Taking a square root.

tremendous growth. With the use of negative numbers and (the all-important number) zero it became possible to talk of all the arithmetic operations on numbers. Decimal notation made arithmetic operations 'child's play'.

Much of Euclidean geometry became absurdly simple if one used coordinates.

In order to utilise this Descartes devised the following scheme. By fixing a point, the *origin*, on a line it becomes possible to talk of a directed distance as a positive or negative number depending on whether the end point is to one or the other side of the origin.

Similarly, he assigned a pair of numbers to every point of the Euclidean plane. First one chooses a pair of orthogonal lines called the *axes*. The intersection point of these lines is called the *origin*. The directed distance from the origin to the foot of the perpendicular from our given point to the first axis is called the *abscissa*; the directed distance from the origin to the foot of the perpendicular from our given point to the second axis is called the *ordinate*[5] (*Figure 4*). (I have stated everything in words here to show

[5]So powerful is this pair of numbers working together with some (by now common) algebra and arithmetic that we use the term 'coordinated' for well-organised activity!

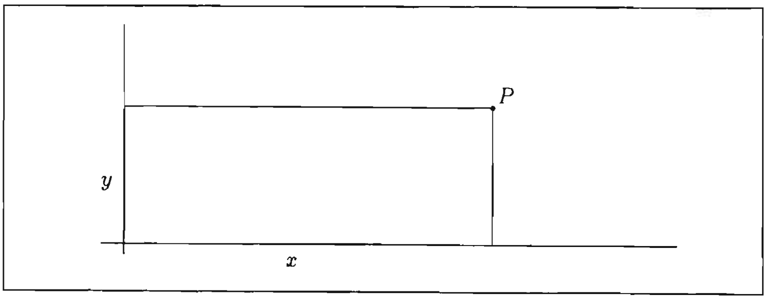

Figure 4 The coordinate plane.

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how cumbersome this original — pre-algebraic — method of writing things was; this continued in Europe for quite some time in spite of the fact that the Indo-Arabic mathematicians had already introduced variables!)

On the one hand, it follows from Euclidean geometry that a point in the plane is uniquely determined by its Cartesian coordinates (Exercise: Prove this). On the other hand, much of Euclidean geometry became absurdly simple if one used coordinates (at least to those mathematicians who knew their arithmetic and algebra) — more importantly, the truth of various statements could be deduced by calculation (Exercise: Deduce all your favourite theorems and riders in Euclidean geometry using Cartesian coordinates). The tricky definition of a circle in Euclidean geometry gave way to the much clearer point of view that a circle is the locus specified by an equation $(x-a)^2 + (y-b)^2 = r^2$ where (a,b) are the co-ordinates of the centre and r the radius (Exercise: try to give *this* definition without using symbols!). While common-sense and intuition seem to take a back seat and algebraic manipulation comes to the fore, this is all to the good from the point of view of Euclid's deductive method.

This was not all. Like all good mathematics, the Cartesian coordinates also opened the door to newer geometrical ideas. Firstly, it became possible to talk about the locus associated with *any* (algebraic or functional) equation involving the two co-ordinates, e.g.

$$y^2 = x^3 - x$$

In other words, the study of plane curves was begun. Secondly, it was no longer necessary to "construct" all the geometrical figures that were studied (Exercise: Try to find a sensible way of tracing out the curve given by the equation above). The equations defined the figures and then the mathematician could "analyse" them. This led to the term *analytic* geometry of Descartes as opposed to the *synthetic* geometry of Euclid. Cartesian coordinates also opened up the possibility of studying geometrical relations between non-spatial entities—one can draw a *graph* showing a geometrical relation between (say) the amount of gum chewed and the linear distance traversed[6]. Finally, a most important consequence was that one could study geometry in dimensions other than two and three. This idea flowered in the hands of Riemann.

[6]See earlier footnote 4 to see why this is important.

Summary

Euclidean geometry in its original form has only a marginal role to play in modern mathematics. It is almost totally supplanted by Cartesian or analytic geometry. Why then do we still learn it? To give us a way of building our geometrical skills while we learn enough algebra and arithmetic to use coordinates. Moreover, it is probably not easy to *discover* new results in Euclidean geometry while thinking about it purely algebraically.

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A number of questions remained unanswered even with the simplicity introduced by the coordinate approach. Will the circle be squared? Will parallel lines meet? Can curves be straightened? Wait for the exciting next instalment!

Address for correspondence
Kapil H Paranjape,
Indian Statistical Institute,
8th Mile, Mysore Road,
Bangalore 560 059, India.