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Off-shell scattering by an approximated additive interaction

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Abstract. We present all the partial wave descriptions of the nucleon–nucleus system by proposing a new additive phenomenological potential with emphasis on off-energy-shell scattering. For most of the general treatment of the physical processes, the off-shell transition matrices are most expedient quantities because they carry as much information as the potential. As the off-shell Jost solution is an indispensable ingredient for deriving transition matrices, we initially construct this function by taking into account the ordinary differential equation method. Finally, we execute certain tests on our expressions with respect to various limiting conditions and present numerical results using the MATLAB programme. Numerical results are in sensible conformity with the previous works.

Keywords. Additive interaction; on- and off-shell Jost solutions; transition matrix; (p–d) scattering; phases and cross-sections.

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1. Introduction

The dynamics and interactions of non-relativistic particles are commonly explained utilising time-independent Schrödinger wave equation with a physical potential model. The analytical solutions to this equation with physically solvable potentials are crucial to our knowledge of a quantum system's physical environment. Only a few standard potentials allow for exact solutions to quantum mechanical wave equation with all partial waves and energies. Coulomb, harmonic oscillator, square well and r^{-4} potentials are examples of such potentials. The Hulthén potential [1–5] is an exponential-type potential for which only the s-wave has an exact solution at all energies. In general, perturbation calculations in the form of approximation approaches are used to solve the radial Schrödinger's equation with $\ell > 0$ to get analytical results needed to study the properties of different physical systems. The Nikiforov–Uvarov approach [6–8], supersymmetric quantum mechanics [9], the screened centrifugal barrier [10–12] and other approximation techniques require some approximation techniques. For $\ell > 0$, the Schrödinger's equation has been used to approximate closed-form analytical solutions for several types of exponential potentials. The Hulthén potential has been brilliantly recommended for describing the model of two-particle interactions in atomic, molecular and nuclear domains. The two-nucleon interaction off the energy shell is an essential ingredient for calculating the properties of multinucleon systems. This interaction generally corresponds to the phenomenological potential with adjustable parameters to reproduce low-energy parameters extracted from nucleon–nucleon elastic scattering experiments in the non-relativistic domain. In the literature, many such potentials have been constructed which have the ability to reproduce almost the same phase shifts. Since a potential is associated only indirectly to physical observables, any confirmation for choosing one in preference to others must be adduced circuitously and at the cost of substantial computation. We propose an alternative description of the two-nucleon interaction as prescribed in our previous articles [13,14] with a screened centrifugal barrier for the partial wave treatment of nuclear scattering off the energy shell. Many researchers have advocated the off-energy-shell treatment of the Coulomb, Hulthén, Manning–Rosen, Morse and Coulomb-like potentials [15–32]. However, for $\ell > 0$, pure quantum mechanical calculation of the off-shell quantities for nuclear Hulthén plus atomic Hulthén potential has yet to be explained. In some previous works [10,24,29,32] the researchers prescribed the pure quantum mechanical treatment of off-shell wave equation and derived

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T-matrix for exponential-type of potentials. The transition matrix, often known as the T-matrix, is a perfect way to start calculating the number of observables because it is inseparably linked to the experiment. We recently developed off-shell quantities [13] for the s-wave associated with the Hulthén plus Hulthén potential in their maximal reduced form and established them as numerically computable.

The screened atomic Hulthén potential [1] is given by

$$V_A(r) = U_0 \frac{\mathrm{e}^{-r/b}}{1-\mathrm{e}^{-r/b}}, \tag{1}$$

where U_0 and b are the strength and inverse range of the interaction. The nuclear Hulthén potential [1,13] is defined as

$$V_N(r) = -U_1 \frac{\mathrm{e}^{-r/b}}{1-\mathrm{e}^{-r/b}} + \frac{\ell(\ell+1)\mathrm{e}^{-2r/b}}{b^2\left[1-\mathrm{e}^{-r/b}\right]^2}, \tag{2}$$

where $U_1 = (\beta^2 - \alpha^2)$, $b = (\beta - \alpha)^{-1}$, α is adjusted from the binding energy [13] and β is a regulating entity. The effective potential reads as

$$V_{\text{eff}}(r) = V_N(r) + V_A(r). \tag{3}$$

Following the approaches of refs [13] and [14], we make a similar investigation to construct ℓ-state solution of the radial wave equation for such an effective interaction. Undoubtedly, this study will serve as a useful reference for conceptually interpreting the quantum system with all partial wave states for short-range potential. The current article focusses on the derivation of on-shell Jost functions using a differential equation approach, as well as irregular or Jost, physical or outgoing wave solutions for all partial waves using a direct integration approach. We develop an analytical expression for T-matrix off the energy shell using off-shell solutions and perform many checks on our expression in terms of its limiting behaviours. The methodology for getting on-shell Jost function is described in §2 and the off-shell Jost solution is discussed in §3. Section 4 presents the construction of off-shell transition matrices and associated limiting features. The results and discussion are presented in §5 and our conclusions are presented in §6.

2. On-shell Jost solution and function

The Schrödinger wave equation for the effective potential given in eq. (3) can be written as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - V_{\text{eff}}(r)\right]\phi_\ell(k,r) = 0, \tag{4}$$

where k denotes the centre of mass momentum, which is associated to the centre of mass energy $E = (\hbar^2 k^2/2\mu)$ and μ is the reduced mass. The wave function $\phi_\ell(k,r)$ satisfies the regular boundary condition.

In order to obtain the solution of eq. (4), we take the following transformation:

$$\phi_\ell(k,r) = b^{\ell+1}[1-\mathrm{e}^{-r/b}]^{\ell+1}\mathrm{e}^{ikr} R_\ell(k,r) \tag{5}$$

and using eqs (1), (2) and (3), we can rewrite eq. (4) as

$$\begin{aligned} &b^2\mathrm{e}^{r/b}(1-\mathrm{e}^{-r/b})R_\ell''(k,r) + \{2(\ell+1)b + 2ikb^2\mathrm{e}^{r/b} \\ &\quad\times(1-\mathrm{e}^{-r/b})\}R_\ell'(k,r) + \{2ikb(\ell+1) - (\ell+1) \\ &\quad+(U_1-U_0)b^2\}R_\ell(k,r) = 0. \end{aligned} \tag{6}$$

For simplification, we change a new variable of the form $(1-\mathrm{e}^{-r/b}) = X$, and substituting into eq. (6) leads to the following hypergeometric equation:

$$\begin{aligned} &X(1-X)\frac{\mathrm{d}^2 R_\ell}{\mathrm{d}X^2} \\ &\quad+\{2(\ell+1) - (1-2ikb+2(\ell+1))X\} \\ &\quad\times\frac{\mathrm{d}R_\ell}{\mathrm{d}X} - \{(\ell+1) - 2ikb(\ell+1) \\ &\quad+(U_0-U_1)b^2\}R_\ell = 0. \end{aligned} \tag{7}$$

On comparing eq. (7) with the following benchmark equation [33–35]

$$\begin{aligned} &X(1-X)\frac{\mathrm{d}^2 R_\ell}{\mathrm{d}X^2} + \{A_3 - (1+A_1+A_2)X\}\frac{\mathrm{d}R_\ell}{\mathrm{d}X} \\ &-A_1A_2R_\ell = 0, \end{aligned} \tag{8}$$

one gets the solution as

$$R_\ell(k,r) = {}_2F_1(A_1, A_2; A_3; X), \tag{9}$$

where A_1, A_2 and A_3 are given by

$$\begin{aligned} A_1 &= \ell + 1 - ikb \\ &\quad+\sqrt{\ell(\ell+1) - b^2k^2 + (U_1-U_0)b^2}, \end{aligned} \tag{10}$$

$$\begin{aligned} A_2 &= \ell + 1 - ikb \\ &\quad-\sqrt{\ell(\ell+1) - b^2k^2 + (U_1-U_0)b^2} \end{aligned} \tag{11}$$

and

$$A_3 = 2(\ell+1). \tag{12}$$

Substituting eq. (9) into eq. (5), we can write down the regular solution as

$$\begin{aligned} \phi_\ell(k,r) &= b^{\ell+1}(1-\mathrm{e}^{-r/b})^{\ell+1}\mathrm{e}^{ikr} \\ &\quad\times{}_2F_1(A_1, A_2; A_3; 1-\mathrm{e}^{-r/b}). \end{aligned} \tag{13}$$

On the other hand, to have the on-shell Jost or irregular solution we apply the following two relations [33–35] in eq. (13):

$$\begin{aligned} {}_2F_1(P_1, Q_1; R_1; Z) &= \frac{\Gamma(R_1)\Gamma(R_1-P_1-Q_1)}{\Gamma(R_1-P_1)\Gamma(R_1-Q_1)} \\ &\quad\times{}_2F_1(P_1, Q_1; P_1+Q_1-R_1+1; 1-Z) \end{aligned}$$

$$+(1-Z)^{R_1-P_1-Q_1}\frac{\Gamma(R_1)\Gamma(P_1+Q_1-R_1)}{\Gamma(P_1)\Gamma(Q_1)}\times{}_2F_1(R_1-P_1,R_1-Q_1;R_1-P_1-Q_1+1;1-Z) \tag{14}$$

and

$${}_2F_1(P_1,Q_1;R_1;Z)=(1-Z)^{R_1-P_1-Q_1}\times{}_2F_1(R_1-P_1,R_1-Q_1;R_1;Z). \tag{15}$$

Finally, with some manipulation, eq. (13) yields

$$\phi_\ell(k,r)=\frac{1}{2ik}\left[\frac{\Gamma(2\ell+2)\Gamma(1+2ikb)}{\Gamma(A_1^*)\Gamma(A_2^*)}\times b^\ell(1-e^{-r/b})^{-\ell}e^{ikr}{}_2F_1(1-2ikb-A_1,1-2ikb-A_2;1-2ikb;e^{-r/b})-\frac{\Gamma(2\ell+2)\Gamma(1-2ikb)}{\Gamma(A_1)\Gamma(A_2)}e^{-ikr}\times{}_2F_1(1+2ikb-A_1^*,1+2ikb-A_2^*;1+2ikb;e^{-r/b})\right]. \tag{16}$$

Based on the criterion [36]

$$\phi_\ell(k,r)=\frac{1}{2ik}[\mathcal{J}_-(k)f_\ell^+(k,r)-\mathcal{J}_+(k)f_\ell^-(k,r)], \tag{17}$$

we can write down the Jost function $\mathcal{J}_+(k)=(\mathcal{J}_-(k))^*$ [36,37] and the Jost solution $f_\ell(k,r)$ or $f_\ell^+(k,r)=(f_\ell^-(k,r))^*$ as follows:

$$f_\ell^+(k,r)=(1-e^{-r/b})^{-\ell}e^{ikr}\times{}_2F_1(1-2ikb-A_1,1-2ikb-A_2;1-2ikb;e^{-r/b}) \tag{18}$$

and

$$\mathcal{J}_+(k)=b^\ell\frac{\Gamma(2\ell+2)\Gamma(1-2ikb)}{\Gamma(A_1)\Gamma(A_2)}. \tag{19}$$

In particular, the Jost solution is a complex quantity. In eq. (19) $\mathcal{J}_+(k)$ with $k=i\kappa_B$ becomes zero at the pole of the Gamma functions $\Gamma(A_1)$ or $\Gamma(A_2)$ which determines the binding energy κ_B of the related system. Also the scattering phase shift is equal to the negative of the phase of the Jost function. Therefore, the two-fold nature of the Jost function, reproducibility of the low-energy scattering parameters including binding energies, has great importance to justify the acceptability of a potential model. When

$$A_1=\ell+1-ikb+\sqrt{\ell(\ell+1)-b^2k^2+(U_1-U_0)b^2}=-n;\ n=0,1,2,\ldots$$

for $k=i\kappa_B$ one has

$$\kappa_B=\frac{b(U_1-U_0)}{2(\ell+1)}-\frac{1}{2b},\quad \text{for}\quad n=0, \tag{20}$$

where

$$\kappa_B=\frac{\sqrt{2\mu E_B}}{\hbar}.$$

The binding energy of the related system is denoted by E_B.

3. Off-shell Jost solution

The off-shell Jost solution corresponding to the off-shell momenta ξ is given by [30–32]

$$f_\ell(k,\xi,r)=(k^2-\xi^2)\int_r^\infty G_\ell^{(I)}(r,r')\,\hat{h}_\ell^{(+)}(\xi r')\mathrm{d}r', \tag{21}$$

with the irregular Green's function [36] for the effective potential

$$G_\ell^{(I)}(r,r')=\frac{1}{\mathcal{J}_+(k)}[\phi_\ell(k,r')f_\ell(k,r)-\phi_\ell(k,r)f_\ell(k,r')] \tag{22}$$

and the quantity

$$\hat{h}_\ell^{(+)}(\xi r)=\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi r)^L L!(\ell-L)!}e^{i\xi r},$$

is the Riccati–Hankel function.

We can rewrite eq. (21) as

$$f_\ell(k,\xi,r)=Y_1(k,\xi,r)-Y_2(k,\xi,r), \tag{23}$$

where

$$Y_1(k,\xi,r)=(k^2-\xi^2)\int_0^\infty G_\ell^{(I)}(r,r')\,\hat{h}_\ell^{(+)}(\xi r')\mathrm{d}r' \tag{24}$$

and

$$Y_2(k,\xi,r)=(k^2-\xi^2)\int_0^r G_\ell^{(I)}(r,r')\,\hat{h}_\ell^{(+)}(\xi r')\mathrm{d}r'. \tag{25}$$

Substituting eq. (22) into eq. (24) leads to

$$Y_1(k,\xi,r)=\frac{(k^2-\xi^2)}{\mathcal{J}_+(k)}\{f_\ell(k,r)Y_3(k,\xi,r')-\phi_\ell(k,r)Y_4(k,\xi,r')\}, \tag{26}$$

where

$$Y_3(k,\xi,r')=\int_0^\infty \phi_\ell(k,r')\,\hat{h}_\ell^{(+)}(\xi r')\mathrm{d}r' \tag{27}$$

and

$$Y_4(k,\xi,r')=\int_0^\infty f_\ell(k,r')\,\hat{h}_\ell^{(+)}(\xi r')\mathrm{d}r'. \tag{28}$$

Making use of eq. (13) and substituting the approximation $r^L \approx b^L(1-\mathrm{e}^{-r/b})^L$ in the expansion of $\hat{h}_\ell^{(+)}(\xi r)$ along with the following standard integral relation [33–35,38,39]

$$\int_0^s z^{\rho-1}(s-z)^{\sigma-1}{}_2F_1(d,e;f;cz)\mathrm{d}z = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}s^{\rho+\sigma-1}{}_3F_2(d,e,\rho;f,\rho+\sigma;cs), \tag{29}$$

$Y_3(k,\xi,r')$ is written as

$$Y_3(k,\xi,r') = \sum_{L=0}^{\ell} \frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!}b^{\ell+2-L} \times \frac{\Gamma(\ell+2-L)\Gamma(-i(k+\xi)b)}{\Gamma(\ell+2-L-i(k+\xi)b)} \times {}_3F_2(A_1,A_2,\ell+2-L;A_3,\ell+2-L-i(k+\xi)b;1). \tag{30}$$

Similarly, making use of eqs (18) and (29) and substituting the approximation $r^L \approx b^L(1-\mathrm{e}^{-r/b})^L$ in the expansion of $\hat{h}_\ell^{(+)}(\xi r)$, $Y_4(k,\xi,r')$ is obtained as

$$Y_4(k,\xi,r') = \sum_{L=0}^{\ell} \frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!}b^{1-L} \times \frac{\Gamma(1-\ell-L)\Gamma(-i(k+\xi)b)}{\Gamma(1-\ell-L-i(k+\xi)b)} \times {}_3F_2(1-A_1^*,1-A_2^*,-i(k+\xi)b;1-2ikb,1-\ell-L-i(k+\xi)b;1). \tag{31}$$

The next work is to evaluate the integration involved in $Y_2(k,\xi,r')$ by substituting eqs (13), (18) and (22) into eq. (25). With certain algebraic manipulations and rearrangements, one has

$$Y_2(k,\xi,r') = \frac{(k^2-\xi^2)}{\mathcal{J}_+(k)}\mathrm{e}^{ikr} \times \sum_{L=0}^{\ell} \frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!}b^{\ell+1-L} \times \Big[(1-\mathrm{e}^{-r/b})^{-\ell} \times {}_2F_1(1-A_1^*,1-A_2^*;1-2ikb;\mathrm{e}^{-r/b}) \times \int_0^r (1-\mathrm{e}^{-r'/b})^{\ell+1-L}\mathrm{e}^{i(k+\xi)r'} \times {}_2F_1(A_1,A_2;A_3;1-\mathrm{e}^{-r'/b})\mathrm{d}r' - (1-\mathrm{e}^{-r/b})^{\ell+1}{}_2F_1(A_1,A_2;A_3;1-\mathrm{e}^{-r/b}) \times \int_0^r (1-\mathrm{e}^{-r'/b})^{-\ell-L}\mathrm{e}^{i(k+\xi)r'} \times {}_2F_1(1-A_1^*,1-A_2^*;1-2ikb;\mathrm{e}^{-r'/b})\mathrm{d}r'\Big]. \tag{32}$$

Changing the quantity $(1-\mathrm{e}^{-r'/b}) = z$ and using the analytic continuation of the Gaussian hypergeometric function given in eq. (14) along with the standard integral relation [38]

$$f_\sigma(d,e;f;z) = \frac{1}{f-1}\Big[{}_2F_1(d,e;f;z) \times \int_0^z s^{\sigma-1}(1-s)^{d+e-f}{}_2F_1(d-f+1,e-f+1;2-f;s)\mathrm{d}s - z^{1-f}{}_2F_1(d-e+1,e-f+1;2-f;z) \times \int_0^z s^{\sigma+f-2}(1-s)^{d+e-f}{}_2F_1(d,e;f;s)\mathrm{d}s\Big] \tag{33}$$

leads to the convenient expression for $Y_2(k,\xi,r')$ as

$$Y_2(k,\xi,r') = (k^2-\xi^2)\sum_{L=0}^{\ell} \frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!} \times b^{2-L}[1-\mathrm{e}^{-r/b}]^{\ell+1}\mathrm{e}^{ikr} \times \sum_{n=0}^{\infty} \frac{\Gamma(n+1-ib(k-\xi))}{\Gamma(1-ib(k-\xi))n!} \times f_{n+1-\ell-L}(A_1,A_2;A_3;1-\mathrm{e}^{-r/b}). \tag{34}$$

In deriving eq. (34) the following expansion [38]

$$(1-z)^{-[1-i(k-\xi)b]} = \sum_{n=0}^{\infty} \frac{\Gamma(n+1-ib(k-\xi))}{\Gamma(1-ib(k-\xi))}\frac{z^n}{n!} \tag{35}$$

has been used judiciously.

Therefore, from eqs (23)–(34) one gets the desired expression of the off-shell Jost solution for the resultant potential as

$$f_\ell(k,\xi,r) = \frac{(k^2-\xi^2)}{\mathcal{J}_+(k)}\Big[(1-\mathrm{e}^{-r/b})^{-\ell}\mathrm{e}^{ikr}{}_2F_1(1-A_1^*,1-A_2^*;1-2ikb;\mathrm{e}^{-r/b}) \sum_{L=0}^{\ell} \frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!} \times b^{\ell+2-L}\frac{\Gamma(\ell+2-L)\Gamma(-i(k+\xi)b)}{\Gamma(\ell+2-L-i(k+\xi)b)} \times {}_3F_2(A_1,A_2,\ell+2-L;A_3,\ell+2-L-i(k+\xi)b;1) - (1-\mathrm{e}^{-r/b})^{\ell+1}\mathrm{e}^{ikr}{}_2F_1(A_1,A_2;A_3;1-\mathrm{e}^{-r/b}) \times \sum_{L=0}^{\ell} \frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!}b^{2+\ell-L} \times \frac{\Gamma(1-\ell-L)\Gamma(-i(k+\xi)b)}{\Gamma(1-\ell-L-i(k+\xi)b)} \times {}_3F_2(1-A_1^*,1-A_2^*,-i(k+\xi)b;1-2ikb,$$

$$1-\ell-L-i(k+\xi)b;\,1)\Big]$$

$$+(k^2-\xi^2)\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!}b^{2-L}$$

$$\times[1-\mathrm{e}^{-r/b}]^{\ell+1}\mathrm{e}^{ikr}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-ib(k-\xi))}{\Gamma(1-ib(k-\xi))n!}$$

$$\times f_{n+1-\ell-L}(A_1,A_2;A_3;1-\mathrm{e}^{-r/b}). \tag{36}$$

4. Off-shell *T*-matrix

To obtain the expression for off-shell T-matrix, the first task is to identify the corresponding off-shell physical or outgoing wave solution $\psi_\ell^{(+)}(k,\xi,r)$ by using the following relation [25,40]:

$$\psi_\ell^{(+)}(k,\xi,r)=\frac{\pi\xi}{2}T_\ell(k,\xi,k^2)f_\ell(k,r)$$

$$+\frac{1}{2i}[f_\ell(k,\xi,r)-f_\ell(k,-\xi,r)] \tag{37}$$

with

$$T_\ell(k,\xi,k^2)=\frac{f_\ell(k,\xi)-f_\ell(k,-\xi)}{i\pi\xi\mathcal{J}_+(k)}, \tag{38}$$

the half off-shell transition or T-matrix. Without explicit dependence on the potential, the off-shell physical or outgoing wave solution and off-shell transition or T-matrix have the following relationship [10,25,40]:

$$T_\ell(p,\xi,k^2)=\frac{2(k^2-\xi^2)}{\pi\xi\,p}\int_0^\infty \mathrm{d}r\,\hat{j}_\ell(pr)\psi_\ell^{(+)}(k,\xi,r)$$

$$-S_\ell(\xi,p,k^2), \tag{39}$$

where

$$S_\ell(\xi,p,k^2)=\frac{2(\xi^2-q^2)}{\pi\xi\,p}\int_0^\infty \mathrm{d}r\,\hat{j}_\ell(\xi r)\hat{j}_\ell(pr)$$

$$=0 \text{ for } \ell=0,$$

$$=\frac{(k^2-\xi^2)}{p^3\xi}\delta(p-\xi)\text{ for } \ell>0. \tag{40}$$

In view of eqs (36), (37), (39) and with the help of eq. (29) and the following standard integral relation [38]

$$\int_0^s z^{\delta-1}(s-z)^{\mu-1}f_\sigma(d,e;f;az)\mathrm{d}z$$

$$=\frac{\Gamma(\delta+\sigma)\Gamma(\mu)}{\sigma(\sigma+f-1)\Gamma(\delta+\sigma+\mu)}a^\sigma s^{\delta+\sigma+\mu-1}$$

$$\times{}_4F_3(1,\sigma+d,\sigma+e,\delta+\sigma;$$

$$\sigma+1,\sigma+f,\delta+\sigma+\mu;as) \tag{41}$$

with Re $\sigma>0$, Re $\sigma+a>1$, Re $\mu>0$, Re $\delta>0$, $|as|<1$, the off-shell transition matrix or T-matrix $T_\ell(p,\xi,k^2)$ for all ℓ reads as

$$T_\ell(p,\xi,k^2)=-\frac{1}{2\pi p\xi}[S_1(p,\xi,k^2)-S_2(p,-\xi,k^2)$$

$$-S_3(-p,\xi,k^2)+S_4(-p,-\xi,k^2)], \tag{42}$$

where

$$S_1(p,\xi,k^2)=(k-p)(k-\xi)\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!}$$

$$\times b^{\ell+1-L}\frac{1}{\mathcal{J}_+(k)}\frac{\Gamma(1-\ell-L)\Gamma(1-i(k+\xi)b)}{\Gamma(1-\ell-L-i(k+\xi)b)}$$

$$\times{}_3F_2(1-A_1^*,1-A_2^*,-i(k+\xi)b;1-2ikb,$$

$$1-\ell-L-i(k+\xi)b;1)$$

$$\times\sum_{M=0}^{\ell}\frac{(i)^{2M-\ell}(\ell+M)!}{(2ip)^M M!(\ell-M)!}b^{-M}$$

$$\times\frac{\Gamma(\ell+2-M)\Gamma(1-i(k+p)b)}{\Gamma(\ell+2-M-i(k+p)b)}$$

$$\times{}_3F_2(A_1,A_2,\ell+2-M;A_3,$$

$$\ell+2-M-i(k+p)b;1)$$

$$+(k-p)(k^2-\xi^2)\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!}$$

$$\times b^{2-L}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-ib(k-\xi))}{\Gamma(1-ib(k-\xi))}$$

$$\times\sum_{M=0}^{\ell}\frac{(i)^{2M-\ell+1}(\ell+M)!}{(2ip)^M M!(\ell-M)!}b^{-M}$$

$$\times\frac{\Gamma(n+3-M-L)}{(n+1-L-\ell)(n+2-L+\ell)}$$

$$\times\frac{\Gamma(1-i(k+p)b)}{\Gamma\,(n+3-M-L-i(k+p)b)}$$

$$\times{}_4F_3(1,n+1-L-\ell+A_1,n+1-L-\ell+A_2,$$

$$n+3-M-L;n+2-L-\ell,n+3-L+\ell,$$

$$n+3-M-L-i(k+p)b;1). \tag{43}$$

From eq. (43) one may get the compact expressions for $S_2(p,-\xi,k^2)$, $S_3(p,-\xi,k^2)$ and $S_4(p,-\xi,k^2)$ by replacing $p\to -p$ and $\xi\to-\xi$. Thus, eq. (43) together with eq. (42) creates the off-shell transition matrix. For the limiting condition $p\to k$ our off-shell version yields the correct half-shell limit which is given by

$$T_\ell(k,\xi,k^2)=\left(\frac{k}{\xi}\right)^\ell\left[\frac{f_\ell(k,\xi)-f_\ell(k,-\xi)}{i\pi\xi\mathcal{J}_+(k)}\right], \tag{44}$$

where the off-shell Jost function

$$f_\ell(k,\xi) = \frac{i(k-\xi)\xi^\ell}{(2\ell+1)!!}\sum_{L=0}^{\ell}\frac{(i)^{2L-\ell}(\ell+L)!}{(2i\xi)^L L!(\ell-L)!}$$
$$\times b^{\ell+1-L}\frac{\Gamma(\ell-L+2)\,\Gamma(1-i(k+\xi)b)}{\Gamma(\ell-L+2-i(k+\xi)b)}$$
$$\times {}_3F_2(A_1, A_2,$$
$$\ell+2-L;\, A_3, \ell+2-L-i(k+\xi)b;\, 1). \qquad (45)$$

Equation (45) is in agreement with [13] for s-wave ($\ell = 0$). Also, the results in eqs (42) and (45) reproduce correct Hulthén limit [24] for s-wave ($\ell = 0$) and [10] for all partial waves respectively, when $U_0 = 0$. We also examined the Coulomb limits of our generated expressions and verified it with [25].

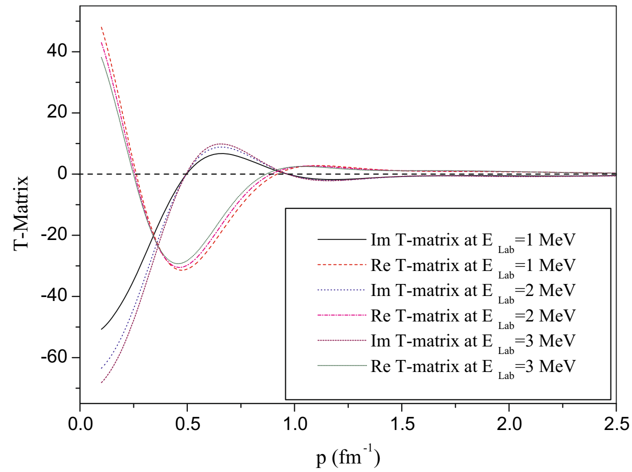

Figure 2. T-matrix $T_\ell(p,\xi,k^2)$ as a function of p for the (p–d) system with $q = 0.5\,\text{fm}^{-1}$.

5. Results and discussion

In figures 1 and 2 we present the off-shell T-matrices using eq. (42) at three different laboratory energies (E_{Lab} =1, 2 and 3 MeV) with two different values of off-shell momentum ($\xi = 0.25$ fm^{-1} and 0.5 fm^{-1}) for $^2S_{1/2}$ (p–d) system. The parameters for $^2S_{1/2}$ (p–d) state which fit the correct binding energy are $\alpha = 0.499$ fm^{-1}, β =1.08 fm^{-1} and we use $\hbar^2/2\mu$ =31.1025 MeV fm^2 and $U_0 b$ =0.04629 fm^{-1} for our numerical computation.

Figure 1 shows the off-shell T-matrix $T_\ell(p,\xi,k^2)$ with $\xi = 0.25\,\text{fm}^{-1}$ of the (p–d) system as a function of the off-shell momentum p for the $^2S_{1/2}$ state at the laboratory energies (E_{Lab} =1, 2 and 3 MeV) and figure 2 shows the off-shell T-matrix $T_\ell(p,\xi,k^2)$ with $\xi = 0.5\,\text{fm}^{-1}$ of the (p–d) system as a function of the off-shell momentum p for the $^2S_{1/2}$ state at the laboratory energies (E_{Lab} =1, 2 and 3 MeV). These graphs show that both Re $T_\ell(p,\xi,k^2)$ and Im $T_\ell(p,\xi,k^2)$ oscillate but approach zero as p becomes large. Interestingly, for low values of laboratory energies, our potential shows large off-shell effects which is in conformity with the observations of earlier works corresponding to various kinds of potentials [20,32,41,42]. It is observed in the literature that all realistic nucleon–nucleon interaction models demonstrate comparable on-shell properties despite the fact that they often result from different approaches to nucleon–nucleon dynamics. But the situation is not so apparent with respect to the off-shell behaviour of the nucleon–nucleon interactions. The corresponding confirmation is observed in three- or more-particles problems concerning the nucleon–nucleon sub-system. Irrational off-shell characteristics have also been observed with the separable nucleon–nucleon models. Among the separable potentials, the Graz model [20,26,42] provides a reasonable overall description of the nucleon–nucleon data. It is motivating to note that the construction of exact analytical expressions for off-shell quantities related to our potential model constitutes essential tools for solving more refined three-body problems.

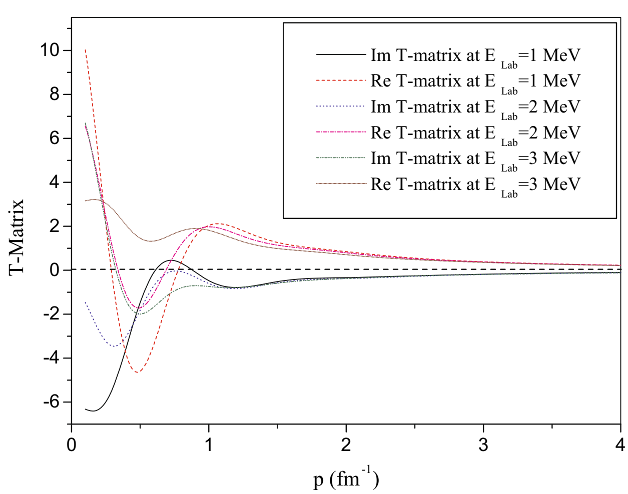

Figure 1. T-matrix $T_\ell(p,\xi,k^2)$ as a function of p for the (p–d) system with $q = 0.25\,\text{fm}^{-1}$.

These indicate that the off-shell behaviour of the proposed model interaction is quite acceptable. We have also proved that our expression for T-matrix reproduced correct half-shell limit, thereby the phase shifts at the respective energies.

Table 1 shows the phase parameters of the particular system calculated from the half-shell T-matrix which are in reasonable agreement with those of Chen *et al* [43] and Ishikawa [44]. However, the discrepancy in very low-energy phase shift data, particularly for

Table 1. Scattering phase shifts for the (p–d) system.

E_{Lab} (MeV)	Phase shift $^2S_{1/2}$ state (degree) (present work)	Phase shift $^2S_{1/2}$ state (degree) Ref. [43]	Phase shift $^2S_{1/2}$ state (degree) Ref. [44]
0.075	−2.188	−0.113	–
0.15	−3.257	−0.537	−0.527
0.3	−5.019	−1.96	–
0.45	−6.593	−3.73	–
0.6	−8.069	−5.62	–
0.75	−9.475	−7.53	–
0.9	−10.825	−9.40	–
1.0	−11.610	−10.6	−10.5
1.05	−12.124	−11.2	–
1.2	−13.378	−13.0	–
1.35	−14.589	−14.6	–
1.5	−15.762	−16.2	−16.2
2.0	−21.012	−21.1	−21.0
3	−25.753	−28.8	−28.7

$E_{\text{Lab}} = 0.075$–0.75 MeV might be due to the improper accountability of the resultant forces produced in actual experimentation.

In charged hadron scattering, the electromagnetic forces play a dominant role over the nuclear ones up to a few hundred keV. Our model calculation might overestimate the Coulomb force compared to the actual one in real situation. This may be the probable reason as to why the low-energy phase parameters reproduce slightly larger negative values than the standard data [43,44]. In view of small discrepancies between the results of these phase shift analysis [43,44] and of our calculation at low energies, we desire to explore to what extent our model calculations will be able to yield realistic cross-section data. At low and intermediate energies, the coupled channel approach to potential scattering with central nucleon–nucleon potential model has proved to be quite successful for estimating the differential cross-sections (excitation functions) for elastic scattering. Several phenomenological two-nucleon models, CD-Bonn, Argonne-V18 (AV18), Nijmegen-I, Nijmegen-II and Reid93 [45–49], exist in the literature which describe two-nucleon scattering observables accurately. At low incident beam energies, up to 20 MeV, the differential cross-section of nucleon–deuteron scattering is explained properly using solely two-nucleon potentials. Theoretical and experimental cross-sections for (p–d) scattering were studied by Sagara *et al* [50] and Huttel *et al* [51]. A few years ago, two independent calculations of proton–deuteron elastic scattering observables with rigorous inclusion of the Coulomb force were performed by Deltuva *et al* [52,53]. Our

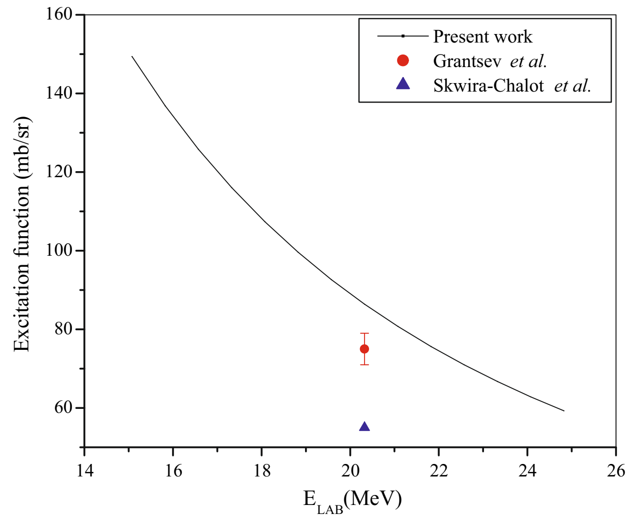

Figure 3. Excitation functions of (p–d) elastic scattering at 12° (C.M.).

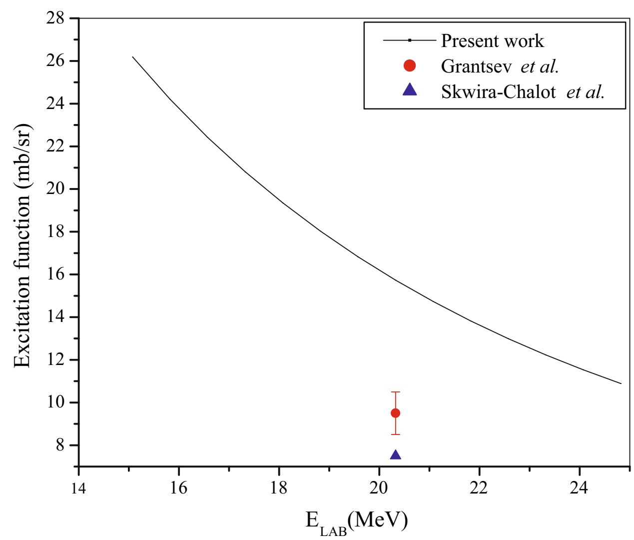

Figure 4. Excitation functions of (p–d) elastic scattering at 23° (C.M.).

cross-section data for (p–d) elastic scattering are portrayed in figures 3 and 4 up to 25 MeV for $\theta_{\text{C.M.}} = 12°$ and 23°, respectively. Our results for (p–d) scattering cross-section agree qualitatively with those of Skwira-Chalot *et al* [54] and other experimental data [55–61] but differ quantitatively. This is quite obvious as we have considered here low-energy s-wave scattering whereas in refs [54–61] the proton beams of 31–190 MeV were used as the projectiles. However, our results show the correct trends of the associated cross-sections. Another probable reason for these discrepancies in phase parameters as well as in cross-section may be attributed to spin–orbit force due to the nucleon magnetic moments that play a significant role in such calculations.

6. Conclusions

The off-shell solutions and transition matrices have all been derived with closed form expressions, with a focus on their limiting characteristics. These findings are obtained using various methods in the r-space approach. The approaches we used can be adapted in several ways. For example, our method may be more appealing to theoretical physicists as it is equally applicable to exponential type of potentials. The elements of the T-matrix are measurable quantities and the interaction potential can be generated if the experimental values of the transition matrices are available. The off-shell T-matrix plays a vital role for understanding various physical processes. These off-shell versions of the T-matrices are the essential components of the Fadeev [62] equation for three-body computation. The importance of the investigation of the off-shell T-matrix consists in the fact that the elements of the T-matrix are measurable quantities and these off-shell elements are generally obtained from the analysis of the p–p bremsstrahlung, (p, 2p) reaction, nuclear matter, three-body bound and scattering problems [63,64]. Calculation of a direct reaction, such as the inelastic scattering of nucleons by nuclei requires the off-shell T-matrix element and also for the calculation of binding energy of nuclear matter as well as finite nuclei. The deuteron photodisintegration cross-section was also found to be sensitive to the off-shell T-matrix [65]. We analysed inelastic scattering of nucleon–nucleus systems using new model of interaction which is in accordance with the proper binding energy and scattering phase shifts of the system concerned. The partial-wave off-shell T-matrices carry as much information as the potential. Therefore, for most general treatment of the physical processes, the half and off-shell transition matrices are in fact the more expedient representations of the interaction than the potential itself. The transition matrices are used to calculate the binding energy in nuclear matter as well as to generate the shell-model spectrum. As a result, the T-matrix calculation allows us to make the most of the available information about the two-nucleon systems.

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