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Continuous limit, higher-order rational solutions and relevant dynamical analysis for Belov–Chaltikian lattice equation with 3×3 Lax pair

TING ZHANG, XIAO-YONG WEN* and XUE-KE LIU

School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, People's Republic of China
*Corresponding author. E-mail: xiaoyongwen@163.com

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Abstract. Belov–Chaltikian (BC) lattice equation, which is related to the research of lattice analogues of W-algebras, is under consideration in this work. This equation may be viewed as an extension of the Volterra lattice equation. Firstly, we correspond BC lattice equation to several continuous equations under the continuous limit. Secondly, based on the known 3×3 matrix form Lax pair of this discrete equation, we construct its discrete generalised $(m, 3N-m)$-fold Darboux transformation for the first time and successfully popularise this technique from 2×2 Lax pair to 3×3 Lax pair. Finally, by applying the resulting Darboux transformation, we get its higher-order rational solutions and analyse their singular trajectories and dynamics using the graphics and limit-state analysis.

Keywords. Belov–Chaltikian lattice equation; continuous limit; discrete generalised (m, 3N-m)-fold Darboux transformation; rational solution; singular trajectory analysis.

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1. Introduction

In recent years, nonlinear differential difference equations (NDDEs) have attracted increasing interest of the scientists due to their applications in various fields, such as the population dynamics, nonlinear lattice dynamics and propagation of electrical signals in circuits [1–3]. NDDEs also have diverse algebraic and geometric properties, such as Lax pair, N-soliton solutions, Hamiltonian structures, conservation laws and so on [1–7]. For example, Volterra lattice equation

$$u_{n,t} = u_n(u_{n+1} - u_{n-1}), \tag{1}$$

is an important NDDE modelling biological process [1], where $u_n = u(n, t)$ is the function of the space variable n and time variable t. In eq. (1), u_n denotes the number of nth species, which increases by colliding with $(n+1)$th species and decreases by colliding with $(n-1)$th species [1], as shown in figure 1.

Besides that, eq. (1) is also related to the study of the spectrum of Langmuir wave in plasma physics (see also ref. [1] and reference therein). In 1993, Belov and Chaltikian studied the lattice analogues of W-algebras and first proposed the following NDDE [8]:

$$\begin{cases} u_{n,t} = u_n(u_{n+1} - u_{n-1}) - (w_{n+1} - w_n), \\ w_{n,t} = w_n(u_{n+1} - u_{n-2}), \end{cases} \tag{2}$$

which is later called Belov–Chaltikian (BC) lattice equation, where $u_n = u(n, t)$, $w_n = w(n, t)$ are functions of the discrete space variable n and time variable t [8]. If $w_n = 0$, eq. (2) can be reduced to eq. (1). So eq. (2) is also thought of as a generalisation of eq. (1). There are some known research results about BC lattice equation (2). Belov and Chaltikian [8] gave its Hamiltonian structure and proved the integrality of eq. (2). Hikami and Inoue [9] studied the lattice W_3 algebra and showed that the lattice eq. (2) is related to the lattice W_3 algebra. In ref. [10], the soliton solutions have been concluded by using the bilinear form of the Bäcklund transform method and nonlinear superposition formula of eq. (2). In refs [11–13], the researchers investigated similarity reduction, generalised symmetries, integrability, recursion operators, factorisation and master symmetries of eq. (2). It is clearly demonstrated that the factorisation operator is a Hamiltonian operator and eq. (2) is a double Hamiltonian system. Yang *et al* [14] presented the

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exact solutions of eq. (2) using the ADM-Padé technique. Tsuchida [15] gave a new Lax representation in the operator form for eq. (2) by the pair of linear equations. In ref. [16], Darboux transformation (DT) of eq. (2) has been constructed based on its new Lax representation in matrix form as follows:

$$E\phi_n = U_n\phi_n = \begin{pmatrix} 0 & \lambda & 0 \\ -u_n & 1 & w_n \\ \lambda & 0 & 0 \end{pmatrix}\phi_n, \tag{3}$$

$$\phi_{n,t} = V_n\phi_n = \begin{pmatrix} u_{n-1} & 1 & 0 \\ \dfrac{-u_n}{\lambda} & \dfrac{1}{\lambda}+u_n & \dfrac{w_n}{\lambda} \\ 1 & 0 & u_{n-2} \end{pmatrix}\phi_n, \tag{4}$$

where λ is a spectral parameter independent of time t, the shift operator E meets the conditions $Ef(n,t) = f(n+1,t)$, $E^{-1}f(n,t) = f(n-1,t)$, $\phi_n = (\varphi_n, \psi_n, \chi_n)^{\mathrm{T}}$ is a basic solution of (3) and (4) (T represents transpose of the vector or matrix). It is easily verified that the consistency condition $U_{n,t} = (EV_n)U_n - U_nV_n$ of (3) and (4) produces eq. (2), and the authors have obtained its exact solutions by applying the resulting DT. Xu and Xue [17] constructed the Hamiltonian structure and infinite conservation laws of eq. (2) lattice hierarchy. Qin *et al* [18] derived the N-soliton solutions in terms of Wronskian determinant for eq. (2) by using the Bell-polynomial approach. In ref. [19], the quasiperiodic solutions of eq. (2) lattice hierarchy have been obtained.

Exact solutions are of great theoretical significance for explaining the physical phenomena expressed by nonlinear equations. Some methods seeking exact solutions, especially soliton solutions, such as the DT method [20], Hirota bilinear method [21] and Riemann–Hilbert method [22,23], have been proposed and developed. Among them, the DT method is known to be an effective way to build exact solutions [20] and has been extended to solve Lax integrable NDDEs with the 2×2 matrix spectral problem. Then some N-fold DTs also have been developed to solve the discrete 3×3 matrix spectral problem [6,16,24,25]. Subsequently in refs [7,26–28], a discrete generalised $(m, N-m)$-fold DT, which can be taken as a generalisation of the N-fold DT, has been proposed based on 2×2 Lax pair, and the main advantage of this technique is that it can yield not only soliton solutions, but also rational solutions and their mixed solutions. Although this method has been extended to a generalised Toda lattice equation with 3×3 Lax pair to give some rational solutions [29], our initial research only gives a higher-order rational solution, which is not comprehensive and systematic. To the best of our knowledge, there are still no relevant research on continuous limit, discrete generalised $(m, 3N-m)$-fold DT and rational solutions for eq. (2). Therefore, we will do further research on eq. (2) using the continuous limit technique and discrete generalised $(m, 3N-m)$-fold DT method in the present work.

The structure of this paper is as follows. In §2, the continuous limit technique is used to correspond the discrete eq. (2) to the continuous equations. In §3, based on the known 3×3 Lax pairs (3) and (4), the discrete generalised $(m, 3N-m)$-fold DT of eq. (2) is constructed for the first time. In §4, some rational solutions are obtained by applying the resulting discrete generalised DT and their singular trajectories and relevant dynamics are investigated using the asymptotic analysis technique. Some conclusions are summed up in the last section.

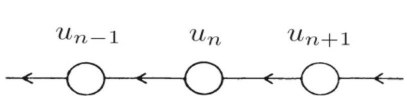

Figure 1. A simple Volterra system (see also the third figure in ref. [1]).

2. Continuous limit of eq. (2)

Admittedly, an important research topic in soliton theory and integrable systems [30–32] is the study of the continuous limit of the NDDEs. In what follows, we apply the continuum limit to eq. (2).

When $w_n = 0$, eq. (2) is just Volterra lattice equation (1). If u_n is assumed to be

$$\begin{aligned} u_n &= -3\varepsilon^2 u(-n+6\varepsilon t, -\varepsilon^3 t) - 3 + o(\varepsilon^3) \\ &\equiv -3\varepsilon^2 u(x,\tau) - 3 + o(\varepsilon^3), \end{aligned} \tag{5}$$

then eq. (1) can be rewritten as

$$(u_\tau + 6uu_x + u_{xxx})\varepsilon^5 + o(\varepsilon^6) = 0, \tag{6}$$

where ε is a very small arbitrary parameter. If we write τ for t and ignore $O(\varepsilon^6)$ of eq. (6), we observe that eq. (1) can be treated as the well-known KdV equation. In other words, Volterra lattice equation (1) can be taken as the discrete KdV equation.

As for eq. (2), if the continuous conditions are chosen as

$$\begin{cases} u_n &= \varepsilon - \varepsilon u[(n+t)\varepsilon^2, \varepsilon^2 t] + o(\varepsilon^2) \\ &\equiv \varepsilon^2 - u(x,\tau) + o(\varepsilon^2), \\ w_n &= \varepsilon^2 + \varepsilon w[(n+t)\varepsilon^2, \varepsilon^2 t] + o(\varepsilon^2) \\ &\equiv w(x,\tau) + o(\varepsilon^2), \end{cases} \tag{7}$$

then eq. (2) can be transformed into

$$\begin{cases} (u_\tau + u_x - w_x)\varepsilon^3 + O(\varepsilon^4) = 0, \\ (w_\tau - w_x)\varepsilon^3 + O(\varepsilon^4) = 0, \end{cases} \tag{8}$$

which is a linear system if we neglect $O(\varepsilon^4)$ of eq. (8).

When the limit conditions

$$\begin{cases} u_n = \varepsilon^2 - u[(n+t)\varepsilon^3, \varepsilon^3 t] + o(\varepsilon) \\ \quad \equiv \varepsilon^2 - u(x,\tau) + o(\varepsilon), \\ w_n = w[(n+t)\varepsilon^3, \varepsilon^3 t] + o(\varepsilon) \\ \quad \equiv w(x,\tau) + o(\varepsilon), \end{cases} \tag{9}$$

are applied to eq. (2), then eq. (2) can be transformed into

$$\begin{cases} (u_\tau + u_x + 2uu_x + w_x)\varepsilon^3 + O(\varepsilon^4) = 0, \\ (w_\tau + w_x - 3wu_x)\varepsilon^3 + O(\varepsilon^4) = 0, \end{cases} \tag{10}$$

which is a new nonlinear system if we write τ for t and ignore $O(\varepsilon^4)$ in eq. (10).

If the limit conditions

$$\begin{cases} u_n = \varepsilon^2 - u[(n+\varepsilon t)\varepsilon^2, \varepsilon^2 t] + o(\varepsilon) \\ \equiv \varepsilon^2 - u(x,\tau) + o(\varepsilon), \\ w_n = 1 + w[(n+\varepsilon t)\varepsilon^2, \varepsilon^2 t] + o(\varepsilon) \\ \equiv w(x,\tau) + o(\varepsilon), \end{cases} \tag{11}$$

are applied to eq. (2), then eq. (2) is transformed into

$$\begin{cases} (u_\tau + 2uu_x - w_x)\varepsilon^2 + O(\varepsilon^3) = 0, \\ (w_\tau - 3u_x - wu_x)\varepsilon^2 + O(\varepsilon^3) = 0, \end{cases} \tag{12}$$

which is also a new nonlinear system if we write τ for t and ignore $O(\varepsilon^3)$ of eq. (12).

Remark 1. Through continuous limit, we have deduced two new continuous nonlinear equations (10) and (12), which are worthy of further study.

3. Discrete generalised $(m, 3N - m)$-fold DT

In the present section, we will construct the discrete generalised $(m, 3N - m)$-fold DT of eq. (2) based on its known 3×3 Lax pairs (3) and (4). First of all, we construct the subsequent gauge transformation

$$\tilde{\phi}_n = T_n \phi_n, \tag{13}$$

which will change (3) and (4) into the expressions

$$\tilde{\phi}_{n+1} = \tilde{U}_n \tilde{\phi}_n = T_{n+1} U_n T_n^{-1} \tilde{\phi}_n,$$
$$\tilde{\phi}_{n,t} = \tilde{V}_n \tilde{\phi}_n = (T_{n,t} + V_n T_n) T_n^{-1} \tilde{\phi}_n, \tag{14}$$

where $\tilde{U}_n$, $\tilde{V}_n$ keep the identical forms as U_n, V_n except for substituting the new $\tilde{u}_n$, $\tilde{w}_n$ for the old u_n, w_n. To guarantee the validity and exactness of the discrete generalised $(m, 3N - m)$-fold DT, we have to construct a specific Darboux matrix T_n as

$$T_n = \begin{pmatrix} (1 - B_n^{(1)}) + \sum_{j=1}^{N} A_n^{(j)} \lambda^j & \sum_{j=1}^{N} B_n^{(j)} \lambda^j & \sum_{j=0}^{N-1} C_n^{(j)} \lambda^j \\ \sum_{j=0}^{N-1} D_n^{(j)} \lambda^j & 1 + \sum_{j=1}^{N} E_n^{(j)} \lambda^j & \sum_{j=0}^{N-1} F_n^{(j)} \lambda^j \\ -H_n^{(1)} + \sum_{j=1}^{N} G_n^{(j)} \lambda^j & \sum_{j=1}^{N} H_n^{(j)} \lambda^j & A_{n-1}^{(N)} \lambda^N + \sum_{j=0}^{N-1} I_n^{(j)} \lambda^j \end{pmatrix}, \tag{15}$$

where N is a positive integer number representing the order of DT and all coefficients in front of λ^j $(0 \le j \le N)$ are functions of the variables n, t and can be ensured by

$$\begin{cases} T_n^{(0)}(\lambda_i)\phi_{i,n}^{(0)}(\lambda_i) = 0, \\ T_n^{(0)}(\lambda_i)\phi_{i,n}^{(1)}(\lambda_i) + T_n^{(1)}(\lambda_i)\phi_{i,n}^{(0)}(\lambda_i) = 0, \\ T_n^{(0)}(\lambda_i)\phi_{i,n}^{(2)}(\lambda_i) + T_n^{(1)}(\lambda_i)\phi_{i,n}^{(1)}(\lambda_i) \\ +T_n^{(2)}(\lambda_i)\phi_{i,n}^{(0)}(\lambda_i) = 0, \\ \cdots\cdots, \\ \sum_{j=0}^{v_i} T_n^{(j)}(\lambda_i)\phi_{i,n}^{(v_i - j)}(\lambda_i) = 0, \end{cases}$$

where

$$\phi_n^{(k)}(\lambda_i) = \frac{1}{k!} \frac{\partial^k}{\partial \lambda_i^k} \phi_n(\lambda_i)$$

is derived by Taylor expansion of $\phi_{i,n}(\lambda_i + \varepsilon) = \phi_{i,n}^{(0)}(\lambda_i) + \phi_{i,n}^{(1)}(\lambda_i)\varepsilon + \phi_{i,n}^{(2)}(\lambda_i)\varepsilon^2 + \phi_{i,n}^{(3)}(\lambda_i)\varepsilon^3 + \cdots$ around $\varepsilon = 0$, while $T_n^{(j)}$ are decided by the binomial expansion of $T(\lambda_i + \varepsilon) = T_n^{(0)} + T_n^{(1)}\varepsilon + \cdots + T_n^{(v_i)}\varepsilon^{v_i}$ and v_i satisfies $3N = m + \sum_{j=1}^{m} v_i$. Based on the above detailed process, we can conclude that eq. (2) has the discrete generalised $(m, 3N - m)$-fold DT theorem as follows:

Theorem 1. *Let* $\phi_{i,n}(\lambda_i) = (\varphi_{i,n}(\lambda_{i,n}), \psi_{i,n}(\lambda_i), \chi_{i,n}(\lambda_i))^{\mathrm{T}}$ *be* m *solutions of* (3) *and* (4) *corresponding to* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *with the seed solutions* u_n, w_n *of eq.* (2)*, then the generalised* $(m, 3N - m)$*-fold DT from* u_n, w_n *to* $\tilde{u_n}$, $\tilde{w_n}$ *is given as*

$$\tilde{u}_n = -\frac{F_{n+1}^{(N-1)} - E_{n+1}^{(N)} u_n}{A_n^{(N)}} + \frac{G_n^{(N)} E_{n+1}^{(N)}}{A_n^{(N)} I_n^{(N)}} w_n,$$
$$\tilde{w}_n = \frac{E_{n+1}^{(N)}}{I_n^{(N)}} w_n, \tag{16}$$

where

$$A_n^{(N)}=\frac{\Delta A_n^{(N)}}{\Delta_1},\quad E_{n+1}^{(N)}=\frac{\Delta E_{n+1}^{(N)}}{\Delta_2},\quad F_{n+1}^{(N-1)}=\frac{\Delta F_{n+1}^{(N-1)}}{\Delta_2},\quad G_n^{(N)}=\frac{\Delta G_n^{(N)}}{\Delta_1},\quad I_n^{(N)}=\frac{\Delta A_{n-1}^{(N)}}{\Delta_1}, \tag{17}$$

with $\Delta_1=\det([\Delta_1^{(1)},\ \Delta_1^{(2)},\dots,\Delta_1^{(m)}])^{\mathrm{T}}$, $\Delta_2=\det([\Delta_2^{(1)},\ \Delta_2^{(2)},\dots,\Delta_2^{(m)}])^{\mathrm{T}}$, $\Delta_1^{(i)}=(\Delta_{1,j,s}^{(i)})_{(v_i+1)\times 3N}$, $\Delta_2^{(i)}=(\Delta_{2,j,s}^{(i)})_{(v_i+1)\times 3N}$, in which $\Delta_{1,j,s}^{(i)},\ \Delta_{2,j,s}^{(i)}$ $(1\le j\le v_i+1,\ 0\le s\le 3N,\ 0\le i\le m)$ are decided as

$$\Delta_{1,j,s}^{(i)}=\begin{cases}\sum_{k=0}^{j-1}C_{N-s+1}^{k}\lambda_i^{N-s-k+1}\varphi_{i,n}^{(j-1-k)} & \text{for } 1\le j\le v_i+1,\ 1\le s\le N,\\ \sum_{k=0}^{j-1}C_{2N-s+1}^{k}\lambda_i^{2N-s-k+1}\psi_{i,n}^{(j-1-k)} & \text{for } 1\le j\le v_i+1,\ N+1\le s\le 2N-1,\\ \sum_{k=0}^{j-1}C_{2N-s+1}^{k}\lambda_i^{2N-s-k+1}\psi_{i,n}^{(j-1-k)}-\varphi_{i,n}^{j-1} & \text{for } 1\le j\le v_i+1,\ s=2N,\\ \sum_{k=0}^{j-1}C_{3N-s}^{k}\lambda_i^{3N-s-k}\chi_{i,n}^{(j-1-k)} & \text{for } 1\le j\le v_i+1,\ 2N+1\le s\le 3N,\end{cases}$$

$$\Delta_{2,j,s}^{(i)}=\begin{cases}\sum_{k=0}^{j-1}C_{N-s}^{k}\lambda_i^{N-s-k}\varphi_{i,n}^{(j-1-k)} & \text{for } 1\le j\le 1+v_i,\ 1\le s\le N,\\ \sum_{k=0}^{j-1}C_{2N-s+1}^{k}\lambda_i^{2N-s-k+1}\psi_{i,n}^{(j-1-k)} & \text{for } 1\le j\le 1+v_i,\ N+1\le s\le 2N,\\ \sum_{k=0}^{j-1}C_{3N-s}^{k}\lambda_i^{3N-s-k}\chi_{i,n}^{(j-1-k)} & \text{for } 1\le j\le 1+v_i,\ 2N+1\le s\le 3N,\end{cases}$$

where $\Delta A_n^{(N)}$ is given by replacing the first column of determinant Δ_1 by the column vector $(x^{(1)},\ x^{(2)},\dots,x^{(m)})^{\mathrm{T}}$ with $x^{(i)}=(x_j^{(i)})_{(v_i+1)\times 1}$, in which $x_j^{(i)}=\sum_{k=0}^{j-1}C_{N-s}^{k}\lambda_i^{N-s-k}\varphi_{i,n}^{(j-1-k)}$ $(1\le j\le v_i+1,\ 1\le s\le 3N)$, $\Delta E_n^{(N)}$ and $\Delta F_n^{(N-1)}$ are given by replacing the $(N+1)$th and the $(2N+1)$th column of determinant Δ_2 by the column vector $(y^{(1)},\ y^{(2)},\dots,y^{(m)})^{\mathrm{T}}$ with $y^{(i)}=(y_j^{(i)})_{(v_i+1)\times 1}$, in which $y_j^{(i)}=-\sum_{k=0}^{j-1}C_{2N-s}^{k}\lambda_i^{2N-s-k}\psi_{i,n}^{(j-1-k)}$ $(1\le j\le v_i+1,\ 1+N\le s\le 3N)$ and $-\sum_{k=0}^{j-1}C_{3N-s}^{k}\lambda_i^{3N-s-k}\psi_{i,n}^{(j-1-k)}(1\le j\le v_i+1,1+2N\le s\le 3N)$. Similarly, $\Delta G_n^{(N)}$ is given by substituting the first column of determinant Δ_1 by the column vector $(z^{(1)},\ z^{(2)},\dots,z^{(m)})^{\mathrm{T}}$ with $z^{(i)}=(z_j^{(i)})_{(v_i+1)\times 1}$, in which $z_j^{(i)}=-A_{n-1}^{(N)}(\sum_{k=0}^{j-1}C_{N-s+1}^{k}\lambda_i^{N-s-k+1}\chi_{i,n}^{(j-1-k)})$ $(1\le j\le v_i+1,\ 1\le s\le 3N)$. Here, $\Delta A_{n-1}^{(N)}$ is obtained from $\Delta A_n^{(N)}$ by substituting n with $n-1$. $\Delta E_{n+1}^{(N-1)}$ and $\Delta F_{n+1}^{(N)}$ are obtained from $\Delta E_n^{(N-1)}$ and $\Delta F_n^{(N)}$ by substituting n with $n+1$.

Remark 2. Here we call expressions (2) and (16) using m spectral parameters the discrete generalised $(m,3N-m)$-fold DT, in which m is the number of spectral parameters used, N denotes the order number of DT and $3N-m$ is the sum of the order number of Taylor expansion of $\phi_{i,n}$ used. For a better understanding of the discrete generalised $(m,3N-m)$-fold DT in the above theorem, the readers can refer to refs [26–29].

Below, we will use several special cases of Theorem 1 and produce some rational solutions of eq. (2).

4. Rational solutions, singular trajectory and relevant dynamical analysis

In this section, we will get some rational solutions of eq. (2) by applying the discrete generalised $(m,3N-m)$-fold DT. Substituting the initial solutions $u_n=w_n=3$ into Lax pairs (3) and (4) with $\lambda=\lambda_k$ leads to one fundamental solution as

$$\phi_n=\begin{pmatrix}\varphi_n\\ \psi_n\\ \chi_n\end{pmatrix}=\begin{pmatrix}C_{k,1}\tau_1^n e^{\rho_1 t+\delta(\varepsilon)}+C_{k,2}\tau_2^n e^{\rho_2 t+\delta(\varepsilon)}+C_{k,3}\tau_3^n e^{\rho_3 t-\delta(\varepsilon)}\\ \frac{1}{\lambda_1}(C_{k,1}\tau_1^{n+1}e^{\rho_1 t+\delta(\varepsilon)}+C_{k,2}\tau_2^{n+1}e^{\rho_2 t+\delta(\varepsilon)}+C_{k,3}\tau_3^{n+1}e^{\rho_3 t-\delta(\varepsilon)})\\ \lambda_k(C_{k,1}\tau_1^{n-1}e^{\rho_1 t+\delta(\varepsilon)}+C_{k,2}\tau_2^{n-1}e^{\rho_2 t+\delta(\varepsilon)}+C_{k,3}\tau_3^{n-1}e^{\rho_3 t-\delta(\varepsilon)})\end{pmatrix}, \tag{18}$$

where $\delta(\varepsilon) = (-1+9\lambda_1)^{\frac{2}{3}} \sum_{j=0}^{3N-1} e_j \varepsilon^{3j}$, e_j $(j = 0, 1, \ldots, 3N-1)$, ε is the same as the previous definition, τ_1, τ_2 and τ_3 are three different roots of the equation $\tau^3-\tau^2+3\tau\lambda_k-3\lambda_k^2=0$, $\rho_1 = 3+\frac{\tau_1}{\lambda_k}$, $\rho_2 = 3+\frac{\tau_2}{\lambda_k}$, $\rho_3 = 3+\frac{\tau_3}{\lambda_k}$, and $C_{k,1}$, $C_{k,2}$, $C_{k,3}$ are arbitrary constants. Let $\lambda = \lambda_k+\varepsilon^3$, $\lambda_k = \frac{1}{9}$. The vector function ϕ_n has the following Taylor expansion around $\varepsilon = 0$ given by

$$\phi_n(\varepsilon^3) = \phi_n^{(0)} + \phi_n^{(1)}\varepsilon^3 + \phi_n^{(2)}\varepsilon^6 + \phi_n^{(3)}\varepsilon^9 + \phi_n^{(4)}\varepsilon^{12} + \phi_n^{(5)}\varepsilon^{15} + \cdots. \quad (19)$$

We can obtain different rational solutions due to different parameters $C_{k,1}$, $C_{k,2}$, $C_{k,3}$. Next, the vector function ϕ_n at $\lambda = \lambda_1$ is considered with two kinds of Taylor series expansions:

- **The first type expansion**.
  When $C_{1,1} = 0$, $C_{1,2} = -C_{1,3} = \frac{I}{\varepsilon}$, where I is an imaginary unit, $\phi_n^{(j)}$ can be given as

$$\phi_n^{(0)} = \begin{pmatrix} \varphi_n^{(0)} \\ \psi_n^{(0)} \\ \chi_n^{(0)} \end{pmatrix} = -\mathrm{e}^{6t} 3^{-n+\frac{1}{6}} \begin{pmatrix} 3s \\ 9s+9 \\ s-1 \end{pmatrix},$$

$$\phi_n^{(1)} = \begin{pmatrix} \varphi_n^{(1)} \\ \psi_n^{(1)} \\ \chi_n^{(1)} \end{pmatrix}, \quad \phi_n^{(2)} = \begin{pmatrix} \varphi_n^{(2)} \\ \psi_n^{(2)} \\ \chi_n^{(2)} \end{pmatrix},$$

in which

$$\begin{aligned}
\varphi_n^{(1)} = {} & \frac{1}{8}\mathrm{e}^{6t} 3^{-n+\frac{1}{6}} (9s^4 + 54s^3 + 162s^2t - 117s^2 \\
& + 1242st + 243t^2 + 54s + 486t - 108s^2e_0 \\
& - 108se_0 - 324te_0), \\
\psi_n^{(1)} = {} & \frac{1}{8}\mathrm{e}^{6t} 3^{-n+\frac{1}{6}} (27s^4 + 270s^3 + 486s^2t + 297s^2 \\
& + 4698st + 729t^2 + 702s + 5670t + 648 \\
& - 324s^2e_0 - 972se_0 - 972te_0 - 648e_0), \\
\chi_n^{(1)} = {} & \frac{1}{8}\mathrm{e}^{6t} 3^{-n+\frac{1}{6}} (3s^4 + 6s^3 + 54s^2t - 75s^2 \\
& + 306st + 81t^2 + 66s - 198t - 36s^2e_0 \\
& + 36se_0 + 108te_0),
\end{aligned}$$

$$\begin{aligned}
\varphi_n^{(2)} = {} & -\frac{27}{560}\mathrm{e}^{6t} 3^{-n+\frac{1}{6}} (s^7 + 21s^6 + 63s^5t - 35s^5 \\
& + 1680s^4t + 945s^3t^2 - 525s^4 + 6615s^3t \\
& + 21735s^2t^2 + 2835st^3 + 1834s^3 - 2562s^2t \\
& + 95130st^2 + 28350t^3 - 2016s^2 + 26460st \\
& + 74844t^2 + 720s + 12960t - 42e_0s^5 \\
& - 420e_0s^4 - 1260e_0s^3t + 840e_0^3s + 1050e_0s^3 \\
& - 16380e_0s^2t - 5670e_0st^2 + 420e_0s^2 \\
& - 18900e_0st - 37800e_0t^2 - 1008e_0s \\
& - 12096e_0t), \\
\psi_n^{(2)} = {} & -\frac{81}{560}\mathrm{e}^{6t} 3^{-n+\frac{1}{6}} (s^7 + 28s^6 + 63s^5t + 112s^5 \\
& + 1995s^4t + 945s^3t^2 - 140s^4 + 13965s^3t \\
& + 24570s^2t^2 + 2835st^3 + 1939s^3 \\
& + 31773s^2t + 141435st^2 + 31185t^3 \\
& + 2632s^2 + 84756st + 198324t^2 + 5508s \\
& + 89316t + 5040 - 42e_0s^5 - 630e_0s^4 \\
& - 1260e_0s^3t + 840e_0^3s - 1050e_0s^3 \\
& - 20160e_0s^2t - 5670e_0st^2 + 840e_0^3 \\
& - 1890e_0s^2 - 55440e_0st - 43470e_0t^2 \\
& - 6468e_0s - 56196e_0t - 5040e_0), \\
\chi_n^{(2)} = {} & -\frac{9}{560}\mathrm{e}^{6t} 3^{-n+\frac{1}{6}} (s^7 + 14s^6 + 63s^5t - 140s^5 \\
& + 1365s^4t + 945s^3t^2 - 280s^4 + 525s^3t \\
& + 18900s^2t^2 + 2835st^3 + 2779s^3 - 16737s^2t \\
& + 54495st^2 + 25515t^3 - 4774s^2 + 23604st \\
& - 5166t^2 + 2400s - 7200t),
\end{aligned}$$

where $s = n+3t$. Other $\phi_n^{(j)}$ $(j = 3, 4, 5, \ldots)$ are not displayed here. For this case, we can derive different exact solutions if we choose $C_{1,1} = 1$, $C_{1,2} = -C_{1,3} = \frac{I}{\varepsilon}$, but this case is complicated, and we will not consider it.

- **The second type expansion**.
  If $C_{1,1} = 1$, $C_{1,2} = C_{1,3} = 0$, $\phi_n^{(j)}$ can be expressed as

$$\phi_n^{(0)} = \begin{pmatrix} \varphi_n^{(0)} \\ \psi_n^{(0)} \\ \chi_n^{(0)} \end{pmatrix} = \mathrm{e}^{6t} 3^{-n} \begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix},$$

$$\phi_n^{(1)} = \begin{pmatrix} \varphi_n^{(1)} \\ \psi_n^{(1)} \\ \chi_n^{(1)} \end{pmatrix} = -\frac{31}{2}\mathrm{e}^{6t} 3^{-n} \begin{pmatrix} 3s^3 + 9s^2 + 27st - 12s + 90t + 18se_0 \\ 9s^3 + 54s^2 + 81st + 45s + 351t + 54 + 54se_0 + 54e_0 \\ s^3 + 9st - 7s + 21t + 6se_0 - 6e_0 \end{pmatrix}, \phi_n^{(2)} = \begin{pmatrix} \varphi_n^{(2)} \\ \psi_n^{(2)} \\ \chi_n^{(2)} \end{pmatrix}$$

in which

$$\varphi_n^{(2)} = \frac{1}{80}e^{6t}3^{-n+3}(s^6+30e_0s^4+15s^5+45s^4t +180e_0^2s^2+180e_0s^3+540e_0s^2t-35s^4 +870s^3t+405s^2t^2+120e_0^3+180e_0^2s +540e_0^2t-390e_0s^2+4140e_0st+810e_0t^2 -135s^3+2025s^2t+5805st^2+405t^3 +180e_0s+1620e_0t+394s^2+408st +11430t^2-240s+2880t),$$

$$\psi_n^{(2)} = \frac{1}{80}e^{6t}3^{-n+4}(s^6+30e_0s^4+21s^5+45s^4t +180e_0^2s^2+300e_0s^3+540e_0s^2t+55s^4 +1050s^3t+405s^2t^2+120e_0^3+540e_0^2s +540e_0^2t+330e_0s^2+5220e_0st+810e_0t^2 +15s^3+4905s^2t+6615st^2+405t^3+360e_0^2 +780e_0s+6300e_0t+664s^2+8328st +17640t^2+720e_0+684s+10908t+720),$$

$$\chi_n^{(2)} = \frac{1}{80}e^{6t}3^{-n+1}(1215t^3+1362s^2-1152s +3456t-945s^2t+14985st^2+1980e_0s -6876st+1620e_0s^2t+9180e_0st+90e_0s^4 +135s^4t+540e_0^2s^2+180e_0s^3+2070s^3t +1215s^2t^2-540e_0^2s+1620e_0^2t-2250e_0s^2 +2430e_0t^2-5940e_0t+18090t^2+3s^6 +27s^5-285s^4+360e_0^3+45s^3),$$

where $s=n+3t$. Other $\phi_n^{(j)}(j=3,4,5,\ldots)$ are not displayed here. For this case, we find that the same rational solutions are given if we choose $C_{1,2}=1, C_{1,1}=C_{1,3}=0$ or $C_{1,1}=C_{1,2}=0, C_{1,3}=1$ or $C_{1,1}=C_{1,2}=1, C_{1,3}=0$ or $C_{1,1}=0,\ C_{1,2}=C_{1,3}=1$ or $C_{1,2}=0, C_{1,1}=C_{1,3}=1$ or $C_{1,1}=C_{1,2}=C_{1,3}=1$. So we only list one case.

Now, we will apply the discrete generalised $(m, 3N-m)$-fold DT to deduce rational solutions of eq. (2). We only consider one case: $N=1$.

Case 1. *The first-order rational solutions*: As $m=3$, $N=1$, the discrete generalised $(3, 3N-3)$-fold DT can reduce to the $(3,0)$-fold DT. We need three spectral parameters $\lambda_1, \lambda_2, \lambda_3$. For λ_1, the above first type expansion in (18) is used, whereas for λ_2, λ_3, we do not make Taylor series expansion of (18). Then, we will get the first-order rational solutions of eq. (2) as

$$\tilde{u}_n = -\frac{F_{n+1}^{(0)}-3E_{n+1}^{(1)}}{A_n^{(1)}}+\frac{3G_n^{(1)}E_{n+1}^{(1)}}{A_n^{(1)}I_n^{(1)}},$$
$$\tilde{w}_n = \frac{3E_{n+1}^{(1)}}{I_n^{(1)}}, \tag{20}$$

where

$$A_n^{(1)}=\frac{\Delta A_n^{(1)}}{\Delta_1},\ E_{n+1}^{(1)}=\frac{\Delta E_{n+1}^{(1)}}{\Delta_2},$$
$$F_{n+1}^{(0)}=\frac{\Delta F_{n+1}^{(0)}}{\Delta_2},\ G_n^{(1)}=\frac{\Delta G_n^{(1)}}{\Delta_1},\ I_n^{(1)}=\frac{\Delta A_{n-1}^{(1)}}{\Delta_1},$$

in which

$$\Delta A_n^{(1)}=\begin{vmatrix} -\varphi_n^{(0)} & -\varphi_n^{(0)}+\lambda_1\psi_n^{(0)} & \chi_n^{(0)} \\ -\varphi_n(\lambda_2) & -\varphi_n(\lambda_2)+\lambda_2\psi_n(\lambda_2) & \chi_n(\lambda_2) \\ -\varphi_n(\lambda_3) & -\varphi_n(\lambda_3)+\lambda_3\psi_n(\lambda_3) & \chi_n(\lambda_3)\end{vmatrix},$$

$$\Delta E_{n+1}^{(1)}=\begin{vmatrix} \varphi_{n+1}^{(0)} & -\psi_{n+1}^{(0)} & \chi_{n+1}^{(0)} \\ \varphi_{n+1}(\lambda_2) & -\psi_{n+1}(\lambda_2) & \chi_{n+1}(\lambda_2) \\ \varphi_{n+1}(\lambda_3) & -\psi_{n+1}(\lambda_3) & \chi_{n+1}(\lambda_3)\end{vmatrix},$$

$$\Delta_1=\begin{vmatrix} \lambda_1\varphi_n^{(0)} & -\varphi_n^{(0)}+\lambda_1\psi_n^{(0)} & \chi_n^{(0)} \\ \lambda_2\varphi_n(\lambda_2) & -\varphi_n(\lambda_2)+\lambda_2\psi_n(\lambda_2) & \chi_n(\lambda_2) \\ \lambda_3\varphi_n(\lambda_3) & -\varphi_n(\lambda_3)+\lambda_3\psi_n(\lambda_3) & \chi_n(\lambda_3)\end{vmatrix},$$

$$\Delta F_{n+1}^{(0)}=\begin{vmatrix} \varphi_{n+1}^{(0)} & \lambda_1\psi_{n+1}^{(0)} & -\psi_{n+1}^{(0)} \\ \varphi_{n+1}(\lambda_2) & \lambda_2\psi_{n+1}(\lambda_2) & -\psi_{n+1}(\lambda_2) \\ \varphi_{n+1}(\lambda_3) & \lambda_3\psi_{n+1}(\lambda_3) & -\psi_{n+1}(\lambda_3)\end{vmatrix},$$

$$\Delta_2=\begin{vmatrix} \varphi_n^{(0)} & \lambda_1\psi_n^{(0)} & \chi_n^{(0)} \\ \varphi_n(\lambda_2) & \lambda_2\psi_n(\lambda_2) & \chi_n(\lambda_2) \\ \varphi_n(\lambda_3) & \lambda_3\psi_n(\lambda_3) & \chi_n(\lambda_3)\end{vmatrix},$$

$$\Delta G_n^{(1)}=\begin{vmatrix} -A_{n-1}^{(1)}\lambda_1\chi_n^{(0)} & -\varphi_n^{(0)}+\lambda_1\psi_n^{(0)} & \chi_n^{(0)} \\ -A_{n-1}^{(1)}\lambda_2\chi_n(\lambda_2) & -\varphi_n(\lambda_2)+\lambda_2\psi_n(\lambda_2) & \chi_n(\lambda_2) \\ -A_{n-1}^{(1)}\lambda_3\chi_n(\lambda_3) & -\varphi_n(\lambda_3)+\lambda_3\psi_n(\lambda_3) & \chi_n(\lambda_3)\end{vmatrix},$$

while $\Delta A_{n-1}^{(1)}$ is obtained from $\Delta A_n^{(1)}$ by replacing n with $n-1$. For λ_2, we choose $C_{2,1}=1,\ C_{2,2}=C_{2,3}=0$, while for λ_3, we choose $C_{3,2}=1,\ C_{3,1}=C_{3,3}=0$. So just for analysis purposes, we can rewrite solutions (20) as

$$\tilde{u}_n = 3-\frac{12}{(2s+3)(2s+5)},$$
$$\tilde{w}_n = 3-\frac{36}{(2s+1)(2s+5)}, \tag{21}$$

from which we can see that $\tilde{u}_n$ possesses singularities at two paralleled straight lines $2s+3=0$ and $2s+5=0$, while $\tilde{w}_n$ has singularity at two straight lines $2s+1=0$ and $2s+5=0$. We can see clearly that the highest power in the numerator and denominator of solutions (21) is 2. To easily understand these trajectories, their three-dimensional plots and singular trajectory plots are shown in figure 2.

Case 2. *The second-order rational solutions*: For the second-order rational solutions, we discuss the following two cases:

(i) As $m=2, N=1$, the discrete generalised $(2, 3N-2)$-fold DT can reduce to the $(2,1)$-fold DT.

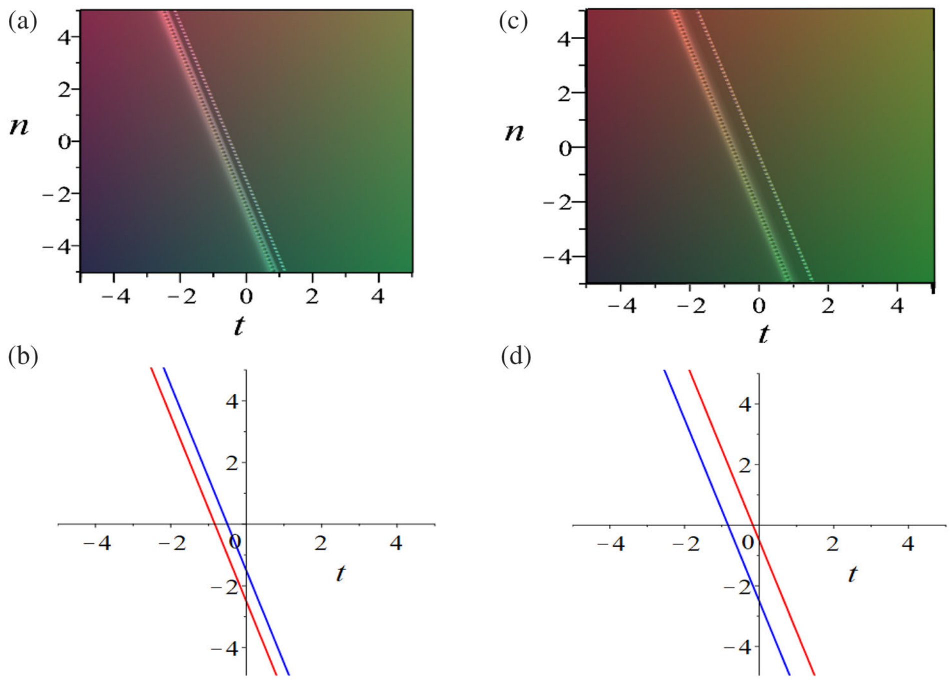

Figure 2. First-order rational solutions (21): (**a**) Three-dimensional plot of $\tilde{u_n}$, (**b**) singular trajectory plot of $\tilde{u_n}$, (**c**) three-dimensional plot of $\tilde{w_n}$ and (**d**) singular trajectory plot of $\tilde{w_n}$.

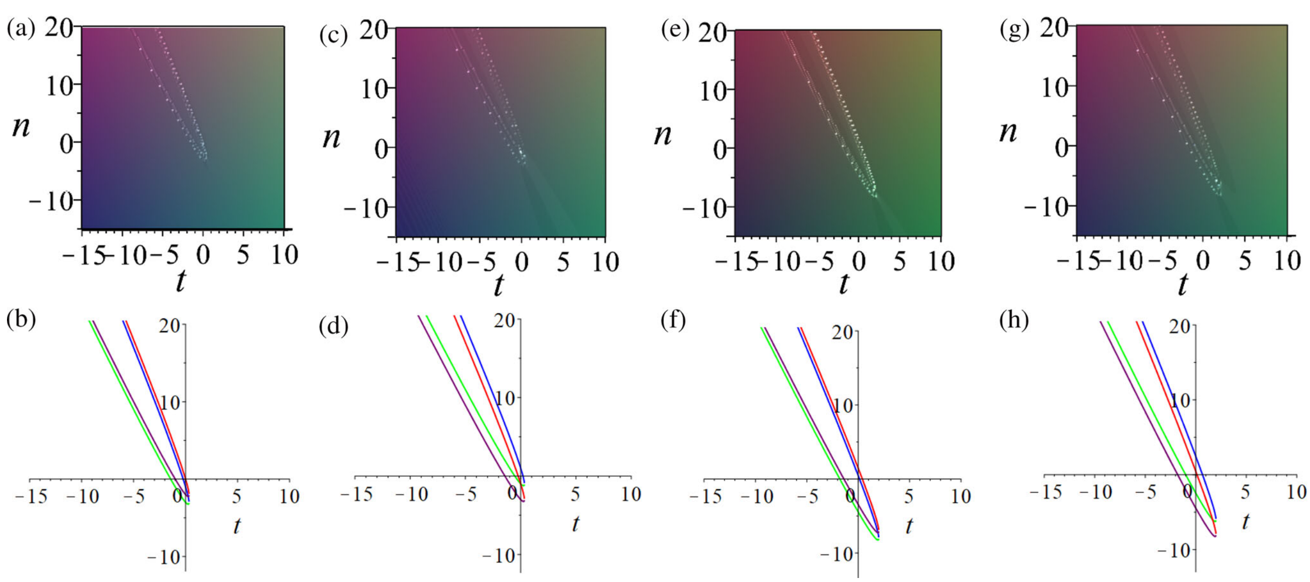

Figure 3. Second-order rational solutions (22): (**a**) Three-dimensional plot of $\tilde{u_n}$ in (22) with $e_0 = 0$, (**b**) singular trajectory plot of $\tilde{u_n}$ in (23) with $e_0 = 0$, (**c**) three-dimensional plot of $\tilde{w_n}$ in (22) with $e_0 = 0$, (**d**) singular trajectory plot of $\tilde{w_n}$ in (23) with $e_0 = 0$, (**e**) three-dimensional plot of $\tilde{u_n}$ in (22) with $e_0 = -\frac{5}{2}$, (**f**) singular trajectory plot of $\tilde{u_n}$ in (23) with $e_0 = -\frac{5}{2}$, (**g**) three-dimensional plot of $\tilde{w_n}$ in (22) with $e_0 = -\frac{5}{2}$ and (**h**) singular trajectory plot of $\tilde{w_n}$ in (23) with $e_0 = -\frac{5}{2}$.

At this time, we need to employ two spectral parameters λ_1, λ_2. For λ_1, we use the second type expansion of (18), whereas for λ_2, we do not make Taylor series expansion of (18). Then the new rational solutions of eq. (2) are expressed as

$$\tilde{u}_n = -\frac{F^{(0)}_{n+1} - 3E^{(1)}_{n+1}}{A^{(1)}_n} + \frac{3G^{(1)}_n E^{(1)}_{n+1}}{A^{(1)}_n I^{(1)}_n},$$
$$\tilde{w}_n = \frac{3E^{(1)}_{n+1}}{I^{(1)}_n}, \tag{22}$$

where

$$A^{(1)}_n = \frac{\Delta A^{(1)}_n}{\Delta_1},$$
$$E^{(1)}_{n+1} = \frac{\Delta E^{(1)}_{n+1}}{\Delta_2},$$
$$F^{(0)}_{n+1} = \frac{\Delta F^{(0)}_{n+1}}{\Delta_2},$$
$$G^{(1)}_n = \frac{\Delta G^{(1)}_n}{\Delta_1},$$
$$I^{(1)}_n = \frac{\Delta A^{(1)}_{n-1}}{\Delta_1},$$

in which

$$\Delta A^{(1)}_n = \begin{vmatrix} -\varphi^{(0)}_n & -\varphi^{(0)}_n + \lambda_1\psi^{(0)}_n & \chi^{(0)}_n \\ -\varphi^{(1)}_n & -\varphi^{(1)}_n + \lambda_1\psi^{(1)}_n + \psi^{(0)}_n & \chi^{(1)}_n \\ -\varphi_n(\lambda_2) & -\varphi_n(\lambda_2) + \lambda_2\psi_n(\lambda_2) & \chi_n(\lambda_2) \end{vmatrix}, \quad \Delta E^{(1)}_{n+1} = \begin{vmatrix} \varphi^{(0)}_{n+1} & -\psi^{(0)}_{n+1} & \chi^{(0)}_{n+1} \\ \varphi^{(1)}_{n+1} & -\psi^{(1)}_{n+1} & \chi^{(1)}_{n+1} \\ \varphi_{n+1}(\lambda_2) & -\psi_{n+1}(\lambda_2) & \chi_{n+1}(\lambda_2) \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} \lambda_1\varphi^{(0)}_n & -\varphi^{(0)}_n + \lambda_1\psi^{(0)}_n & \chi^{(0)}_n \\ \lambda_1\varphi^{(1)}_n + \varphi^{(0)}_n & -\varphi^{(1)}_n + \lambda_1\psi^{(1)}_n + \psi^{(0)}_n & \chi^{(1)}_n \\ \lambda_2\varphi_n(\lambda_2) & -\varphi_n(\lambda_2) + \lambda_2\psi_n(\lambda_2) & \chi_n(\lambda_2) \end{vmatrix}, \quad \Delta F^{(0)}_{n+1} = \begin{vmatrix} \varphi^{(0)}_{n+1} & \lambda_1\psi^{(0)}_{n+1} & -\psi^{(0)}_{n+1} \\ \varphi^{(1)}_{n+1} & \lambda_1\psi^{(1)}_{n+1} + \psi^{(0)}_{n+1} & -\psi^{(1)}_{n+1} \\ \varphi_{n+1}(\lambda_2) & \lambda_2\psi_{n+1}(\lambda_2) & -\psi_{n+1}(\lambda_2) \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \varphi^{(0)}_n & \lambda_1\psi^{(0)}_n & \chi^{(0)}_n \\ \varphi^{(1)}_n & \lambda_1\psi^{(1)}_n + \psi^{(0)}_n & \chi^{(1)}_n \\ \varphi_n(\lambda_2) & \lambda_2\psi_n(\lambda_2) & \chi_n(\lambda_2) \end{vmatrix}, \quad \Delta G^{(1)}_n = \begin{vmatrix} -A^{(1)}_{n-1}\lambda_1\chi^{(0)}_n & -\varphi^{(0)}_n + \lambda_1\psi^{(0)}_n & \chi^{(0)}_n \\ -A^{(1)}_{n-1}\lambda_1\chi^{(1)}_n - A^{(1)}_{n-1}\chi^{(0)}_n & -\varphi^{(1)}_n + \lambda_1\psi^{(1)}_n + \psi^{(0)}_n & \chi^{(1)}_n \\ -A^{(1)}_{n-1}\lambda_2\chi_n(\lambda_2) & -\varphi_n(\lambda_2) + \lambda_2\psi_n(\lambda_2) & \chi_n(\lambda_2) \end{vmatrix},$$

while $\Delta A^{(1)}_{n-1}$ is obtained from $\Delta A^{(1)}_n$ by substituting n with $n-1$. For λ_2, we take $C_{2,1} = 1,\ C_{2,2} = C_{2,3} = 0$. Similarly, we can rewrite solutions (22) as

$$\tilde{u}_n = 3 + \frac{18t - 6s^2 - 24s - 27 + 12e_0}{(3t + s^2 + 4s + 3 + 2e_0)(3t + s^2 + 2s + 2e_0)},$$

$$\tilde{w}_n = 3 - \frac{54t - 18s^2 + 36s + 9 + 36e_0}{(3t + s^2 - 1 + 2e_0)(3t + s^2 + 4s + 3 + 2e_0)}. \tag{23}$$

Remarkably, in this case, the highest power of the numerator and denominator for solutions (23) is 4. Let the denominator of $\tilde{u}_n$ in (23) be zeroes, we know that $\tilde{u}_n$ has four singular curves $s + 1 - \sqrt{-2e_0 - 3t + 1} = 0$, $s+1+\sqrt{-2e_0 - 3t + 1} = 0, s+2-\sqrt{-2e_0 - 3t + 1} = 0$ and $s + 2 + \sqrt{-2e_0 - 3t + 1} = 0$, while $\tilde{w}_n$ owns four singular curves $s + 2 + \sqrt{-2e_0 - 3t + 1} = 0$, $s+2-\sqrt{-2e_0 - 3t + 1} = 0, s+\sqrt{-2e_0 - 3t + 1} = 0$ and $s - \sqrt{-2e_0 - 3t + 1} = 0$. What needs to be explained here is that in rational solutions (23), we have an arbitrary parameter e_0, which can control the position of the singular curves of the rational solutions, which means that we can change e_0 to move the rational solution to wherever we need. To profoundly comprehend rational solutions $\tilde{u}_n$ and $\tilde{w}_n$, their three-dimensional plots and singular trajectory plots are displayed in figure 3.

(ii) For solutions (22), if we use the above first type expansion of (18), the other conditions are the same as (i) and we get another new rational solutions of eq. (2) as

$$\tilde{u}_n = 3 - \frac{P_1 + P_2}{P_3 P_4},$$
$$\tilde{w}_n = 3 - \frac{P_5 + P_6}{P_7 P_8}, \tag{24}$$

where

$$
\begin{aligned}
P_1 &= 324t^3 + (540s^2 + 1080s + 1026)t^2 \\
&\quad + (36s^4 + 288s^3 + 864s^2 + 1548s + 774)t, \\
P_2 &= 12s^6 + 120s^5 + 426s^4 + 588s^3 + 156s^2 \\
&\quad -222s - 108, \\
P_3 &= -9t^2 + 6s^2t + 12st - 9t + s^4 + 8s^3 + 20s^2 \\
&\quad +19s + 6, \\
P_4 &= -9t^2 + 6s^2t - 15t + s^4 + 4s^3 + 2s^2 - s, \\
P_5 &= 972t^3 + (1620s^2 + 1944s + 1134)t^2 \\
&\quad + (108s^4 + 864s^3 + 2592s^2 + 972s - 1458)t, \\
P_6 &= -36s^6 - 216s^5 - 198s^4 + 756s^3 + 1152s^2 \\
&\quad -486s - 972, \\
P_7 &= -9t^2 + 6s^2t + 12st - 9t + s^4 + 8s^3 + 20s^2 \\
&\quad +19s + 6, \\
P_8 &= -9t^2 + 6s^2t - 12st - 9t + s^4 - 4s^2 + 3s.
\end{aligned}
$$

Remarkably, we can clearly see that the highest power in the numerator and denominator of solutions (31) is 8. If the denominator of solutions (31) is equal to 0, it is difficult to directly solve the singular curves of solutions (24). So we make an asymptotic analysis of solutions (24). Let

$$
\gamma_1 = s - \sqrt{(3\sqrt{2} - 3)t}, \quad t > 0,
$$
$$
\gamma_2 = s - \sqrt{(3 + 3\sqrt{2})(-t)}, \quad t < 0.
$$

When $s = \gamma_1 + \sqrt{(-3 + 3\sqrt{2})t}, t \to +\infty$:

$$
\begin{aligned}
\tilde{u}_n &\to u^+ = 3 - \frac{6}{(-2\gamma_1 - 3)\sqrt{2} + 2\gamma_1^2 + 6\gamma_1 + 5}, \\
\tilde{w}_n &\to w^+ = 3 - \frac{18}{(-2\gamma_1 - 2)\sqrt{2} + 2\gamma_1^2 + 4\gamma_1 + 1}
\end{aligned}
\tag{25}
$$

where u^+, v^+ respectively represent the limit-state expressions of $\tilde{u}_n, \tilde{v}_n$ as $t \to +\infty$.

When $s = \gamma_2 + \sqrt{(-3 - 3\sqrt{2})t}, t \to -\infty$:

$$
\begin{aligned}
\tilde{u}_n &\to u^- = 3 - \frac{6}{(2\gamma_2 + 3)\sqrt{2} + 2\gamma_2^2 + 6\gamma_2 + 5}, \\
\tilde{w}_n &\to w^- = 3 - \frac{18}{(2\gamma_2 + 2)\sqrt{2} + 2\gamma_2^2 + 4\gamma_2 + 1},
\end{aligned}
\tag{26}
$$

where u^-, v^- respectively represent the limit-state expressions of $\tilde{u}_n, \tilde{v}_n$ as $t \to -\infty$.

It can be easily observed that $\tilde{u}_n$ has singularities at four curves $-2\gamma_1 - 2 + \sqrt{2} = 0$, $-2\gamma_1 - 4 + \sqrt{2} = 0$, $2\gamma_2 + 4 + \sqrt{2} = 0$, $2\gamma_2 + 2 + \sqrt{2} = 0$, and $\tilde{w}_n$ possesses singularities at four curves $-2\gamma_1 + \sqrt{2} = 0$, $-2\gamma_1 - 4 + \sqrt{2} = 0$, $2\gamma_2 + \sqrt{2} = 0$, $2\gamma_2 + 4 + \sqrt{2} = 0$. The collision structures of solutions (24) are complex in the area $n^2 + t^2 \leq 25$, and we do not discuss their states in this section. So we only plot the singular trajectory structures when $|t|$ is relatively large as displayed in figure 4, from which we can clearly see that the singularities of the three-dimensional plot are completely consistent with these singular trajectories via asymptotic analysis, which also shows the correctness of our asymptotic analysis results.

Case 3. *The third-order rational solutions.* As $m = 1$, $N = 1$, the discrete generalised $(1, 3N - 1)$-fold DT can reduce to $(1, 2)$-fold DT. At this time, we only need to use one spectral parameter λ_1, and we may utilise the first or second type expansion of (18). Here, the third-order rational solutions of eq. (2) with the second type expansion of (18) are given as

$$
\begin{aligned}
\tilde{u}_n &= -\frac{F_{n+1}^{(0)} - 3E_{n+1}^{(1)}}{A_n^{(1)}} \\
&\quad + \frac{3G_n^{(1)} E_{n+1}^{(1)}}{A_n^{(1)} I_n^{(1)}}, \quad \tilde{w}_n = \frac{3E_{n+1}^{(1)}}{I_n^{(1)}},
\end{aligned}
\tag{27}
$$

where

$$
A_n^{(1)} = \frac{\Delta A_n^{(1)}}{\Delta_1},
$$
$$
E_{n+1}^{(1)} = \frac{\Delta E_{n+1}^{(1)}}{\Delta_2},
$$
$$
F_{n+1}^{(0)} = \frac{\Delta F_{n+1}^{(0)}}{\Delta_2},
$$

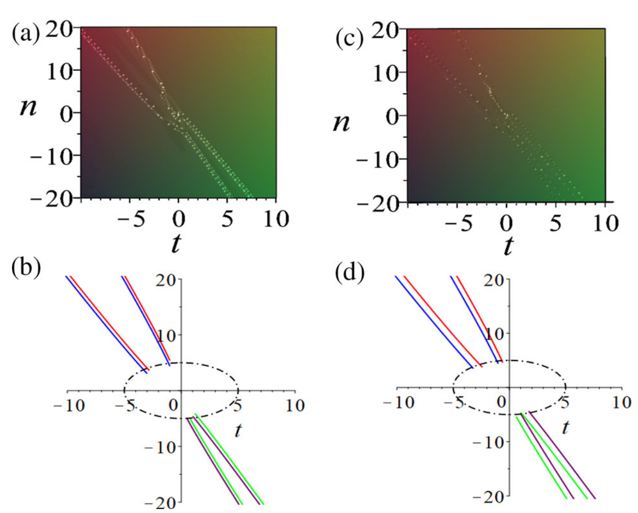

Figure 4. Second-order rational solutions (24): (**a**) Three-dimensional plot of $\tilde{u_n}$ in (24), (**b**) singular trajectory plot of $\tilde{u_n}$ in (25) and (26), (**c**) three-dimensional plot of $\tilde{w_n}$ in (24) and (**d**) singular trajectory plot of $\tilde{w_n}$ in (25) and (26).

$$G_n^{(1)} = \frac{\Delta G_n^{(1)}}{\Delta_1},$$

$$I_n^{(1)} = \frac{\Delta A_{n-1}^{(1)}}{\Delta_1},$$

in which

$$\Delta A_n^{(1)} = \begin{vmatrix} -\varphi_n^{(0)} & -\varphi_n^{(0)} + \lambda_1 \psi_n^{(0)} & \chi_n^{(0)} \\ -\varphi_n^{(1)} & -\varphi_n^{(1)} + \lambda_1 \psi_n^{(1)} + \psi_n^{(0)} & \chi_n^{(1)} \\ -\varphi_n^{(2)} & -\varphi_n^{(2)} + \lambda_1 \psi_n^{(2)} + \psi_n^{(1)} & \chi_n^{(2)} \end{vmatrix},$$

$$\Delta E_{n+1}^{(1)} = \begin{vmatrix} \varphi_{n+1}^{(0)} & -\psi_{n+1}^{(0)} & \chi_{n+1}^{(0)} \\ \varphi_{n+1}^{(1)} & -\psi_{n+1}^{(1)} & \chi_{n+1}^{(1)} \\ \varphi_{n+1}^{(2)} & -\psi_{n+1}^{(2)} & \chi_{n+1}^{(2)} \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} \lambda_1 \varphi_n^{(0)} & -\varphi_n^{(0)} + \lambda_1 \psi_n^{(0)} & \chi_n^{(0)} \\ \lambda_1 \varphi_n^{(1)} + \varphi_n^{(0)} & -\varphi_n^{(1)} + \lambda_1 \psi_n^{(1)} + \psi_n^{(0)} & \chi_n^{(1)} \\ \lambda_1 \varphi_n^{(2)} + \varphi_n^{(1)} & -\varphi_n^{(2)} + \lambda_1 \psi_n^{(2)} + \psi_n^{(1)} & \chi_n^{(2)} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \varphi_n^{(0)} & \lambda_1 \psi_n^{(0)} & \chi_n^{(0)} \\ \varphi_n^{(1)} & \lambda_1 \psi_n^{(1)} + \psi_n^{(0)} & \chi_n^{(1)} \\ \varphi_n^{(2)} & \lambda_1 \psi_n^{(2)} + \psi_n^{(1)} & \chi_n^{(2)} \end{vmatrix},$$

$$\Delta F_{n+1}^{(0)} = \begin{vmatrix} \varphi_{n+1}^{(0)} & \lambda_1 \psi_{n+1}^{(0)} & -\psi_{n+1}^{(0)} \\ \varphi_{n+1}^{(1)} & \lambda_1 \psi_{n+1}^{(1)} + \psi_{n+1}^{(0)} & -\psi_{n+1}^{(1)} \\ \varphi_{n+1}^{(2)} & \lambda_1 \psi_{n+1}^{(2)} + \psi_{n+1}^{(1)} & -\psi_{n+1}^{(2)} \end{vmatrix},$$

$$\Delta G_n^{(1)} = \begin{vmatrix} -A_{n-1}^{(1)} \lambda_1 \chi_n^{(0)} & -\varphi_n^{(0)} + \lambda_1 \psi_n^{(0)} & \chi_n^{(0)} \\ -A_{n-1}^{(1)} \lambda_1 \chi_n^{(1)} - A_{n-1}^{(1)} \chi_n^{(0)} & -\varphi_n^{(1)} + \lambda_1 \psi_n^{(1)} + \psi_n^{(0)} & \chi_n^{(1)} \\ -A_{n-1}^{(1)} \lambda_1 \chi_n^{(2)} - A_{n-1}^{(1)} \chi_n^{(1)} & -\varphi_n^{(2)} + \lambda_1 \psi_n^{(2)} + \psi_n^{(1)} & \chi_n^{(2)} \end{vmatrix},$$

while $\Delta A_{n-1}^{(1)}$ is obtained from $\Delta A_n^{(1)}$ by replacing n with $n-1$. So just for analysis purposes, let us rewrite solutions (27) as

$$\tilde{u}_n = 3 + \frac{M}{N}, \quad \tilde{w}_n = 3 + \frac{P}{Q}, \tag{28}$$

with

$$\begin{aligned} M = {} & 36450t^5 + (60750s^2 + 85050s)t^4 \\ & + (8100s^4 + 48600s^3 + 234900s^2 \\ & + 408240s + 145800)t^3 \\ & + (-1620s^6 - 8100s^5 + 8100s^4 + 92340s^3 \\ & + 145800s^2 + 87480s + 23328)t^2 \\ & + (-270s^8 - 3240s^7 \\ & - 16740s^6 - 50328s^5 - 96390s^4 - 112320s^3 \\ & - 68040s^2 - 15552s)t - 18s^{10} - 270s^9 - 1620s^8 \\ & - 4860s^7 - 7074s^6 - 2430s^5 \\ & + 6120s^4 + 7560s^3 + 2592s^2, \\ N = {} & [135t^3 + (45s^2 + 225s + 450)t^2 \\ & + (15s^4 + 90s^3 + 135s^2 - 12s - 72)t \\ & + s^6 + 11s^5 + 45s^4 + 85s^3 \\ & + 74s^2 + 24s][135t^3 + (45s^2 + 135s + 270)t^2 \\ & + (15s^4 + 30s^3 - 45s^2 - 72s)t + s^6 + 5s^5 \\ & + 5s^4 - 5s^3 - 6s^2], \\ P = {} & 109350t^5 + (182250s^2 + 109350s + 218700)t^4 \\ & + (24300s^4 + 145800s^3 + 704700s^2 + 447120s \\ & + 126360)t^3 + (-4860s^6 + 1620s^5 + 137700s^4 \\ & + 244620s^3 + 239760s^2 + 288360s + 115344)t^2 \\ & + (-810s^8 - 5400s^7 - 11340s^6 + 4536s^5 \\ & + 73710s^4 + 116640s^3 + 3240s^2 - 76896s - 25920)t \\ & - 54s^{10} - 450s^9 - 720s^8 + 2700s^7 + 7938s^6 \\ & - 1890s^5 - 19980s^4 - 9000s^3 + 12816s^2 + 8640s, \\ Q = {} & [135t^3 + (45s^2 + 45s + 180)t^2 \\ & + (15s^4 - 30s^3 - 45s^2 + 48s + 12)t \\ & + s^6 - s^5 - 5s^4 + 5s^3 + 4s^2 \\ & - 4s][135t^3 + (45s^2 + 225s + 450)t^2 \\ & + (15s^4 + 90s^3 + 135s^2 - 12s - 72)t \\ & + s^6 + 11s^5 + 45s^4 \\ & + 85s^3 + 74s^2 + 24s], \end{aligned}$$

from which we can clearly see that the highest power in the numerator and denominator of solutions (28) is 12. If we use the above first type expansion of (18), and the other conditions are the same as Case 1, the highest power in the numerator and denominator of solutions (28) is 18, which means that the solutions (28) are very complex so that we will not discuss this case. Furthermore, we also perform asymptotic analysis of solutions (28). For convenience, let

$$\zeta = s - \sqrt{[(80 + 30\sqrt{6})^{\frac{1}{3}} + \frac{10}{(80 + 30\sqrt{6})^{\frac{1}{3}}} + 5](-t)}, \quad t < 0,$$

we can calculate the limits of $\tilde{u_n}$ and $\tilde{w_n}$ as $t \to -\infty$:

$$\begin{aligned} \tilde{u}_n &\to u^- = 3 - \frac{10800}{N_1 N_2}, \\ \tilde{w}_n &\to w^- = 3 - \frac{32400}{N_1 N_3}, \end{aligned} \tag{29}$$

where

$$\begin{aligned} N_1 = {} & 3(80 + 30\sqrt{6})^{\frac{2}{3}}\sqrt{6} - 8(80 + 30\sqrt{6})^{\frac{2}{3}} \\ & - 10(80 + 30\sqrt{6})^{\frac{1}{3}} - 60\zeta - 110, \\ N_2 = {} & 3(80 + 30\sqrt{6})^{\frac{2}{3}}\sqrt{6} - 8(80 + 30\sqrt{6})^{\frac{2}{3}} \\ & - 10(80 + 30\sqrt{6})^{\frac{1}{3}} - 60\zeta - 50, \\ N_3 = {} & 3(80 + 30\sqrt{6})^{\frac{2}{3}}\sqrt{6} - 8(80 + 30\sqrt{6})^{\frac{2}{3}} \\ & - 10(80 + 30\sqrt{6})^{\frac{1}{3}} - 60\zeta + 10. \end{aligned}$$

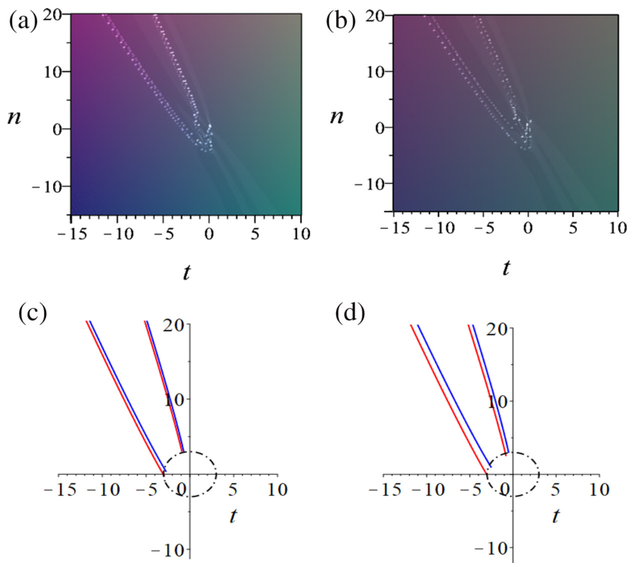

Figure 5. Third-order rational solutions (28): (**a**) Three-dimensional plot of $\tilde{u_n}$ in (28). (**b**) singular trajectory plot of $\tilde{u_n}$ in (29), (**c**) three-dimensional plot of $\tilde{w_n}$ in (28) and (**d**) singular trajectory plot of $\tilde{w_n}$ in (29).

We also find that the solutions $\tilde{u_n}$ and $\tilde{w_n}$ of eq. (28) tend to their backgrounds as $t \rightarrow +\infty$ and so we just discuss the case of $t < 0$ here. Because the interaction structures of solutions are complex in the area $n^2 + t^2 \leq 9$, we do not discuss the states of solutions in this section and only draw the singular trajectory plots of the solutions (28) when $|t|$ is relatively large. What is more, the asymptotic expressions (29) clearly show that the rational solution $\tilde{u_n}$ has two singular curves $3(80 + 30\sqrt{6})^{\frac{2}{3}}\sqrt{6} - 8(80 + 30\sqrt{6})^{\frac{2}{3}} - 10(80 + 30\sqrt{6})^{\frac{1}{3}} - 60\zeta - 110 = 0$ and $3(80 + 30\sqrt{6})^{\frac{2}{3}}\sqrt{6} - 8(80 + 30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} - 60\zeta - 50 = 0$, whereas $\tilde{w_n}$ has two singular curves $3(80 + 30\sqrt{6})^{\frac{2}{3}}\sqrt{6} - 8(80 + 30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} - 60\zeta - 110 = 0$ and $3(80 + 30\sqrt{6})^{\frac{2}{3}}\sqrt{6} - 8(80 + 30\sqrt{6})^{\frac{2}{3}} - 10(80 + 30\sqrt{6})^{\frac{1}{3}} - 60\zeta + 10 = 0$. To profoundly comprehend rational solutions $\tilde{u_n}$ and $\tilde{w_n}$, we draw their three-dimensional plots and singular trajectory plots in figure 5, which also shows that our asymptotic analysis results are correct.

Remark 3. It should be noted that in the above (2, 1)-fold DT and (3, 0)-fold DT, for spectral parameter λ, without using Taylor series expansion, if at least two of $C_{k,1}$, $C_{k,2}$, $C_{k,3}$ are not zeroes, the exponential-rational mixed solutions of eq. (2) will be derived, whose expressions are very complex, and so we will not discuss them here.

5. Conclusions

This article has investigated the BC lattice equation (2) associated with 3×3 matrix form Lax pair. The key innovations are displayed as follows.

- Using the continuous limit technique, the discrete eq. (2) has been mapped to the new continuous equations.
- The discrete generalised DT has been successfully popularised from 2×2 Lax pair to 3×3 Lax pair, and the new discrete generalised $(m, 3N - m)$-fold DT of eq. (2) has been established for the fist time;
- We have given higher-order rational solutions of eq. (2) by means of the special cases of the resulting DT, and analysed their singular trajectories and relevant dynamical behaviours using asymptotic analysis technique.

We expect that the results obtained above have potential applications in understanding some physical phenomena.

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