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Stability of finite difference schemes for two-space dimensional telegraph equation

MAHMUT MODANLI and FATIH OZBAG*

Department of Mathematics, Faculty of Art and Science, Harran University, Sanliurfa 63300, Turkey
*Corresponding author. E-mail: fozbag@harran.edu.tr

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Abstract. This paper is devoted to the study of two-dimensional hyperbolic partial differential telegraph equation. Converting the PDE to an ODE yields exact solution to this problem. Then, using first-order finite difference techniques, we obtain approximate numerical solutions. The numerical solution's error analysis is provided. The stability estimates of finite difference schemes, as well as some numerical tests to check the correctness with regard to the precise solution are provided.

Keywords. Two-dimensional telegraph equation; finite difference scheme; stability.

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1. Introduction

A hyperbolic telegraph partial differential equation in two-space dimensions is of the form

$$z_{\theta\theta}(\theta, u, v) + \lambda z_\theta(\theta, u, v) + \beta z(\theta, u, v) = z_{uu}(\theta, u, v) + z_{vv}(\theta, u, v) + \Psi(\theta, u, v) \quad (1)$$

with initial conditions

$$z(0, u, v) = \omega_1(u, v), \quad z_\theta(0, u, v) = \omega_2(u, v), \quad 0 < u, v < I \quad (2)$$

and boundary conditions

$$z(\theta, 0, v) = \psi_1(\theta, v), \quad z(\theta, I, v) = \psi_2(u, \theta), \quad 0 < \theta < T, \quad z(\theta, u, 0) = \psi_3(\theta, u), \quad z(\theta, u, I) = \psi_4(\theta, u), \quad (3)$$

where λ, β are constants and ω_1, ω_2, ψ_1, ψ_2, ψ_3, ψ_4 are known functions.

The hyperbolic PDEs with constant and variable coefficients are crucial in the formulation of fundamental equations in applied sciences such as engineering, aerospace, chemistry and biological systems [1–6].

After reviewing the literature, we found that a lot of effort was expended to solve the one- and also two-dimensional hyperbolic telegraph equations numerically. There are many studies on various numerical schemes for the numerical solution of one-dimensional telegraph equations (see refs [7–15]).

In the last few decades, a lot of attention has been given in the literature on how to solve two-dimensional hyperbolic telegraph equations numerically. In [16], the numerical solution of this type of problems is offered using a meshless technique based on moving least squares (MLS). A modified cubic B-spline differential quadrature method (MCB-DQM) [17] is applied to introduce numerical scheme for the two-dimensional telegraph equation. In [18], a numerical solution of a two-dimensional telegraph equation using three-dimensional Haar wavelet is presented. Based on the radial basis function, direct meshless method (DMM) [19], singular boundary method (SBM) [20], alternating direction implicit methods [21], Crank–Nicolson collocation spectral method [22] and modal Hermite spectral collocation method [23] are used for solving 2D telegraph equations.

This paper considers a first-order finite difference scheme for solving the initial-boundary value problem of the two-dimensional hyperbolic telegraph equation. The most straightforward way for discretising partial differential equations is the finite-difference method. The finite-difference approach is commonly specified on a regular grid, which allows for relatively efficient solution methods. Because the finite-difference approach is specified dimension by dimension, it is simple to raise the 'element order' to achieve higher-order precision. Regular grids are useful for very-large-scale computations on supercomputers often used in

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meteorological, seismological and astrophysical simulations.

The arrangement of the article is as follows: In §2, stability estimation for the first-order difference scheme for the two-dimensional hyperbolic telegraph equation is established. For the stability estimation of the problem, main theorems are given for the abstract form of the problem and its finite difference scheme. In §3, a numerical example is presented to check the accuracy and support the theoretical statements. The error analysis of the difference scheme is presented in a table and graphs of the precise and numerical solutions are plotted.

2. Stability

The abstract version of problem (1) in Hilbert space $\mathcal{L}_2[0, I]$ is as follows:

$$\begin{cases} \dfrac{\mathrm{d}^2 z(\theta)}{\mathrm{d}\theta^2} + \lambda \dfrac{\mathrm{d}z(\theta)}{\mathrm{d}\theta} + Bz(\theta) + Cz(\theta) + pz(\theta) \\ \quad = \Psi(\theta), \quad 0 \le \theta \le T, \\ z(0) = \omega, \quad z'(0) = \psi, \end{cases} \tag{4}$$

where $B = B^u$ and $C = C^v$ are self-adjoint positive defined operators by the formulas and corresponding domains

$$\begin{aligned} &B^u z(u) = -z_{uu} + \delta_1 z(u), \\ &D(B^u) = \{z(u) : z, z_u, z_{uu} \in \mathcal{L}_2[0, I], z(0) = z(I), \\ &z'(0) = z'(I)\} \end{aligned} \tag{5}$$

and

$$\begin{aligned} &C^v z(u) = -z_{vv} + \delta_2 z(v), \\ &D(C^v) = \{z(v) : z, z_v, z_{vv} \in \mathcal{L}_2[0, I], z(0) = z(I), \\ &\qquad z'(0) = z'(I)\}, \end{aligned} \tag{6}$$

where $\Psi(\theta) = \Psi(\theta, u, v)$ is the known abstract function and $z(\theta) = z(\theta, u, v)$ is the unknown abstract function on $[0, I]$. If the operators B and C meet the criteria listed above, the partial differential equation (1) becomes the ordinary differential equation (4). As a result, the approach employed is known as the operator method [14,15,24].

A unique solution of problem (4) is as follows:

$$\begin{aligned} z(\theta) = {} & \mathrm{e}^{-\frac{\lambda}{2}\theta} C(\theta)\omega + \frac{\lambda}{2}\mathrm{e}^{-\frac{\lambda}{2}\theta} s(\theta)\omega + \mathrm{e}^{-\frac{\lambda}{2}\theta} s(\theta)\psi \\ & + \int_0^\theta \mathrm{e}^{-\frac{\lambda}{2}(\theta - s)} s(\theta - s)\Psi(s)\mathrm{d}s. \end{aligned} \tag{7}$$

Taking

$$A = B + C + p - \frac{\lambda^2}{4},$$

$$C(\theta) = \frac{\mathrm{e}^{iA^{1/2}} + \mathrm{e}^{-iA^{1/2}}}{2}$$

and

$$s(\theta) = A^{-\frac{1}{2}} \frac{\mathrm{e}^{iA^{1/2}} - \mathrm{e}^{-iA^{1/2}}}{2i}$$

and using the method [14], formula (7) can be obtained easily.

Now, we can give the following main theorem.

Theorem 2.1. *Let $\omega \in D(B), D(C)$, $\psi \in D(B^{1/2})$, $D(C^{1/2})$ and $\Psi(\theta)$ be a continuously differentiable function on [0,T]. Then there exists a unique solution of problem (4) and its stability estimate is*

$$\begin{aligned} &\max_{0 \le \theta \le T} \left\| \frac{\mathrm{d}^2 z(\theta)}{\mathrm{d}\theta^2} \right\|_H + \max_{0 \le \theta \le T} \|Bz(\theta)\|_H \\ &\quad + \max_{0 \le \theta \le T} \|Cz(\theta)\|_H \le K\big[\|B\omega\|_H + \|C\omega\|_H \\ &\quad + \|B^{1/2}\psi\|_H + \|C^{1/2}\psi\|_H + \|\Psi(0)\|_H \\ &\quad + \max_{0 \le \theta \le T} \|\Psi(\theta)\|_H\big], \end{aligned}$$

where K does not depend on ω, ψ and $\Psi(\theta)$.

Proof. Using the procedure [14], the proof can be easily completed. □

Now, we will obtain the first order of the difference scheme in t of problem (4).

$$\begin{cases} \dfrac{z_{k+1} - 2z_k + z_{k-1}}{\tau^2} + \lambda \dfrac{z_k - z_{k-1}}{\tau} + Bz_k \\ \quad + Cz_k + pz_k = \Psi_k, \\ z_0 = \omega, \quad \dfrac{z_1 - z_0}{\tau} = \left(B + C + p - \dfrac{\lambda^2}{4}\right)\tau, \\ z_1 = \dfrac{1}{1 + \frac{\lambda}{2}\tau}\psi. \end{cases} \tag{8}$$

For formulas (8), the following theorem is satisfied.

Theorem 2.2. *The stability inequality*

$$\begin{aligned} &\max_{1 \le k \le N} \|z_k\|_H \\ &\quad \le K\Bigg[\max_{1 \le k \le N-1} \|A^{-1/2}\Psi_k\|_H \\ &\qquad + \max_{1 \le k \le N-1} \|A^{-1/2}\psi\|_H + \|\omega\|_H\Bigg] \end{aligned} \tag{9}$$

holds where K does not depend on τ, ω, ψ and Ψ_k, $1 \le k \le N - 1$.

The proof of this theorem can be shown using the method of [14].

3. Computational illustrations

In this section, a test problem is numerically solved to demonstrate the effectiveness of the approach. To evaluate the performance of the method, we calculate the maximum norm errors. Hence, we deal with the initial-boundary value problem (10) of the two-dimensional hyperbolic telegraph equation as a computational examination:

$$\begin{cases} z_{\theta\theta}(\theta,u,v)+2z_{\theta}(\theta,u,v)+z(\theta,u,v) \\ \quad =z_{uu}(\theta,u,v)+z_{vv}(\theta,u,v)+\Psi(\theta,u,v), \\ \Psi(\theta,u,v)=2\mathrm{e}^{-\theta}\sin u\sin v, \\ \quad 0<u,v<\pi, 0<\theta<1, \\ z(0,u,v)=\sin u\sin v, \\ z_{\theta}(0,u,v)=-\sin u\sin v, \quad 0\le u,v\le\pi, \\ z(\theta,0,v)=z(\theta,\pi,v) \\ =z(\theta,u,0)=z(\theta,u,\pi)=0, \quad 0\le\theta\le 1, \end{cases} \tag{10}$$

where the exact solution of problem (10) is $z(\theta,u,v)=\mathrm{e}^{-\theta}\sin u\sin v$.

In order to calculate the numerical solution of problem (10), the first step is to construct the first-order accuracy difference scheme. We take the grid space as

$$\begin{aligned} &W_{\tau,h}=[0,1]_{\tau}\times[0,\pi]_h=\{(\theta_k,u_n,v_n): \\ &\theta_k=k\tau, 0\le k\le N, N\tau=1 \\ &u_n=nh, 0\le n\le M, Mh=\pi \\ &v_m=mh, 0\le m\le M, Mh=\pi\}. \end{aligned}$$

For problem (10), the next step is establishing the first order of accuracy difference scheme.

$$\begin{cases} \dfrac{z_{n,m}^{k+1}-2z_{n,m}^{k}+z_{n,m}^{k-1}}{\tau^2}+2\dfrac{z_{n,m}^{k}-z_{n,m}^{k-1}}{\tau}+z_{n,m}^{k} \\ \quad =\dfrac{z_{n+1,m}^{k}-2z_{n,m}^{k}+z_{n-1,m}^{k}}{h^2} \\ \qquad +\dfrac{z_{n,m+1}^{k}-2z_{n,m}^{k}+z_{n,m-1}^{k}}{h^2}+\Psi_n^k, \\ u_n=nh, v_m=mh, \theta_k=k\tau, 0\le k\le N, \\ \quad 0\le n,m\le M, \\ N\tau=1, Mh=\pi, \\ z_{n,m}^{0}=\sin u\sin v, \quad \dfrac{z_{n,m}^{1}-z_{n,m}^{0}}{\tau}=-\sin u\sin v, \\ \quad 0\le n,m\le M, \\ z_{0,m}^{k}=z_{M,m}^{k}=z_{n,0}^{k}=z_{n,M}^{k}=0, 0\le k\le N. \end{cases} \tag{11}$$

Equation (11) can be written as a linear system with the size of $(N+1)(M+1)\times(N+1)(M+1)$. The matrix form of this system is as follows:

$$\begin{cases} AZ_{n+1}+BZ_n+CZ_{n-1}=D\omega_n, \ 0\le n\le N, \\ Z_0=\bar{0}, \quad Z_M=\bar{0}. \end{cases} \tag{12}$$

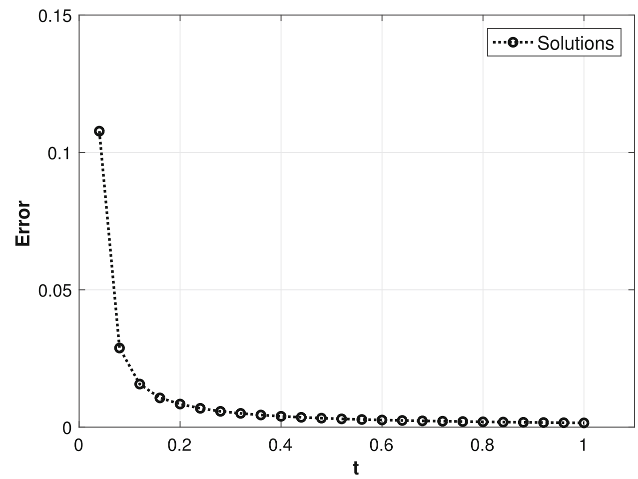

Figure 1. Error plot of problem (10).

Here, A, B, C and D are $(N+1)(M+1)\times(N+1)(M+1)$ matrices

$$A=C=\begin{bmatrix} \bar{0} & \bar{0} & \cdots & \bar{0} & \bar{0} \\ \bar{0} & A_1 & \cdots & \bar{0} & \bar{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{0} & \bar{0} & \cdots & A_1 & \bar{0} \\ \bar{0} & \bar{0} & \cdots & \bar{0} & \bar{0} \end{bmatrix} \tag{13}$$

where

$$A_1=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\bar{0}=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

and

$$a=-\frac{1}{h^2}.$$

B is defined as

$$B=\begin{bmatrix} B_1 & \bar{0} & \bar{0} & \bar{0} & \cdots & \bar{0} \\ B_2 & B_3 & B_4 & \bar{0} & \cdots & \bar{0} \\ \bar{0} & B_2 & B_3 & B_4 & \cdots & \bar{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & B_2 & B_3 & B_4 \\ -B_1 & B_1 & \cdots & \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \tag{14}$$

Table 1. Error analysis.

Difference scheme, $\tau = 1/N$, $h = \pi/M$						
$N = M$	10	20	30	40	50	60
Max. error	0.0084	0.0039	0.0026	0.0019	0.0015	0.0013
CPU time	0.0279	0.2316	1.8947	10.5406	49.6951	169.9135

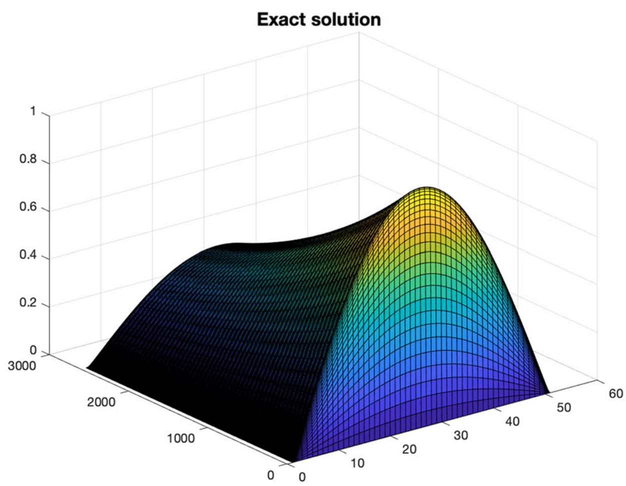

Figure 2. In (11), for $N = 50$ and $M = 50$, the result of the exact solution.

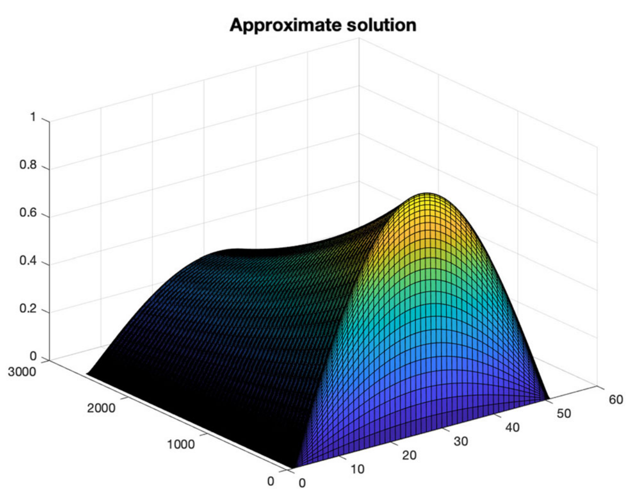

Figure 3. In (11), for $N = 50$ and $M = 50$, the result of the approximate solution.

where

$$B_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & b & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a & c & a & 0 & \cdots & 0 \\ 0 & a & c & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a & c & a \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & d & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$b = \frac{1}{\tau^2} - \frac{2}{\tau},$$
$$c = -\frac{2}{\tau^2} + \frac{2}{\tau} + \frac{4}{h^2} + 1,$$
$$d = \frac{1}{\tau^2}.$$

D is the identity matrix and ω_n, Z_s are $(N+1)(M+1) \times 1$ column vector as

$$Z_s = \begin{bmatrix} Z^0_{s,0} \\ Z^0_{s,1} \\ \vdots \\ Z^0_{s,M} \\ Z^1_{s,1} \\ \vdots \\ Z^1_{s,M} \\ \vdots \\ Z^N_{s,M} \end{bmatrix}, s = n-1, n, n+1$$

and

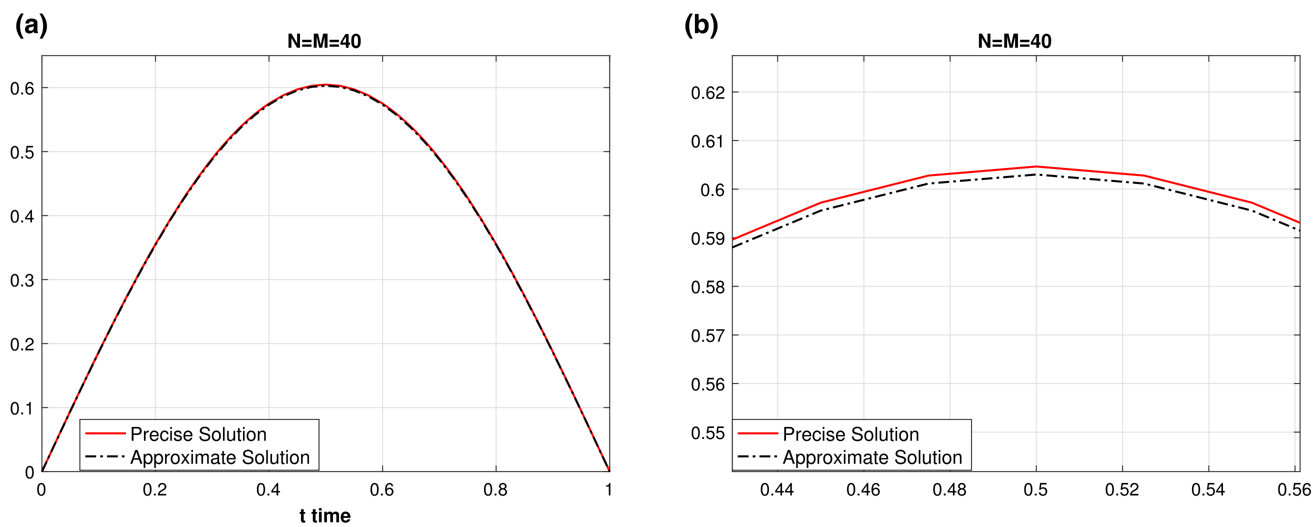

Figure 4. **(a)** Plot of maximum norm errors and exact solution of problem (10) and **(b)** zoomed plot near (0.5, 0.6).

$$\omega_n = \begin{bmatrix} \Psi^1_{n,0} \\ \Psi^1_{n,1} \\ \vdots \\ \Psi^1_{n,M} \\ \Psi^2_{n,1} \\ \vdots \\ \Psi^2_{n,M} \\ \vdots \\ \Psi^{N-1}_{n,M} \end{bmatrix}.$$

To solve the systems of eqs (12), with regard to n and m, we employed a modified Gauss elimination approach. To obtain the solutions of eqs (12) we use the MATLAB program. The numerical solutions are calculated for different grid points of N, M. The following formula is used to calculate the errors:

$$\epsilon = \max_{\substack{1\le k\le N-1 \\ 1\le n,m\le M-1}} |z(\theta, u, v) - z(\theta_k, u_n, v_m)|,$$

where $z(\theta_k, u_n, v_m)$ is the numerical solution. We then present figure 1 that shows the maximum norm error as θ changes. From figure 1, one can note that as the grid nodes grow, errors of numerical solutions decay. This demonstrates the accuracy of the constructed scheme.

Also, for some fixed grid points, numerical results of the error calculations are recoded and given in table 1 to support figure 1.

From table 1, it can be seen that the maximum norm error decays as the grid nodes grow. Note that as we take N and M bigger, the size of the matrix equation (12) becomes huge.

Furthermore, graphs of the precise and numerical solution of example (10) are given. Figures 2 and 3 are plotted to demonstrate how similar the solutions are to each other.

In figure 4, we take $N = M = 40$ and give a 2D line plot of the exact and numerical solutions of (10) for a fixed mesh point. As we compare them, we see that they almost coincide. To see this, we zoom in at the point (0.5, 0.6).

4. Conclusion

As a conclusion, an initial-boundary value problem of two-space dimensional telegraph equations was considered. For the stability estimation of the problem, main theorems were given for the abstract form of the problem and its finite difference scheme. Then, a test problem was numerically solved with the difference scheme method to demonstrate the accuracy and effectiveness of the approach. Error table as well as a graph of the exact and numerical solution were given. From table 1, it is seen that the margin of error gets smaller as the gap gets thinner. Since the problem we are working with is three-dimensional, the calculation for very large values could not be made because the computer memory was exceeded. Also, table 1 confirms that the central processing unit time is growing faster for large N and M values since the computer used is not powerful enough. Therefore, supercomputers are needed for further calculations. Moreover, the solutions were compared to show that the results are almost the same. MATLAB program was used for all calculations and plottings.

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