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Lie symmetry analysis and exact solution of (2 + 1)-dimensional nonlinear time-fractional differential-difference equations

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Abstract. The invariant analysis of time-fractional nonlinear differential-difference equations and determination of their exact solutions using the Lie symmetry method is not discussed in the literature. In this paper, we present a systematic method to derive Lie point symmetries to nonlinear time-fractional differential-difference equations and illustrate its applicability through the physically important class of (2 + 1)-dimensional time-fractional Toda lattice equations with Riemann–Liouville fractional derivative. We have shown the similarity reduction of the time-fractional nonlinear partial differential-difference equation into nonlinear fractional ordinary differential-difference equation in Erdélyi-Kober fractional derivative with a new independent variable. We derive their new exact solutions wherever possible utilising the Lie point symmetries. Our study reveals that the (2 + 1)-dimensional nonlinear time-fractional Toda lattice equations admit the infinite-dimensional symmetry algebra.

Keywords. Lie symmetry analysis; symmetry algebra; exact solutions; nonlinear time-fractional differential-difference equations; time-fractional Toda lattice equations; Riemann–Liouville fractional derivative.

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1. Introduction

It is well known that the nonlinear differential-difference equations (DΔEs) model physically significant problems, such as the vibration of particles, ladder-type electric circuits, spin lattices, etc., which arise in various fields of science and engineering (see [1] and references therein). The DΔEs arise in the numerical analysis of the nonlinear partial differential equations under the semi-discretisation scheme and also appear whenever any discrete natural phenomena are studied. Recent investigations reveal that fractional-order space and time derivatives, respectively, are used to efficiently capture the non-local interactions and memory effects in the complex system [2,3]. In recent years, many methods have been developed to study fractional partial differential equations (FPDEs). Out of these, Lie symmetry analysis and invariant subspace method are efficient tools to systematically study the invariance properties of nonlinear FPDEs and derive their exact solution [4–10]. Norwegian mathematician Sophus Lie introduced the method of symmetry reduction based on the invariance of the differential equation under the one-parameter continuous group of transformations, and several research groups developed it further [11–15]. Maeda [16] generalised the method to derive the continuous point symmetries of pure difference equations. Recently, the Lie group analysis of differential equations has been extended to discrete systems governed by differential-difference and pure difference equations by Levi and Winternitz [17,18], Quispel *et al* [19] and others [20–23].

To the best of the authors' knowledge, the invariant analysis of the nonlinear time-fractional differential-difference equations using the Lie symmetry method is not done in the literature. The main aim of this article is to generalise the intrinsic method proposed by Levi and Winternitz [17,18] for differential-difference equations to time-fractional differential-difference equations involving Riemann–Liouville fractional derivative. In this article, we derive the Lie point symmetries, similarity reductions and new exact solutions associated with a certain class of (2 + 1)-dimensional time-fractional Toda lattice equations having the form

$$ {}_0\partial_t^{\gamma} v_x(n) = \mathrm{e}^{v(n-1)-v(n)} - \mathrm{e}^{v(n)-v(n+1)}, \qquad (1)$$

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$$ {}_0\partial_t^\gamma v_x(n) = \mathrm{e}^{v(n+1)-v(n)}\big[v(n+1)+v(n)\big]_x - \mathrm{e}^{v(n)-v(n-1)}\big[v(n)+v(n-1)\big]_x, \quad (2) $$

$$ {}_0\partial_t^\gamma v_x(n) = v_x(n)\big[\mathrm{e}^{v(n+1)-v(n)} - \mathrm{e}^{v(n)-v(n-1)}\big], \quad (3) $$

$$ {}_0\partial_t^\gamma v_x(n) = ({}_0\partial_t^\gamma v(n)+1)(v(n-1)-2v(n) + v(n+1)), \quad (4) $$

where $v(n) = v(n,x,t)$, $v(n\pm1) = v(n\pm1,x,t)$, $0 < \gamma < 1$.

The outline of the article is as follows. In §2, to be self-contained, we give brief details of the derivation of Lie point symmetries of (2 + 1)-dimensional time-fractional partial differential-difference equations (FPDΔE) with Riemann–Liouville fractional derivative. Also, we illustrate its effectiveness through a class of (2 + 1)-dimensional time-fractional nonlinear Toda lattice equations. In §3, we give a brief discussion of our results and §4 gives concluding remarks.

2. Lie symmetry analysis of (2 + 1)-dimensional FPDΔEs

In this section, we present brief details of the Lie symmetry analysis for the (2+1)-dimensional FPDΔE with the continuous variable $v(n,x,t)$ depending on the discrete independent variable $n \in \mathbb{Z}$ and the continuous independent variables x, t, having the form

$$ \partial_t^\gamma v_x(n) = F\Big(n, x, t, v(l)|_{l=n-a}^{n+b}, v_x(l)|_{l=n-p_1}^{n+q_1}, \partial_t^\gamma v(l)|_{l=n-p_2}^{n+q_2}, v_{xx}(n)|_{l=n-p_{11}}^{n+q_{11}}, \partial_t^{\gamma+1} v(l)|_{l=n-p_{22}}^{n+q_{22}}\Big), \quad (5) $$

where ∂_t^γ denotes the Riemann–Liouville fractional time derivative of order γ, $0 < \gamma < 1$ as [24],

$$ {}_0\partial_t^\gamma h(t) = \begin{cases} \dfrac{1}{\Gamma(m-\gamma)}\dfrac{\partial^m}{\partial t^m}\displaystyle\int_0^t (t-u)^{m-\gamma-1}h(u)\mathrm{d}u, \\ \quad \text{if } m-1<\gamma<m, m\in\mathbb{N} \\ \dfrac{\partial^m h}{\partial t^m}, \text{if } \gamma = m \in \mathbb{N} \end{cases} \quad (6) $$

and $a, b, p_1, q_1, p_2, q_2, p_{11}, q_{11}, p_{22}, q_{22}$ are finite non-negative integers. The one-parameter continuous infinitesimal transformation are

$$ \hat{x} = x + \epsilon\xi(x,t,v(n)) + O(\epsilon^2), $$
$$ \hat{t} = t + \epsilon\tau(x,t,v(n)) + O(\epsilon^2), $$
$$ \hat{v}(n) = v(n) + \epsilon\psi(n,x,t,v(n)) + O(\epsilon^2). \quad (7) $$

The functions ξ, τ and ψ are the infinitesimals of the transformations for the variables x, t and $v(n)$, respectively. The extended infinitesimal transformations of the derivatives of integer and fractional order in (5) are

$$ \frac{\partial\hat{v}(n)}{\partial\hat{x}} = \frac{\partial v(n)}{\partial x} + \epsilon\psi^x(n,x,t,v(n)) + O(\epsilon^2), $$
$$ \frac{\partial^2\hat{v}(n)}{\partial\hat{x}^2} = \frac{\partial^2 v(n)}{\partial x^2} + \epsilon\psi^{xx}(n,x,t,v(n)) + O(\epsilon^2), $$
$$ \frac{\partial^\gamma\hat{v}(n)}{\partial\hat{t}^\gamma} = \frac{\partial^\gamma v(n)}{\partial t^\gamma} + \epsilon\psi_\gamma^t(n,x,t,v(n)) + O(\epsilon^2), $$
$$ \frac{\partial^{\gamma+1}\hat{v}(n)}{\partial\hat{t}^\gamma\partial\hat{x}} = \frac{\partial^{\gamma+1}v(n)}{\partial t^\gamma\partial x} + \epsilon\psi_{\gamma+1}^{xt}(n,x,t,v(n)) + O(\epsilon^2), \quad (8) $$

where $\psi^x, \psi^{xx}, \psi_\gamma^t, \psi_{\gamma+1}^{xt}$ are the extended set of infinitesimals of order $1, 2, \gamma, \gamma+1$ respectively. The infinitesimal operator is defined by

$$ W = \xi(x,t,v(n))\frac{\partial}{\partial x} + \tau(x,t,v(n))\frac{\partial}{\partial t} + \sum_{l=n-a}^{n+b}\psi(l,x,t,v(l))\frac{\partial}{\partial v(l)}. \quad (9) $$

The explicit expression for ψ_γ^t reads [25,26] as

$$ \psi_\gamma^t = D_t^\gamma(\psi(n) - \xi v_x(n) - \tau v_t(n)) + \xi D_t^\gamma v_x(n) + \tau D_t^{\gamma+1}v(n)). \quad (10) $$

For the computational purpose, we rewrite the above expression using the generalised Leibniz rule [25] as

$$ \psi_\gamma^t = D_t^\gamma(\psi(n)) + \sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{j-\gamma}{j+1}D_t^{j+1}\tau\,{}_0D_t^{\gamma-j}v(n) - \sum_{j=1}^{\infty}\binom{\gamma}{j}D_t^j(\xi)\,{}_0D_t^{\gamma-j}v_x(n). $$

Now the expression for $\psi_{\gamma+1}^{xt}$ yields,

$$ \begin{aligned} \psi_{\gamma+1}^{xt} &= D_t^\gamma(\psi^x(n)) + \sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{j-\gamma}{j+1}D_t^{j+1}\tau\,{}_0D_t^{\gamma-j}v_x(n) - \sum_{j=1}^{\infty}\binom{\gamma}{j}D_t^j(\xi)\,{}_0D_t^{\gamma-j}v_{xx}(n) \\ &= D_t^\gamma(\psi_x(n) + (\psi_{v(n)}(n) - \xi_{v(n)}v_x(n) - \xi_x)v_x(n) - (\tau_{v(n)}v_x(n) + \tau_x)v_t(n) \\ &\quad + \xi\,{}_0D_t^\gamma v_{xx}(n) + \sum_{j=0}^{\infty}\binom{\gamma}{j} \end{aligned} $$

$$\times\left[\frac{j-\gamma}{j+1}D_t^{j+1}\tau\ {}_0D_t^{\gamma-j}v_x(n) - D_t^j(\xi)\ {}_0D_t^{\gamma-j}v_{xx}(n)\right]. \quad (11)$$

Therefore, the extended prolongation of the infinitesimal operator according to the intrinsic method reads as

$$W^{(\gamma,2)} = \xi(x,t,v(n))\frac{\partial}{\partial x} + \tau(x,t,v(n))\frac{\partial}{\partial t} + \sum_{l=n-a}^{n+b}\psi(l)\frac{\partial}{\partial v(l)} + \sum_{l=n-p_1}^{n+q_1}\psi^x(l)\frac{\partial}{\partial v_x(l)} + \sum_{l=n-p_2}^{n+q_2}\psi_\gamma^t(l)\frac{\partial}{\partial(\partial_t^\gamma v(l))} + \sum_{l=n-p_{11}}^{n+q_{11}}\psi^{xx}(l) \times\frac{\partial}{\partial v_{xx}(l)} + \sum_{l=n-p_{22}}^{n+q_{22}}\psi_{\gamma+1}^{tt}(l)\frac{\partial}{\partial(\partial_t^{\gamma+1}v(l))} + \sum_{l=n-a_{12}}^{n+b_{12}}\psi_{\gamma+1}^{xt}(l)\frac{\partial}{\partial(\partial_t^\gamma v_x(l))}. \quad (12)$$

DEFINITION 1

(*Infinitesimal criterion for the invariance of FPD∆Es*) Let $W^{(\gamma,2)}$ defined in (12) be the extended infinitesimal generator of (9). Then the one-parameter Lie group of infinitesimal point transformations (7) and (8) are admitted by FPD∆E (5) iff

$$W^{(\gamma,2)}G|_{G=0} = 0, \quad (13)$$

where $G = \partial_t^\gamma v_x(n) - F$ is given in (5).

Theorem 2. *If the extended infinitesimal generator* $W^{(\gamma,2)}$ *given in* (12) *of the generator* W *given in* (9) *which leaves the FPD∆E* (5) *invariant, then the infinitesimal operator* W *reads as*

$$W = \xi(x)\frac{\partial}{\partial x} + \tau(t)\frac{\partial}{\partial t} + \psi(n,x,t)\frac{\partial}{\partial v(n)}, \quad (14)$$

where $\tau(t)|_{t=0} = 0$ *and* $\psi_{v(n)v(n)}(n) = 0$.

Proof 1. Let us assume that FD∆Es (5) is invariant under the one-parameter infinitesimal transformations (7) and (8). Applying the invariant condition (13) leads to

$$\xi(x,t,v(n))F_x + \tau(x,t,v(n))F_t + \sum_{l=n-a}^{n+b}\psi(l)F_{v(l)} + \sum_{l=n-p_1}^{n+q_1}\psi^x(l)F_{v_x(l)} + \sum_{l=n-p_2}^{n+q_2}\psi_\gamma^t(l)F_{\partial_t^\gamma v(l)} + \sum_{l=n-p_{11}}^{n+q_{11}}\psi^{xx}(l)F_{v_{xx}(l)} + \sum_{l=n-p_{22}}^{n+q_{22}}\psi_{\gamma+1}^{tt}(l)F_{\partial_t^{\gamma+1}v(l)} + \sum_{l=n-a_{12}}^{n+b_{12}}\psi_{\gamma+1}^{xt}(l)F_{\partial_t^\gamma v_x(l)} = 0.$$

Making use of ψ_γ^{xt}, ψ^x, ψ^{xx}, ... in (15) yields

$$\xi(x,t,v(n))F_x + \tau(x,t,v(n))F_t + \sum_{l=n-a}^{n+b}\psi(l)F_{v(l)} + \sum_{l=n-p_1}^{n+q_1}\left[D_x(\psi(l)) - D_x(\xi)v_x(l) - D_x(\tau)v_t(l)\right] \times F_{v_x(l)} + \sum_{l=n-p_2}^{n+q_2}\left[D_t^\gamma(\psi(l)) + \sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{j-\gamma}{j+1}D_t^{j+1}\tau \times {}_0D_t^{\gamma-j}v(l) - \sum_{j=1}^{\infty}\binom{\gamma}{j}D_t^j(\xi){}_0D_t^{\gamma-j}v_x(l)\right]F_{\partial_t^\gamma v(l)} + \sum_{l=n-p_{11}}^{n+q_{11}}\left[D_x(\psi^x(l)) - D_x(\xi)v_{xx}(l) - D_x(\tau)v_{xt}(l)\right] \times F_{v_{xx}(l)} + \sum_{l=n-p_{22}}^{n+q_{22}}\left[D_t^\gamma(\psi^t(l)) + \sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{j-\gamma}{j+1}D_t^{j+1}\tau \times {}_0D_t^{\gamma-j}v_t(l) - \sum_{j=1}^{\infty}\binom{\gamma}{j}D_t^j(\xi){}_0D_t^{\gamma-j}v_{tt}(l)\right]F_{\partial_t^{\gamma+1}v(l)} + \sum_{l=n-a_{12}}^{n+b_{12}}\left[D_t^\gamma(\psi^x(l)) + \sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{j-\gamma}{j+1}D_t^{j+1}\tau D_t^{\gamma-j}v_x(l) - \sum_{j=1}^{\infty}\binom{\gamma}{j}D_t^j(\xi)D_t^{\gamma-j}v_{xx}(l)\right]F_{\partial_t^\gamma v_x(l)} = 0.$$

Comparing all the coefficients of the fractional integrals and derivatives of $v(n)$ to zero, we have

$$\begin{aligned}
&D_t^{\gamma-j}v_{xx}(n) : D_t^j(\xi) = 0,\ j = 1,2,\dots\\
&D_t^{\gamma-j}v_x(n)v_t(n) : D_t^j(\tau_{v(n)}) = 0,\ j = 0,1,2,\dots\\
&D_t^{\gamma-j}v_x^2(n) : D_t^j(\xi_{v(n)}) = 0,\ j = 0,1,2,\dots\\
&D_t^{\gamma-j}v_t(n) : D_t^j(\tau_x) = 0,\ j = 0,1,2,\dots
\end{aligned} \quad (15)$$

so that

$$\begin{aligned}
&\xi_t = 0, \quad \xi_{v(n)} = 0\\
&\tau_x = 0, \quad \tau_{v(n)} = 0.
\end{aligned} \quad (16)$$

Equation (16) gives $\xi = \xi(x), \tau = \tau(t)$. The fixed lower limit of the integral in (6) should be invariant with respect to the transformation (7), leading to $\tau(t)|_{t=0} = 0$. Hence, the remaining determining equation reads as

$$\xi(x)F_x + \tau(t)F_t + \sum_{l=n-a}^{n+b} \psi(l)F_{v(l)} + \sum_{l=n-p_1}^{n+q_1} \times\Big[D_x(\psi(l)) - D_x(\xi)v_x(l)\Big]F_{v_x(l)} + \sum_{l=n-p_2}^{n+q_2}\Big[D_t^\gamma(\psi(l)) -\gamma\tau'(t)\,{}_0D_t^\gamma v(l)\Big]F_{\partial_t^\gamma v(l)} + \sum_{l=n-p_{11}}^{n+q_{11}}\Big[D_x(\psi^x(l)) -D_x(\xi)v_{xx}(l)\Big]F_{v_{xx}(l)} + \sum_{l=n-p_{22}}^{n+q_{22}}\Big[D_t^\gamma(\psi^t(l)) -\gamma\tau'(t)\,{}_0D_t^\gamma v_t(l)\Big]F_{\partial_t^{\gamma+1}v(l)} + \sum_{l=n-a_{12}}^{n+b_{12}}\Big[D_t^\gamma(\psi^x(l)) -\gamma\tau'(t)\,{}_0D_t^{\gamma-l}v_x(l)\Big]F_{\partial_t^\gamma v_x(l)} = 0.$$

The total time fractional derivative of ψ reads [26] as

$$D_t^\gamma\psi(n) = \frac{\partial^\gamma\psi(n)}{\partial t^\gamma} + \psi_{v(n)}\frac{\partial^\gamma v(n)}{\partial t^\gamma} - v(n)\frac{\partial^\gamma\psi_{v(n)}}{\partial t^\gamma} + \sum_{j=1}^{\infty}\binom{\gamma}{j}\frac{\partial^\gamma\psi_{v(n)}}{\partial t^\gamma}\,{}_0D_t^{\gamma-j}v(n) + \lambda(n). \tag{17}$$

It is appropriate to mention that the appearance of the four-folded infinite series term $\lambda(n)$ in (17) is due to the lack of chain rule for fractional derivative. Sahadevan and Bakkyaraj [26] assumed that ψ is linear in $v(n)$ and hence $\lambda(n)$ is vanished. Recently, Zhang [27] proved that this assumption is indeed true. Hence, the infinitesimal ψ is linear in $v(n)$ having the form $\psi(n) = \psi_1(n)v(n)+\psi_2(n)$.

Note 1. Since $\psi(n) = \psi_1(n)v(n) + \psi_2(n)$ and for the computational purpose rewrite the expression $\psi^{xt}_{\gamma+1}$ given in eq. (11) using the generalised Leibniz rule (8) as

$$\begin{aligned}\psi^{xt}_{\gamma+1}(n) &= \sum_{j=0}^{\infty}\binom{\gamma}{j}\Big[D_t^j(\psi_{1x}(n))\,{}_0D_t^{\gamma-j}v(n) \\ &+ [D_t^j(\psi_1(n)-\xi_x) + \frac{j-\gamma}{j+1}D_t^{j+1}\tau]D_t^{\gamma-j}v_x(n) \\ &- D_t^j\tau_x D_t^{\gamma-j}v_t(n) - D_t^j\xi_{v(n)}D_t^{\gamma-j}{v_x}^2(n) \\ &- D_t^j\tau_{v(n)}D_t^{\gamma-j}(v_t(n)v_x(n)) \\ &- D_t^j(\xi)_0D_t^{\gamma-j}v_{xx}(n)\Big] + D_t^\gamma(\psi_{2x}(n)) \\ &+ \xi\,{}_0D_t^\gamma v_{xx}(n) - {}_0D_t^\gamma v(n)\psi_{1x}(n).\end{aligned} \tag{18}$$

2.1 *Lie symmetries of time-fractional Toda lattice equation* (1)

Levi and Winternitz [17,18] investigated the invariance properties of eq. (1) when $\gamma = 1$. Suppose that eq. (1) is invariant under the one-parameter transformations (7) and (8). We have the transformed equation

$$\frac{\partial^{\gamma+1}\hat{v}(n)}{\partial\hat{t}^\gamma\partial\hat{x}} = \mathrm{e}^{\hat{v}(n-1)-\hat{v}(n)} - \mathrm{e}^{\hat{v}(n)-\hat{v}(n+1)} \tag{19}$$

provided $u = v(n)$ satisfies (1). Substituting transformations (7) and (8) in (19) yields the invariant equation

$$\psi^{xt}_{\gamma+1}(n) - \mathrm{e}^{v(n-1)-v(n)}[\psi(n-1)-\psi(n)] +\mathrm{e}^{v(n)-v(n+1)}[\psi(n)-\psi(n+1)] = 0. \tag{20}$$

Making use of $\psi = \psi_1(n)v(n)+\psi_2(n)$ and the expression $\psi^{xt}_{\gamma+1}$ in (20), we obtain

$$\begin{aligned}&\sum_{j=0}^{\infty}\binom{\gamma}{j}D_t^j(\psi_{1x}(n))\,(D_t^{\gamma-j}v(n)) + D_t^\gamma(\psi_{2x}(n)) \\ &+(\psi_1(n)-\xi_x-\gamma\tau_t)(\mathrm{e}^{v(n-1)-v(n)} - \mathrm{e}^{v(n)-v(n+1)}) \\ &+\sum_{j=1}^{\infty}\binom{\gamma}{j}(D_t^{\gamma-j}v_x(n))\Big[D_t^j[\psi_1(n)-\xi_x] \\ &+\frac{j-\gamma}{j+1}D_t^{j+1}\tau\Big] - \sum_{j=0}^{\infty}\binom{\gamma}{j}D_t^j(\tau_x)D_t^{\gamma-j}(v_t(n)) \\ &-\sum_{j=0}^{\infty}\binom{\gamma}{j}D_t^j(\xi_{v(n)})(D_t^{\gamma-j}u_x^2(n)) \\ &-\sum_{j=0}^{\infty}\binom{\gamma}{j}D_t^j(\tau_{v(n)})(D_t^{\gamma-j}(v_x(n)v_t(n))) - \sum_{j=1}^{\infty}\binom{\gamma}{j} \\ &\times D_t^j(\xi)D_t^{\gamma-j}v_{xx}(n) - \mathrm{e}^{v(n-1)-v(n)} \\ &\times\Big[\psi(n-1)v(n-1)+\psi_2(n-1)-\psi_1(n)v(n) \\ &-\psi_2(n)\Big] + \mathrm{e}^{v(n)-v(n+1)} \\ &\times\Big[\psi_1(n)v(n)+\psi_2(n)-\psi_1(n+1)v(n+1) \\ &-\psi_2(n+1)\Big] = 0.\end{aligned}$$

Comparing all the powers of fractional integral and derivatives of $v(n)$ to zero, the overdetermined system of linear equations reads as

$$
\begin{aligned}
&D_t^{\gamma-j}v(n): && D_t^j(\psi_{1x}(n))=0,\ j=0,1,2,\ldots\\
&D_t^{\gamma-j}v_{xx}(n): && D_t^j(\xi)=0,\ j=1,2,\ldots\\
&D_t^{\gamma-j}v_x(n)v_t(n): && D_t^j(\tau_{v(n)})=0,\ j=0,1,2,\ldots\\
&D_t^{\gamma-j}v_x^2(n): && D_t^j(\xi_{v(n)})=0,\ j=0,1,2,\ldots\\
&D_t^{\gamma-j}v_t(n): && D_t^j(\tau_x)=0,\ j=0,1,2,\ldots\\
&D_t^{\gamma-j}v_x(n): && D_t^j[\psi_1(n)-\xi_x]+\tfrac{j-\gamma}{j+1}D_t^{j+1}\tau=0,\\
& && j=1,2,\ldots
\end{aligned}
$$

$$
\begin{aligned}
&D_t^{\gamma}(\psi_{2x}(n))+(\psi_1(n)-\xi_x-\gamma\tau_t)(\mathrm{e}^{v(n-1)-v(n)}\\
&\quad-\mathrm{e}^{v(n)-v(n+1)})-\mathrm{e}^{v(n-1)-v(n)}\Big[\psi_1(n-1)v(n-1)\\
&\quad\times\psi_2(n-1)-\psi_1(n)v(n)-\psi_2(n)\Big]\\
&\quad+\mathrm{e}^{v(n)-v(n+1)}\Big[\psi_1(n)v(n)+\psi_2(n)\\
&\quad-\psi_1(n+1)v(n+1)-\psi_2(n+1)\Big]=0. \qquad (21)
\end{aligned}
$$

Solving the above system yields the following explicit expressions for the infinitesimals:

$$
\begin{aligned}
\xi&=a_0x+a_1,\\
\tau&=a_2t,\\
\psi&=[a_0+\gamma a_2]n+g(x)t^{\gamma-1}+h(t), \qquad (22)
\end{aligned}
$$

where a_0, a_1, a_2 are arbitrary constants and $g(x)$ and $h(t)$ are arbitrary functions of x and t, respectively.

2.1.1 *Symmetry algebra of the vector fields.* The existence of the arbitrary functions $g(x)$ and $h(t)$ leads to an infinite-dimensional symmetry algebra. A general element of this symmetry algebra reads as

$$W=W_1+W_2+W_3+W_4(g)+W_5(h),$$

where

$$
\begin{aligned}
W_1&=x\frac{\partial}{\partial x}+n\frac{\partial}{\partial v(n)},\\
W_2&=\frac{\partial}{\partial x},\\
W_3&=t\frac{\partial}{\partial t}+\gamma n\frac{\partial}{\partial v(n)},\\
W_4(g)&=g(x)t^{\gamma-1}\frac{\partial}{\partial v(n)},\\
W_5(h)&=h(t)\frac{\partial}{\partial v(n)}. \qquad (23)
\end{aligned}
$$

Table 1 is the associated commutator table of the vector fields.

We observe that the commutation relations between $W_4(g_1), W_4(g_2)$ and $W_5(h_1), W_5(h_2)$ turn out to be $[W_4(g_1)W_4(g_2)]=0$ and $[W_5(h_1)W_5(h_2)]=0$, respectively, which are not of Virasoro-type whereas the vector fields of eq. (1) when $\gamma=1$, forms centreless Kac–Moody–Virasoro (KMV)-type subalgebras [28–33].

2.1.2 *Similarity variable and symmetry reductions*

Case 1

By solving the Lagrange's characteristic equation,

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}t}{t}=\frac{\mathrm{d}v(n)}{n(1+\gamma)},$$

corresponding to the infinitesimal generator

$$W_1+W_3=x\frac{\partial}{\partial x}+n\frac{\partial}{\partial v(n)}+t\frac{\partial}{\partial t}+\gamma n\frac{\partial}{\partial v(n)},$$

we obtain the following similarity variable and the similarity transformation:

$$\zeta=\frac{x}{t},\quad v(n)=n(\gamma+1)\ln t+k_n(\zeta), \qquad (24)$$

where $k_n(\zeta)=k(n,\zeta)$.

Theorem 3. *The transformation* $v(n)=n(\gamma+1)\ln t+k_n(\zeta)$, $\zeta=\frac{x}{t}$ *converts the nonlinear time-fractional partial differential-difference equation* (1) *to nonlinear ordinary differential-difference equation of fractional order*

$$\mathcal{P}_1^{-\gamma,\gamma}k_n'(\zeta)-(\mathrm{e}^{k_{n-1}(\zeta)-k_n(\zeta)}-\mathrm{e}^{k_n(\zeta)-k_{n+1}(\zeta)})=0,$$

where $\mathcal{P}_\mu^{\tau,\gamma}k(\zeta)$ *is the Erdélyi–Kober fractional differential operator. Erdélyi–Kober fractional differential operator of order* $\gamma>0$ *is defined by* [34,35]

$$\mathcal{P}_\mu^{\tau,\gamma}h(\zeta):=\prod_{j=0}^{r-1}\left(\tau+j-\frac{1}{\mu}\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)(\mathcal{K}_\mu^{\tau+\gamma,m-\gamma}h)(\zeta),$$
$$\zeta>0,\mu>0,\gamma>0,$$

$$r=\begin{cases}[\gamma]+1, & \text{if}\,\gamma\notin\mathbb{N},\\ \gamma & \text{if}\,\gamma\in\mathbb{N},\end{cases} \qquad (25)$$

and $(\mathcal{K}_\mu^{\tau,\gamma}h)(\zeta)$ *is the Erdélyi–Kober fractional integral operator of order* $\gamma>0$ *as*

$$(\mathcal{K}_\mu^{\tau,\gamma}h)(\zeta):=\begin{cases}\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_1^\infty(u-1)^{\gamma-1}u^{-(\tau+\gamma)}h(\zeta u^{\frac{1}{\mu}})\mathrm{d}u, & \text{if}\,\gamma>0,\\ h(\zeta) & \text{if}\,\gamma=0.\end{cases}$$

Proof 2. Let $0<\gamma<1$. The LHS of eq. (1) with transformation (24) yields

$$\frac{\partial^\gamma}{\partial t^\gamma}\left(\frac{1}{t}k_n'(\zeta)\right)$$

Table 1. Commutator table of eq. (1).

	W_1	W_2	W_3	$W_4(g)$	$W_5(h)$
W_1	0	$-W_2$	0	$W_4(xg')$	0
W_2	W_2	0	0	$W_4(g')$	0
W_3	0	0	0	$(\gamma-1)W_4(g)$	$W_5(th')$
$W_4(g)$	$-W_4(xg')$	$-W_4(g')$	$(1-\gamma)W_4(g)$	0	0
$W_5(h)$	0	0	$-W_5(th')$	0	0

$$= \frac{\partial}{\partial t}\left[\frac{1}{\Gamma(1-\gamma)}\int_0^t (t-\nu)^{1-\gamma-1}\frac{1}{\nu}k_n'\left(\frac{x}{\nu}\right)\mathrm{d}\nu\right].$$

$r = t/\nu$ in the above equation yields

$$\frac{\partial^\gamma}{\partial t^\gamma}\left(\frac{1}{t}k_n'(\zeta)\right) = \frac{\partial}{\partial t}\left[\frac{t^{-\gamma}}{\Gamma(1-\gamma)}\int_1^\infty (r-1)^{1-\gamma-1}r^{-(1-\gamma)}k_n'(\zeta r)\mathrm{d}r\right].$$

Making use of the definitions of the Erdélyi–Kober fractional operators, we obtain

$$\frac{\partial^\gamma}{\partial t^\gamma}\left(\frac{1}{t}k_n'(\zeta)\right) = \frac{\partial}{\partial t}\left[t^{-\gamma}(\mathcal{K}_1^{0,1-\gamma}k_n')(\zeta)\right] = \left[t^{-\gamma-1}(\mathcal{P}_1^{-\gamma,\gamma}k_n')(\zeta)\right].$$

Proceeding further, we get the following relation:

$$\mathcal{P}_1^{-\gamma,\gamma}k_n'(\zeta) - (\mathrm{e}^{k_{n-1}(\zeta)-k_n(\zeta)} - \mathrm{e}^{k_n(\zeta)-k_{n+1}(\zeta)}) = 0.$$

□

Case 2
By solving the Lagrange's characteristic equation corresponding to the infinitesimal operator

$$W_1 = x\frac{\partial}{\partial x} + n\frac{\partial}{\partial v(n)},$$

we obtain the following similarity variable and the similarity transformation:

$$\zeta = t,$$
$$v(n) = n(\ln(x) + k_n(\zeta).$$

The symmetry reduction corresponding to the above similarity transformation leads to

$$n\zeta^{-\gamma} = (\Gamma(1-\gamma))(\mathrm{e}^{k_{n-1}(\zeta)-k_n(\zeta)} - \mathrm{e}^{k_n(\zeta)-k_{n+1}(\zeta)}).$$

Solving the above equation, we obtain the solution of eq. (1) as

$$v(n) = n\ln x + a_1(t) - \sum_{j=0}^{n}\ln\left[a_2(t) + j + \left(\frac{t^{-\gamma}}{2\Gamma(1-\gamma)}j^2\right)\right],$$

where $a_1(t)$, $a_2(t)$ are arbitrary functions of t.

Case 3
Corresponding to the infinitesimal generator,

$$W_2 + W_5(t^{\gamma-1}) = \frac{\partial}{\partial x} + t^{\gamma-1}\frac{\partial}{\partial v(n)},$$

we obtain the solution to eq. (1) as

$$v(n) = xt^{\gamma-1} + a_1(t)n + a_2(t),$$

where $a_1(t)$, $a_2(t)$ are arbitrary functions of t.

2.2 *Lie symmetries of the time-fractional Toda-like lattice equation* (2)

The Lie group analysis of eq. (2) is well known for the case $\gamma = 1$ [36]. The invariance of (2) under infinitesimal transformations (7) and (8) reads as

$$\frac{\partial^{\gamma+1}\hat{v}(n)}{\partial\hat{t}^\gamma\partial\hat{x}} = \mathrm{e}^{\hat{v}(n+1)-\hat{v}(n)}[\hat{v}(n+1)+\hat{v}(n)]_x - \mathrm{e}^{\hat{v}(n)-\hat{v}(n-1)}[\hat{v}(n)+\hat{v}(n-1)]_x \quad (26)$$

provided $u = v(n)$ satisfies (2). Substituting transformations (7) and (8) in (26), we obtain the following invariant equation:

$$\psi_{\gamma+1}^{xt}(n) - \mathrm{e}^{v(n+1)-v(n)}(v_x(n+1)+v_x(n))[\psi(n+1) - \psi(n)] + \mathrm{e}^{v(n)-v(n-1)}(v_x(n)+v_x(n-1))[\psi(n) - \psi(n-1)] - \mathrm{e}^{v(n+1)-v(n)}(\psi^x(n+1)+\psi^x(n)) + \mathrm{e}^{v(n)-v(n-1)}[\psi^x(n)-\psi^x(n-1)] = 0.$$

Solving the above system consistently yields the following expressions for the infinitesimals:

$$\xi = g(x)$$
$$\tau = a_0 t$$
$$\psi = -\gamma a_0 n + a_1 t^{\gamma-1} + h(t), \quad (27)$$

where a_0, a_1 are arbitrary constants and $g(x)$ and $h(t)$ are arbitrary functions of x and t, respectively.

2.2.1 *Symmetry algebra of the vector fields.* The existence of the arbitrary functions $g(x)$ and $h(t)$ lead to an infinite-dimensional symmetry algebra. A general element of this symmetry algebra reads as

$$W = W_1(g) + W_2 + W_3 + W_4(h),$$

where

$$W_1(g) = g(x)\frac{\partial}{\partial x}$$
$$W_2 = t\frac{\partial}{\partial t} - \gamma n\frac{\partial}{\partial v(n)}$$
$$W_3 = t^{\gamma-1}\frac{\partial}{\partial v(n)}$$
$$W_4(h) = h(t)\frac{\partial}{\partial v(n)} \tag{28}$$

Table 2 is the commutator table.

Since $[W_1(g_1), W_1(g_2)] = W_1(g_1g_2' - g_2g_1')$ and it is interesting to observe that a centreless Virasoro-type subalgebra is immediately obtained by restricting the arbitrary function $g(x)$ to Laurent polynomials so that we have the commutator $[W_1(x^r), W_1(x^s)] = (s-r)W_1(x^{r+s-1})$. Also we point out that the time-fractional partial differential-difference equation (FPDΔE) (2) does not admit the KMV subalgebra whereas its integer-order counterpart admits the KMV-type subalgebra [36].

2.2.2 *Similarity variable and symmetry reductions*

Case 1
The similarity variable and similarity transformation related to the infinitesimal operator

$$W_1(x) + W_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \gamma n\frac{\partial}{\partial v(n)}$$

are

$$\zeta = \frac{x}{t}, \quad v(n) = (-\gamma n)\ln t + k_n(\zeta). \tag{29}$$

Theorem 4. *The transformation $v(n) = (-\gamma n)\ln t + k_n(\zeta)$, $\zeta = \frac{x}{t}$ reduces the time-fractional Toda lattice equation (2) to the nonlinear FODΔE of the form*

$$\mathcal{P}_1^{-\gamma,\gamma}k_n'(\zeta) - e^{k_{n+1}(\zeta)-k_n(\zeta)}(k_{n+1}'(\zeta) + k_n'(\zeta)) + e^{k_n(\zeta)-k_{n-1}(\zeta)}(k_n'(\zeta) + k_{n-1}'(\zeta)) = 0. \tag{30}$$

Proof 3. The proof is similar to Theorem 3. □

2.3 *Lie symmetries of time-fractional modified Toda lattice equation* (3)

The Lie group analysis of eq. (3) is well known [37] for the case $\gamma = 1$. The invariance of (3) under infinitesimal transformations (7) and (8) reads as

$$\psi_{\gamma+1}^{xt}(n) - \psi^x(n)[e^{\hat{v}(n+1)-\hat{v}(n)} - e^{\hat{v}(n)-\hat{v}(n-1)}] - \hat{v}_x(n)[e^{\hat{v}(n+1)-\hat{v}(n)}](\psi(n+1) - \psi(n)) + \hat{v}_x(n) \times[e^{\hat{v}(n)-\hat{v}(n-1)}](\psi(n) - \psi(n-1)) = 0.$$

Solving eq. (31) leads to the explicit form of the infinitesimals

$$\xi = g(x)$$
$$\tau = a_0 t$$
$$\psi = -\gamma a_0 n + a_1 t^{\gamma-1} + h(t), \tag{31}$$

where a_0, a_1 are arbitrary constants and $g(x)$ and $h(t)$ are arbitrary functions of x and t, respectively.

The existence of the arbitrary functions $g(x)$ and $h(t)$ leads to an infinite-dimensional symmetry algebra. A general element of this symmetry algebra reads as

$$W = W_1(g) + W_2 + W_3 + W_4(h),$$

where

$$W_1(g) = g(x)\frac{\partial}{\partial x}$$
$$W_2 = t\frac{\partial}{\partial t} - \gamma n\frac{\partial}{\partial v(n)}$$
$$W_3 = t^{\gamma-1}\frac{\partial}{\partial v(n)}$$
$$W_4(h) = h(t)\frac{\partial}{\partial v(n)}. \tag{32}$$

Table 3 is the commutator table. Clearly $\{W_1\}$ is the Virasoro-type subalgebra which can be obtained by restricting $g(x)$ to Laurent polynomials so that we have the commutator

$$[W_1(x^r), W_1(x^s)] = (s-r)W_1(x^{r+s-1}).$$

2.3.1 *Similarity transformation and symmetry reductions*

Case 1
The similarity variable and similarity transformation related to the infinitesimal operator

$$W_1(x) + W_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \gamma n\frac{\partial}{\partial v(n)}$$

are

$$\zeta = \frac{x}{t}, \quad v(n) = (-\gamma n)\ln t + k_n(\zeta) \tag{33}$$

Theorem 5. *The transformation $v(n) = (-\gamma n)\ln t + k_n(\zeta)$, $\zeta = x/t$ reduces the FPDΔE (3) to the nonlinear FODΔE of the form*

$$P_1^{-\gamma,\gamma}k_n'(\zeta) - \Gamma(1-\gamma)k_n'(\zeta)[e^{k_{n+1}(\zeta)-k_n(\zeta)}$$

Table 2. Commutator table of eq. (2).

	$W_1(g_2)$	W_2	W_3	$W_4(h)$
$W_1(g_1)$	$W_1(g_1g_2' - g_2g_1')$	0	0	0
W_2	0	0	$(\gamma-1)W_3$	$W_4(th')$
W_3	0	$(1-\gamma)W_3$	0	0
$W_4(h)$	0	$-W_4(th')$	0	0

Table 3. Commutator table of eq. (3).

	$W_1(g_2)$	W_2	W_3	$W_4(h)$
$W_1(g_1)$	$W_1(g_1g_2' - g_2g_1')$	0	0	0
W_2	0	0	$(\gamma-1)W_3$	$W_4(th')$
W_3	0	$(1-\gamma)W_3$	0	0
$W_4(h)$	0	$-W_4(th')$	0	0

$$-e^{k_n(\zeta)-k_{n-1}(\zeta)}] = 0. \tag{34}$$

Case 2
The similarity variable and similarity transformation associated with the infinitesimal operator

$$W_2 = t\frac{\partial}{\partial t} - \gamma n\frac{\partial}{\partial v(n)}$$

read as

$$\zeta = x,$$
$$v(n) = -\gamma n(\ln(t)) + k_n(\zeta). \tag{35}$$

Making use of similarity transformation (35) in (3) yields

$$\frac{1}{\Gamma(1-\gamma)} = [e^{k_{n+1}(\zeta)-k_n(\zeta)} - e^{k_n(\zeta)-k_{n-1}(\zeta)}].$$

Solving the above equation, we obtain the solution of eq. (3) as

$$v(n) = \gamma n(\ln(t)) + a_1(x) + \sum_{j=1}^{n}\ln\left(a_2(x) + \frac{1}{\Gamma(1-\gamma)}j\right),$$

where $a_1(x), a_2(x)$ are arbitrary functions of x.

Case 3
The family of solutions of eq. (3) corresponding to the infinitesimal generator

$$W_1(x) + W_3 = x\frac{\partial}{\partial x} + t^{\gamma-1}\frac{\partial}{\partial v(n)}$$

is

$$v(n) = t^{\gamma-1}(\ln(x)) + a_1(t)n + a_2(t),$$

where $a_1(t), a_2(t)$ are arbitrary functions of t.

2.4 *Lie symmetries of time-fractional Toda lattice equation* (4)

The Lie group analysis of eq. (4) is well known [38] for the case $\gamma = 1$. The invariance of (4) under infinitesimal transformations (7) and (8) leads to the following explicit expressions for infinitesimals:

$$\xi = -a_0x + a_1$$
$$\tau = a_2t$$
$$\psi = a_0v(n) + \left[(g(x)t^{\gamma-1} + a_3t^{\gamma-1})n + h(x)t^{\gamma-1} + \frac{(a_0-\gamma a_2)t^{\gamma}}{\Gamma(1+\gamma)}\right],$$

where a_0, a_1, a_2, a_3 are arbitrary constants and $g(x)$, $h(x)$ are arbitrary functions of x.

2.4.1 *Symmetry algebras of the vector fields.* The existence of the arbitrary functions $g(x), h(x)$ leads to an infinite-dimensional symmetry algebra. A general element of this symmetry algebra reads as

$$W = W_1 + W_2 + W_3 + W_4(g) + W_5 + W_6(h),$$

where

$$W_1 = \frac{\partial}{\partial x}$$
$$W_2 = t\frac{\partial}{\partial t} - \frac{\gamma t^{\gamma}}{\Gamma(1+\gamma)}\frac{\partial}{\partial v(n)}$$
$$W_3 = -x\frac{\partial}{\partial x} + \left(v(n) + \frac{t^{\gamma}}{\Gamma(1+\gamma)}\right)\frac{\partial}{\partial v(n)}$$
$$W_4(g) = g(x)t^{\gamma-1}n\frac{\partial}{\partial v(n)}$$
$$W_5 = t^{\gamma-1}n\frac{\partial}{\partial v(n)}$$

Table 4. Commutator table of eq. (4).

	W_1	W_2	W_3	$W_4(g)$	W_5	$W_6(h)$
W_1	0	0	$-W_1$	$W_4(g')$	0	$W_6(h')$
W_2	0	0	$\frac{-\gamma}{n(\Gamma(1+\gamma))}W_5$	$(\gamma-1)W_4$	$(\gamma-1)W_5$	$(\gamma-1)W_6$
W_3	W_1	$\frac{\gamma}{n(\Gamma(1+\gamma))}W_5$	0	$W_4(-xg'-g)$	$(1-\gamma)W_4$	$W_6(-xh'-h)$
$W_4(g)$	$-W_4(g')$	$(1-\gamma)W_5$	$-W_4(-xg'-g)$	0	0	0
W_5	0	$(1-\gamma)W_5$	W_5	0	0	0
$W_6(h)$	$-W_6(h')$	$(1-\gamma)W_6$	$W_6(-xh'-h)$	0	0	0

$$W_6(h) = h(x)t^{\gamma-1}\frac{\partial}{\partial v(n)}. \tag{36}$$

Table 4 is the commutator table. Interestingly, we point out that the symmetry algebra of the vector fields of both eq. (4) and its integer-order counterpart do not admit the KMV-type subalgebra.

2.4.2 *Similarity variable and symmetry reductions.* The similarity variable and similarity transformation associated with the infinitesimal operator

$$W_2 = t\frac{\partial}{\partial t} - \frac{\gamma}{\Gamma(1+\gamma)}t^{\gamma}\frac{\partial}{\partial v(n)}$$

of the form

$$z = x,$$
$$v(n) = \frac{-t^{\gamma}}{\Gamma(\gamma+1)} + f_n(z),$$

leads to the solution of eq. (4) as

$$v(n) = \frac{-t^{\gamma}}{\Gamma(1+\gamma)} + a_1 n + a_2,$$

where a_1, a_2 are arbitrary constants.

3. Discussion

Bluman and Kumei [39] presented the necessary and sufficient conditions for the linearisability of nonlinear partial differential equations (PDEs) based on the symmetry properties of linear PDEs. They proposed a systematic algorithm to find an invertible mapping that transforms a given nonlinear PDE into a linear PDE. The existence of such mapping is based on the condition that a given PDE admits an infinite-dimensional Lie algebra of an infinitesimal generator. It is well known that the potential Burgers equation $u_t = u_{xx}+u_x^2$ has an infinite-dimensional symmetry algebra $W_\infty = \mu(x,t)\mathrm{e}^{-u}\partial_u$, where $\mu(x,t)$ is any solution to the linear heat equation $u_t = u_{xx}$. If $v = \mathrm{e}^u$ in the potential Burgers equation, it yields the linear heat equation. Levi and Scimiterna [40] investigated a similar result of the linearisability for partial difference equations following the analogy of the continuous case. Also, Matveev and Salle [41] discussed the Darboux transformation to solve the Toda lattice equation, and Nijhof *et al* [42] converted the nonlinear differential-difference equation to a linear integral equation. However, the linearisation of the time-fractional nonlinear differential-difference equation is still unknown. Fractional derivatives do not satisfy the Leibniz rule, chain rule and semigroup property which are the standard properties of classical integer-order derivatives. The non-standard properties of fractional derivatives pose a challenge to deal with fractional differential equations (FDEs). Even if one such invertible mapping exists like in the continuous case, it is not possible to convert the time-fractional nonlinear differential-difference equations into a linear one due to the non-availability of the chain rule for the fractional derivatives.

4. Conclusion

In this article, we have attempted to illustrate the application of the Lie symmetry technique to study the invariant analysis of a class of (2 + 1)-dimensional time-fractional Toda lattice equations and derived their Lie point symmetries. We have reduced the nonlinear FPDΔE into a nonlinear ODΔE of fractional order and constructed their exact solutions utilising the obtained Lie point symmetries wherever possible. From our analysis, we point out that the symmetry algebra of each of the equations is infinite-dimensional and interestingly (2 + 1)-dimensional time-fractional modified Toda lattice equation admits the centreless Virasoro-type subalgebra. It is not clear at the moment about the significance of the existence or the non-existence of Virasoro-type subalgebras to (2 + 1)-dimensional nonlinear FPDΔEs in connection with its integrability. We believe that the present study will help us to further generalise the Lie symmetry method to determine the non-classical, potential symmetries of the nonlinear fractional differential-difference equations.

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