Pramana – J. Phys. (2022) 96:217
https://doi.org/10.1007/s12043-022-02455-3

Treatment of inelastic scattering within the separable interaction model

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MS received 29 April 2021; revised 13 May 2022; accepted 4 July 2022

Abstract. The closed-form analytical expressions for the off-shell solutions for Hulthén-distorted Yamaguchi potential are derived to deal with the charged hadron systems. To construct these solutions, the particular integrals of the non-homogeneous Schrödinger equations are utilised in conjunction with the interacting Green's functions. The Jost functions thus obtained, both on- and off-shell, are exploited to find the half-off-shell T-matrix. The off-shell Jost function exists but off-shell Jost solution for the said potential has not yet been discussed in the literature. The merits of the T-matrix are examined through some model calculations. Exploiting the expressions for on- and half-shell transition matrices, the s-wave elastic and inelastic scattering cross-sections are also estimated. Our results for the proton–proton and proton–oxygen systems are in close agreement with other calculations.

Keywords. Hulthén-distorted Yamaguchi potential; interacting Green's function; off-shell Jost solution; T-matrix; scattering phase shifts; cross-sections.

PACS Nos 03.65.Nk; 21.30.Fe; 13.75.Cs; 24.10.-i; 25.40.Ep

1. Introduction

The off-shell scattering for motion in a Coulomb-distorted nuclear potential has been studied by a number of researchers [1–10]. Literally, the pure Coulomb interaction is an infinitely long-range potential but in reality it is screened at some distance. Invariably, this effect of screening affects the theory and experimental results. To investigate the effect of screening in such a situation, we shall study the off-shell scattering by screened Coulomb-distorted nuclear potential. We consider here the atomic Hulthén potential as a screened/cut-off Coulomb interaction and the nuclear part of the potential is represented by a separable non-local one. In our previous article [11], we constructed analytical expressions for the single and double transforms of the outgoing wave Hulthén Green's function by the form factors of the Yamaguchi potential to compute low-energy elastic scattering phase shifts for the alpha–nucleon and alpha–nucleus systems. In ref. [11], only the on-shell or elastic scattering was treated. Here we shall deal with the inelastic scattering of the charged hadron systems.

One can calculate the off-shell Jost function, without solving the inhomogeneous Schrödinger equation, by using its integral representation in terms of the on-shell regular solution. The derivation of the off-shell Jost solution needs the evaluation of some complicated indefinite integrals. Prescribing the method for handling such indefinite integrals, one of us (UL) [12] derived an expression for the off-shell Jost solution and thereby the Jost function for pure Hulthén potential. In ref. [12] the method for treating some indefinite integrals involving hypergeometric functions has been prescribed nicely. Motivated with this, we try to construct here the off-shell quantities for motion in Hulthén-distorted Yamaguchi potential in the maximal reduced form and make them amenable to numerical treatment. To the best of the authors' knowledge, the results for the off-shell solutions are new in the literature. Utilising the expressions for the on- and off-shell T-matrices, we shall compute elastic and inelastic cross-sections along with the estimation of low-lying excited states of the target nucleus.

The proton–proton scattering is considered to be one of the most important sources of information regarding nuclear forces. It gives the best estimation of the range of the two-nucleon forces. In this direction, numerous theoretical and experimental investigations

Published online: 03 November 2022

have been advocated by several research workers [13–33] in low and very high energy regions. The elastic scattering of protons by the oxygen nucleus has been studied by a number of groups [34–50] to investigate the low-lying energy levels of the compound nucleus F^{17} and also the energy levels of O^{16} due to excitation. The nucleon-induced inelastic cross-sections of oxygen [51,52] have immense importance in the study of the structural difference between the two compound mirror nuclei associated with the proton and neutron reactions on the same target. These are achieved by observing the differences between the induced inelastic cross-sections. When, either angle-integrated or for a specific angle, cross-sections are plotted as functions of incident particle energy or laboratory energy then these plots are called excitation functions. They frequently show a range of narrower and wider peaks. Many of these peaks are originated either by excitations of the target or by resonances.

The plan of the present text is as follows. In §2 we construct exact analytical expressions for the off-shell Jost and physical solutions by direct integration approach to the problem. Section 3 is devoted to calculate half-off-shell T-matrix, Coulomb limit, zero Coulomb limits and cross-sections. Section 4 is related to results and summary.

2. Off-shell Jost and physical solutions

The off-shell Jost $f(\chi,\xi,s)$ and physical $\psi^{(+)}(\chi,\xi,s)$ solutions for motion in a local plus non-local separable potential satisfy the non-homogeneous Schrödinger-like equations [1–5]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\chi^2-V(s)\right]f(\chi,\xi,s) = \int_0^\infty V(s,s')f(\chi,\xi,s')\mathrm{d}s'+(\chi^2-\xi^2)\mathrm{e}^{i\xi s} \quad (1)$$

and

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\chi^2-V(s)\right]\psi^{(+)}(\chi,\xi,s) = \int_0^\infty V(s,s')\psi^{(+)}(\chi,\xi,s')\mathrm{d}s' +(\chi^2-\xi^2)\sin(\xi s), \quad (2)$$

where χ is the centre of mass momentum and ξ is the off-shell one. Here the local potential $V(s)$ is the atomic Hulthén potential which takes care of the charges,

$$V(s)=V_0\frac{\mathrm{e}^{-s/d}}{1-\mathrm{e}^{-s/d}} \quad (3)$$

and the non-local separable one $V(s,s')$ of Yamaguchi [53] is expressed as

$$V(s,s')=\gamma\mathrm{e}^{-\omega(s+s')}. \quad (4)$$

The parameter V_0 defines the strength and d stands for the screening parameter of the Hulthén potential [54]. The other quantities γ and ω represent the strength and inverse range parameters of the separable non-local interaction. The particular integrals of eqs (1) and (2) represent the off-shell Jost and physical solutions for Hulthén plus rank one separable non-local potential

$$f(\chi,\xi,s)=(\chi^2-\xi^2)\int_s^\infty G^{(I)}(s,s')i\xi s'\mathrm{d}s' \quad (5)$$

and

$$\psi^{(+)}(\chi,\xi,s)=(\chi^2-\xi^2)\int_0^\infty G^{(+)}(s,s')\sin(\xi s')\mathrm{d}s' \quad (6)$$

with $G^{(I)}(s,s')$ and $G^{(+)}(s,s')$ the irregular and physical Green's functions [12,55] for Hulthén plus separable potential. To circumvent the difficulties in evaluating the indefinite integrals involved in eqs (5) and (6), we express $G^{(I)}(s,s')$ in terms of pure Hulthén Green's function and its integral transforms. From the Lippmann–Schwinger [9] integral equation, one can easily obtain

$$G^{(I)}(s,s')=G_H^{(I)}(s,s')+\frac{\gamma}{D(\chi)}G_H^{(I)}(\omega,s') \times\int_s^\infty G_H^{(I)}(s,s')\mathrm{e}^{-\omega s'}\mathrm{d}s', \quad (7)$$

where $G_H^{(I)}(\omega,s')$, the Laplace transform of $G_H^{(I)}(s,s')$ is

$$G_H^{(I)}(\omega,s')=\int_0^\infty G_H^{(I)}(s,s')\mathrm{e}^{-\omega s}\mathrm{d}s \quad (8)$$

and $D(\chi)$, the Fredholm determinant related to regular/irregular boundary condition is

$$D(\chi)=1 -\gamma\int_0^\infty\int_s^\infty \mathrm{d}s\,\mathrm{d}s'\mathrm{e}^{-\omega s}G_H^{(I)}(s,s')\mathrm{e}^{-\omega s'}. \quad (9)$$

Following Lippmann–Schwinger [9] approach, however, one can write $G^{(+)}(s,s')$ in the form

$$G^{(+)}(s,s')=G_H^{(+)}(s,s') +\frac{\gamma}{D^{(+)}(\chi)}G_H^{(+)}(\omega,s')G_H^{(+)}(\omega,s), \quad (10)$$

where $G_H^{(+)}(\omega,s')$ and $G_H^{(+)}(\omega,s)$ are the Laplace transforms of $G_H^{(+)}(s,s')$ with respect to s and s' respectively. The Fredholm determinant $D^{(+)}(\chi)$ is associated with the physical boundary condition.

2.1 *Off-shell Jost solution*

Equation (5), together with eqs (7) and (8) yield

$$f(\chi,\xi,s) = f_H(\chi,\xi,s) + \frac{\gamma}{D(\chi)}(\chi^2-\xi^2)G_H^{(I)}(\xi,\omega) \times \int_s^\infty G_H^{(I)}(s,s')\,\mathrm{e}^{-\omega s'}\mathrm{d}s', \quad (11)$$

where

$$G_H^{(I)}(\xi,\omega) = \int_0^\infty \int_s^\infty \mathrm{d}s\,\mathrm{d}s'\mathrm{e}^{i\xi s'}G_H^{(I)}(s,s')\mathrm{e}^{-\omega s'}. \quad (12)$$

The pure Hulthén off-shell Jost solution is given by [12, 56,57]

$$\begin{aligned} f_H(\chi,\xi,s) &= (\chi^2-\xi^2)\int_s^\infty G_H^{(I)}(s,s')\mathrm{e}^{i\xi s'}\mathrm{d}s' \\ &= d\mathrm{e}^{i\xi s}(1-\mathrm{e}^{-s/d}) \\ &\quad\times\Bigg[\frac{i(\xi-\chi)}{f_H(\chi)}{}_3F_2(M,N,-i(\chi+\xi)d; \\ &\quad K, 1-i(\chi+\xi)d;1) \\ &\quad\times{}_2F_1(1+M,1+N;2;1-\mathrm{e}^{-s/d}) \\ &\quad+\frac{f_H(\chi,\xi)}{df_H(\chi)}(1-\mathrm{e}^{-s/d})^{-1} \\ &\quad\times{}_2F_1(M,N;K;\mathrm{e}^{-s/d}) \\ &\quad+d(\chi^2-\xi^2) \\ &\quad\times\sum_{n=0}^\infty\frac{\Gamma(n+1-id(\chi-\xi))}{\Gamma(1-id(\chi-\xi))n!} \\ &\quad\times f_{n+1}(M+1,N+1;2;1-\mathrm{e}^{-s/d})\Bigg]. \end{aligned} \quad (13)$$

The on- and off-shell Jost functions $f_H(\chi)$ and $f_H(\chi,\xi)$ [12,56–59] are

$$f_H(\chi) = \frac{\Gamma(K)}{\Gamma+M)\Gamma(1+N)} \quad (14)$$

and

$$f_H(\chi,\xi) = \frac{\Gamma(K+i(\chi-\xi)d)\Gamma(1+i(\chi-\xi)d)}{\Gamma(1+M+i(\chi-\xi)d)\Gamma(1+N+i(\chi-\xi)d)} \quad (15)$$

with

$$M = -id\chi + id\sqrt{\chi^2+V_0},$$

$$N = -id\chi - id\sqrt{\chi^2+V_0}$$

and $K = 1-2id\chi$. (16)

The function $f_{n+1}(M+1,N+1;2;1-\mathrm{e}^{-s/d})$ represents the particular solution of the inhomogeneous Gaussian hypergeometric equation [60–62]

$$z(1-z)\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \{c_1-(a_1+b_1+1)z\}\frac{\mathrm{d}y}{\mathrm{d}z} - a_1b_1y = z^{\sigma-1}, \quad (17)$$

written as

$$\begin{aligned} f_n(a_1,b_1;c_1;x) &= x^n\sum_{j=0}^\infty\frac{\Gamma(n+a_1+j)\Gamma(n+vb_1+j)\Gamma(n+c_1-1)\Gamma(n)}{\Gamma(n+a_1)\Gamma(n+b_1)\Gamma(n+j+1)\Gamma(n+c_1+j)}x^j \\ &= \frac{x^n}{n(n+c_1-1)} \\ &\quad\times{}_3F_2(1,n+a_1,n+b_1;n+1,n+c_1;x). \end{aligned} \quad (18)$$

One can easily evaluate the value of the indefinite integral

$$G_H^{(I)}(s,\omega) = \int_s^\infty \mathrm{d}s' G_H^{(I)}(s,s')\mathrm{e}^{-\omega s'}$$

involved in eq. (11) by substituting $\xi = i\omega$ in eq. (13) and divide it by $(\chi^2-\xi^2)$. To evaluate the double integral $G_H^{(I)}(\xi,\omega)$ in eq. (12) we proceed as follows: Combination of eqs (11)–(13) yields

$$G_H^{(I)}(\xi,\omega) = T_1(\xi,\omega)+T_2(\xi,\omega)+T_3(\xi,\omega) \quad (19)$$

with

$$T_1(\xi,\omega) = d^2\sum_{n=0}^\infty\frac{\Gamma(n+1-id(\chi-\xi))}{\Gamma(1-id(\chi-\xi))n!} \times I_1(\omega,\chi), \quad (20)$$

$$T_2(\xi,\omega) = \frac{d\mathrm{e}^{-i\pi/2}}{(\chi+\xi)f_H(\chi)}{}_3F_2(M,N,-i(\chi+\xi)d; K,1-i(\chi+\xi)d;1)\,I_2(\omega,\chi), \quad (21)$$

$$T_3(\xi,\omega) = \frac{f_H(\chi,\xi)}{(\chi^2-\xi^2)\,f_H(\chi)}\,I_3(\omega,\chi), \quad (22)$$

$$I_1(\omega,\chi,n) = \int_0^\infty \mathrm{d}s\mathrm{e}^{-(\omega-i\chi)s}(1-\mathrm{e}^{-s/d})f_{n+1} \times(M+1,N+1;2;1-\mathrm{e}^{-s/d}), \quad (23)$$

$$I_2(\omega,\chi) = \int_0^\infty \mathrm{d}s\mathrm{e}^{-(\omega-i\chi)s}(1-\mathrm{e}^{-s/d}) \times{}_2F_1(1+M,1+N;2;1-\mathrm{e}^{-s/d}) \quad (24)$$

and

$$I_3(\omega,\chi) = \int_0^\infty \mathrm{d}s\mathrm{e}^{-(\omega-i\chi)s}{}_2F_1(M,N;K;\mathrm{e}^{-s/d}). \tag{25}$$

To calculate the integral in eq. (23) we substitute $z = (1-\mathrm{e}^{-s/d})$ and get

$$I_1(\omega,\chi) = d\int_0^1 \mathrm{d}z\, z(1-z)^{(\omega-i\chi)d-1} \times f_{n+1}(M+1,N+1;2;z). \tag{26}$$

Use of the following standard integral, transformation formula [63]

$$\int_0^s z^{\delta-1}(s-z)^{\mu-1}f_\sigma(d,e;f;az)\mathrm{d}z = \frac{\Gamma(\delta+\sigma)\Gamma(\mu)}{\sigma(\sigma+f-1)\Gamma(\delta+\sigma+\mu)}a^\sigma s^{\delta+\sigma+\mu-1} \times {}_4F_3(1,\sigma+d,\sigma+e,\delta+\sigma; \sigma+1,\sigma+f,\delta+\sigma+\mu;as) \tag{27}$$

with Re $\sigma > 0$, Re $\sigma + a > 1$, Re $\mu > 0$, Re $\delta > 0$, $|as| < 1$.

$$ {}_3F_2(a_1,b_1,c_1;e_1,f_1;1) = \frac{\Gamma(S)\Gamma(f_1)}{\Gamma(S+a_1)\Gamma(f-a_1)} \times {}_3F_2(a_1,e_1-b_1,e_1-c_1;S+a_1,e_1;1); \quad S = e_1+f_1-a_1-b_1-c_1 \tag{28}$$

and some algebraic manipulation leads to

$$I_1(\omega,\chi) = \frac{\Gamma(n+1)\Gamma(d(\omega-i\chi))}{(\omega+i\chi)\Gamma(n+2+d(\omega-i\chi))} \times X{}_3F_2(1,-M,-N;1+(\omega+i\chi)d,n+2;1). \tag{29}$$

With $z = (1-\mathrm{e}^{-s/d})$ and the standard definite integral [60,62,64,65]

$$\int_0^s z^{\rho-1}(s-z)^{\sigma-1}{}_2F_1(d,e;f;cz)\mathrm{d}z = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}s^{\rho+\sigma-1}{}_3F_2(d,e,\rho;f,\rho+\sigma;cs) \tag{30}$$

in eq. (24) one obtains

$$I_2(\omega,\chi) = \frac{1}{[1+(\omega-i\chi)d](\omega-i\chi)} \times {}_2F_1(1+M,1+N;2+(\omega-i\chi)d;1). \tag{31}$$

To evaluate the integral in eq. (25), we substitute $z = \mathrm{e}^{-s/d}$ together with eqs (28) and (30) to get

$$I_3(\omega,\chi) = \frac{f_H(\chi)}{[1+M](\omega-i\chi)} \times {}_3F_2(1,M,1+N+(\omega-i\chi)d; 2+M,1+(\omega-i\chi)d;1). \tag{32}$$

In view of eqs (20)–(22) in conjunction with eqs (29), (31) and (32) we have

$$T_1(\xi,\omega) = d^3 \times \sum_{n=0}^{\infty} \frac{\Gamma(n+1-id(\chi-\xi))\Gamma((\omega-i\chi)d)}{\Gamma(1-id(\chi-\xi))\Gamma(n+3+(\omega-i\chi)d)} \times {}_3F_2(1,n+2+M,n+2+N;n+3+(\omega-i\chi)d,n+2;1), \tag{33}$$

$$T_2(\xi,\omega) = \frac{d^2\mathrm{e}^{-i\pi/2}\Gamma((\omega-i\chi)d-M-N)\Gamma((\omega-i\chi)d)}{(\chi+\xi)f_H(\chi)\Gamma(1+(\omega-i\chi)d-M)\Gamma(1+(\omega-i\chi)d-N)} \times {}_3F_2(M,N,-i(\chi+\xi)d;K,1-i(\chi+\xi)d;1)\, I_2(\omega,\chi), \tag{34}$$

and

$$T_3(\xi,\omega) = \frac{f_H(\chi,\xi)}{(1+M)(\omega-i\chi)\left(\chi^2-\xi^2\right)} \times {}_3F_2(1,M,1-N+(\omega-i\chi)d; 2+M,1+(\omega-i\chi)d;1). \tag{35}$$

In deriving the above equations we have also made use of eq. (28) and

$$ {}_2F_1(a_1,b_1;c_1;1) = \frac{\Gamma(c_1)\Gamma(c_1-a_1-b_1)}{\Gamma(c_1-a_1)\Gamma(c_1-b_1)}. \tag{36}$$

Thus, eq. (19) together with eqs (33)–(35) produce the desired expression for $G_H^{(I)}(\xi,\omega)$. It is observed that the expression for $G_H^{(I)}(\xi,\omega)$ involves an aesthetically unpleasing sum. To remove this infinite sum we first calculate the sum of $T_1(\xi,\omega)$ and $T_2(\xi,\omega)$. Using eq. (18), the quantity $T_1(\xi,\omega)+T_2(\xi,\omega)$ is rewritten as

$$T_1(\xi,\omega)+T_2(\xi,\omega)=d^3\times\sum_{n=0}^{\infty}\frac{(n+1)\Gamma(n+1-id(\chi-\xi))\Gamma((\omega-i\chi)d)}{\Gamma(1-id(\chi-\xi))\Gamma(n+2+(\omega-i\chi)d)}\times f_{n+1}(M+1,N+1;2+(\omega-i\chi)d;1)+\frac{d^2\mathrm{e}^{-i\pi/2}\Gamma((\omega-i\chi)d-M-N)\Gamma((\omega-i\chi)d)}{(\chi+\xi)f_H(\chi)\Gamma(1+(\omega-i\chi)d-M)\Gamma(1+(\omega-i\chi)d-N)}\times{}_3F_2(M,N,-i(\chi+\xi)d;K,1-i(\chi+\xi)d;1)\,I_2(\omega,\chi). \quad (37)$$

In the above relation we have used eq. (18). Some algebraic manipulation, proper utilisation of eqs (28), (36) and the following transformation formulas [63]

$$f_m(a_1,b_1;c_1;z)=\frac{\Gamma(m)\Gamma(m+c_1-1)\Gamma(a_1)\Gamma(b_1)}{\Gamma(m+a_1)\Gamma(m+b_1)\Gamma(c_1)}\times\sum_{j=0}^{\infty}\frac{\Gamma(m+a_1+j)\Gamma(m+b_1+j)\Gamma(c_1)}{\Gamma(a_1)\Gamma(b_1)\Gamma(m+j+c_1)}\times\frac{z^{m+j}}{\Gamma(m+1+j)}, \quad (38)$$

$${}_3F_2(a_1,a_2,a_3;b_1,b_2;1)=\frac{\Gamma(b_2)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(b_2-a_3)\Gamma(b_1+b_2-a_1-a_2)}\times{}_3F_2(b_1-a_1,b_1-a_2,a_3;b_1,b_1+b_2-a_1-a_2;1) \quad (39)$$

and

$$\sum_{r=0}^{\infty}\frac{(a_1)_r(b_1)_r}{r!(c_1)_r}=\frac{\Gamma(a_1+1+n)\Gamma(b_1+1+n)\Gamma(c_1)}{n!\Gamma(a_1+b_1+n+1)}\times{}_3F_2(a_1,b_1,c_1+n;c_1,a_1+b_1+n+1;1) \quad (40)$$

lead to

$$T_1(\xi,\omega)+T_2(\xi,\omega)=-d^3\frac{\Gamma(1+N)\Gamma((\omega-i\chi)d)}{(1+M)\Gamma(1-id(\chi-\xi))\Gamma(2+(\omega-i\chi)d)}\times\sum_{n=0}^{\infty}\frac{(n+1-id(\chi-\xi))(n+1)}{\Gamma(n+2+N)}\times{}_3F_2(-n,M+1,1+(\omega-i\chi)d-N;M+2,2+(\omega-i\chi)d;1). \quad (41)$$

The ${}_3F_2(*)$ function in the above equation represents a polynomial. Therefore, opening the sum and rearranging the terms we arrive at

$$T_1(\xi,\omega)+T_2(\xi,\omega)=-d^3\frac{N\Gamma(N-id(\chi-\xi))\Gamma((\omega-i\chi)d)}{(1+M)\Gamma(1+N-id(\chi-\xi))\Gamma(2+(\omega-i\chi)d)}\times{}_4F_3(2,M+1,1-i(\chi-\xi)d,1+(\omega-i\chi)d-N;M+2,2+(\omega-i\chi)d,2-i(\chi-\xi)d-N;1). \quad (42)$$

Equations (14), (15), (19), (35) and (42) produce the required results for $G_H^{(I)}(\xi,\omega)$ for Hulthén plus Yamaguchi potential. The expression for $D(\chi)$ given in eq. (9) can be derived by substituting $\xi=i\omega$ in the expression of $G_H^{(I)}(\xi,\omega)$. The off-shell Jost solution $f(\chi,\xi,s)$ for the additive interaction under consideration is obtained by combining the closed form expressions for $f(\chi,\xi,s)$, $G_H^{(I)}(s,\omega)$, $G_H^{(I)}(\xi,\omega)$ and $D(\chi)$ in eq. (11).

2.2 *Off-shell physical solution*

The next task is to calculate the integral transforms $G_H^{(+)}(s,\omega)$ and $G_H^{(+)}(\xi,\omega)$ related to Hulthén physical Green's function. The physical Hulthén Green's function is expressed as [55]

$$G_H^{(+)}(s,s')=-\chi^{-1}\psi_H^{(+)}(\chi,s_<)f_H(\chi,s_>), \quad (43)$$

where

$$\psi_H^{(+)}(\chi,s)=\frac{1}{f_H(\chi)}d\chi\mathrm{e}^{i\chi s}(1-\mathrm{e}^{-s/d})\times{}_2F_1(1+M,1+N;2;1-\mathrm{e}^{-s/d}) \quad (44)$$

and

$$f_H(\chi,s)=\mathrm{e}^{i\chi s}\times{}_2F_1(M,N;K;\mathrm{e}^{-s/d}). \quad (45)$$

The quantities $s_>$ and $s_<$ are the larger and smaller values of s and s'. Here $\psi_H^{(+)}(\chi,s)$ and $f_H(\chi,s)$ are the on-shell physical and irregular solutions of the pure Hulthén potential [54,56–59]. For simplification of calculation we rewrite $G_H^{(+)}(s,s')$ as

$$G_H^{(+)}(s,s') = G_H^{(R)}(s,s') - \chi^{-1}\psi_H^{(+)} \times(\chi,s)f_H(\chi,s'), \tag{46}$$

where $G_H^{(R)}(s,s')$, the regular Hulthén Green's function

$$G_H^{(R)}(s,s') = \psi_H^{(+)}(\chi,s)f_H(\chi,s') - \psi_H^{(+)}(\chi,s')f_H(\chi,s). \tag{47}$$

Thus, eq. (47) in view of eqs (44), (45) yields

$$G_H^{(R)}(s,s') = \lim_{\epsilon\to 0} d\mathrm{e}^{i\chi(s+s')}\big[(1-\mathrm{e}^{-s/d})_2F_1(1+M, 1+N;2;1-\mathrm{e}^{-s/d})_2F_1(M,N;\epsilon;1-\mathrm{e}^{-s'/d}) - {}_2F_1(M,N;\epsilon;1-\mathrm{e}^{-s/d})(1-\mathrm{e}^{-s'/d})_2F_1(1+M, 1+N;2;1-\mathrm{e}^{-s'/d})\big]. \tag{48}$$

The regular Green's function in eq. (48) is expressed in its present form with the help of the following recursion relation [60–62]:

$${}_2F_1(P_1,Q_1;R_1;Z) = \frac{\Gamma(R_1)\Gamma(R_1-P_1-Q_1)}{\Gamma(R_1-P_1)\Gamma(R_1-Q_1)} \times {}_2F_1(P_1,Q_1;P_1+Q_1-R_1+1;1-Z) + (1-Z)^{R_1-P_1-Q_1}\frac{\Gamma(R_1)\Gamma(P_1+Q_1-R_1)}{\Gamma(P_1)\Gamma(Q_1)} \times {}_2F_1(R_1-P_1,R_1-Q_1; R_1-P_1-Q_1+1;1-Z). \tag{49}$$

Substituting eq. (10) in (6) one obtains

$$\psi^{(+)}(\chi,\xi,s) = \psi_H^{(+)}(\chi,\xi,s) + \frac{\gamma(\chi^2-\xi^2)}{2iD^{(+)}(\chi)} \times G_H^{(+)}(s,\omega)\big[G_H^{(+)}(\xi,\omega) - G_H^{(+)}(-\xi,\omega)\big] \tag{50}$$

with

$$\psi_H^{(+)}(\chi,\xi,s) = \frac{(\chi^2-\xi^2)}{2i}\int_0^\infty G_H^{(+)}(s,s')\big[\mathrm{e}^{i\xi s'} - \mathrm{e}^{-i\xi s'}\big]\mathrm{d}s' = \frac{(\chi^2-\xi^2)}{2i}\big[G_H^{(+)}(s,\xi) - G_H^{(+)}(s,-\xi)\big], \tag{51}$$

$$G_H^{(+)}(s,\omega) = \int_0^\infty G_H^{(+)}(s,s')\mathrm{e}^{-\omega s'}\mathrm{d}s', \tag{52}$$

$$G_H^{(+)}(\xi,\omega) = \int_0^\infty\int_0^\infty \mathrm{d}s\,\mathrm{d}s' G_H^{(+)}(s,s')\mathrm{e}^{-\omega s'}\mathrm{e}^{i\xi s} \tag{53}$$

and $D^{(+)}(\chi)$ is the Fredholm determinant expressed as

$$D^{(+)}(\chi) = 1 - \gamma\int_0^\infty\int_0^\infty \mathrm{d}s\,\mathrm{d}s' G_H^{(+)}(s,s')\mathrm{e}^{-\omega s}\mathrm{e}^{-\omega s'}. \tag{54}$$

The quantity $D^{(+)}(\chi)$ is obtained by replacing $\xi = i\omega$ in the expression for $G_H^{(+)}(\xi,\omega)$. In the following, we shall derive the expressions for $G_H^{(+)}(s,\omega)$ and $G_H^{(+)}(\xi,\omega)$ by direct integration. Substituting eqs (44)–(46) and (48) in eq. (52) one has

$$G_H^{(+)}(s,\omega) = \lim_{\epsilon\to 0} d\mathrm{e}^{i\chi s}\Big[(1-\mathrm{e}^{-s/d})_2F_1(1+M, 1+N;2;1-\mathrm{e}^{-s/d}) \times\int_0^s \mathrm{e}^{i\chi s'}{}_2F_1(M,N;\epsilon;1-\mathrm{e}^{-s'/d})\mathrm{d}s' - {}_2F_1(M,N;\epsilon;1-\mathrm{e}^{-s/d}) \times\int_0^s \mathrm{e}^{i\chi s'}(1-\mathrm{e}^{-s'/d})_2F_1(1+M, 1+N;2;1-\mathrm{e}^{-s'/d})\mathrm{d}s'\Big] - \frac{1}{f_H(\chi)}d\chi\mathrm{e}^{i\chi s}(1-\mathrm{e}^{-s/d})\times{}_2F_1(1+M,1+N; 2;1-\mathrm{e}^{-s/d})\int_0^s \mathrm{e}^{i\chi s'}{}_2F_1(M,N;K;\mathrm{e}^{-s'/d})\mathrm{d}s'. \tag{55}$$

Making change of variable $z = (1-\mathrm{e}^{-s'/d})$ and use of eq. (30) and the following standard integral [64]

$$f_\sigma(d,e;f;z) = \frac{1}{f-1}\Big[{}_2F_1(d,e;f;z) \times\int_0^z s^{\sigma-1}(1-s)^{d+e-f}{}_2F_1(d-f+1, e-f+1;2-f;s)\mathrm{d}s - z^{1-f}{}_2F_1(d-e+1,e-f+1;2-f;z) \times\int_0^z s^{\sigma+f-2}(1-s)^{d+e-f}{}_2F_1(d,e;f;s)\mathrm{d}s\Big] \tag{56}$$

eq. (55) leads to

$$G_H^{(+)}(s,\omega) = d\mathrm{e}^{i\chi s}(1-\mathrm{e}^{-s/d}) \times\Big[-\frac{1}{f_H(\chi)(\omega-i\chi)} \times{}_3F_2(M,N,(\omega-i\chi)d;1+i(\omega-i\chi)d,K;1) \times{}_2F_1(1+M,1+N;2;1-\mathrm{e}^{-s/d}) + d\sum_{n=0}^\infty\frac{\Gamma(n+1-(\omega+i\chi)d)}{\Gamma(1-(\omega+i\chi)d)n!} \times f_{n+1}(M+1,N+1;2;1-\mathrm{e}^{-s/d})\Big]. \tag{57}$$

The quantity $G_H^{(+)}(s,\xi)$ is obtained by substituting $\omega = -i\xi$ in eq. (57) and

$$G_H^{(+)}(s,-\xi) = \lim_{\xi\to-\xi} G_H^{(+)}(s,\xi).$$

To calculate $G_H^{(+)}(\xi,\omega)$ we substitute eq. (57) in (53) to have

$$G_H^{(+)}(\xi,\omega) = \int_0^\infty G_H^{(+)}(s,\omega)\mathrm{e}^{i\xi s}\mathrm{d}s = T_1(\xi,\omega) + T_2(\xi,\omega). \tag{58}$$

In Appendix, we derive another equivalent expression for $G_H^{(+)}(\xi,\omega)$ in terms of different ${}_4F_3(*)$ function. The quantity $G_H^{(+)}(-\xi,\omega)$ can be obtained by substituting $\xi = -\xi$ in eq. (58) and the expression for $D^{(+)}(\chi)$ can easily be calculated by putting $\xi = i\omega$ in the above equation. Therefore, eq. (50) together with eqs (51), (54), (57) and (58) produce the desired expression for off-shell physical solution for Hulthén plus rank one non-local separable potential.

3. Half-off-shell T-matrix and limits

The relation between off-shell physical solution, half-off-shell T-matrix and off-shell Jost solutions [9,10,66] is given by

$$\psi^{(+)}(\chi,\xi,s) = \frac{\pi\xi}{2}T(\chi,\xi,\chi^2)f(\chi,s) + \frac{1}{2i}[f(\chi,\xi,s) - f(\chi,\xi,s)], \tag{59}$$

where

$$T(\chi,\xi,\chi^2) = \frac{f(\chi,\xi) - f(\chi,-\xi)}{i\pi\xi f(\chi)}. \tag{60}$$

The off-shell Jost function $f(\chi,\xi)$ for motion in Hulthén plus non-local separable potential under consideration reads as

$$f(\chi,\xi) = \lim_{s\to 0} f(\chi,\xi,s) \tag{61}$$

and the other one $f(\chi,-\xi)$ is the complex conjugate of $f(\chi,\xi)$. The advantage of eq. (41) is two-fold: (i) one can identify the off-shell Jost solution or (ii) the half-off-shell T-matrix. From eqs (11), (15) and (43), the Jost function yields

$$\begin{aligned} f(\chi,\xi) &= \lim_{s\to 0}\left\{ f_H(\chi,\xi,s) + \frac{\gamma}{D(\chi)}(\chi^2-\xi^2)G_H^{(I)}(\xi,\omega)G_H^{(I)}(s,\omega)\right\} \\ &= f_H(\chi,\xi) + \frac{\gamma}{D(\chi)}(\chi^2-\xi^2)G_H^{(I)}(\xi,\omega) \\ &\quad\times \lim_{s\to 0} G_H^{(I)}(s,\omega). \end{aligned} \tag{62}$$

The expression for $G_H^{(I)}(s,\omega)$ can easily be expressed in the form

$$\begin{aligned} G_H^{(I)}(s,\omega) &= \frac{d\mathrm{e}^{i\chi s}(1-\mathrm{e}^{-s/d})}{(\chi^2+\xi^2)} \\ &\times\Big[-\frac{1}{f_H(\chi)(\omega-i\chi)} \\ &\times {}_3F_2(M,N,(\omega-i\chi)d; \\ &1+i(\omega-i\chi)d,K;1)\,{}_2F_1(1+M, \\ &1+N;2;1-\mathrm{e}^{-s/d}) \\ &+d\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\omega+i\chi)d)}{\Gamma(1-(\omega+i\chi)d)n!} \\ &\times f_{n+1}(M+1,N+1;2;1-\mathrm{e}^{-s/d}) \\ &+\frac{f_H(\chi,i\omega)}{df_H(\chi)} \\ &\times(1-\mathrm{e}^{-s/d})^{-1}\,{}_2F_1(M,N;K;\mathrm{e}^{-s/d})\Big] \end{aligned} \tag{63}$$

with

$$f_H(\chi,i\omega) = \frac{\Gamma(1+(\omega-i\chi)d)\Gamma(1+(\omega+i\chi)d)}{\Gamma(1+M+(\omega+i\chi)d)} \Gamma(1+N+(\omega+i\chi)d). \tag{64}$$

In the limit $s\to 0$ the expression in eq. (63) in conjunction with eqs (13) and (36) reads as

$$(\chi^2+\omega^2)\lim_{s\to 0} G_H^{(I)}(s,\omega) = f_H(\chi,i\omega). \tag{65}$$

Thus, eq. (63) together with eqs (9), (15), (19), (35), (42), (63) and (64) yield

$$f(\chi,\xi) = f_H(\chi,\xi) + \frac{\gamma(\chi^2-\xi^2)}{D(\chi)(\chi^2+\xi^2)} \times G_H^{(I)}(\xi,\omega)f_H(\chi,i\omega), \tag{66}$$

where the explicit expression for $D(\chi)$ is

$$\begin{aligned} D(\chi) &= 1+\gamma \\ &\times\Big[d^3\frac{N\Gamma(N+(\omega+i\chi)d-1)\Gamma((\omega-i\chi)d)}{(1+M)\Gamma(1+N+(\omega+i\chi)d)\Gamma(2+(\omega-i\chi)d)} \\ &\times{}_4F_3(2,M+1,1-(\omega+i\chi)d,1+(\omega-i\chi)d-N; \\ &M+2,2+(\omega-i\chi)d,2-(\omega+i\chi)d-N;1) \\ &-d^2\frac{f_H(\chi,i\omega)}{(1+M)(\omega-i\chi)}\times{}_3F_2(1,M, \\ &1-N+(\omega-i\chi)d;2+M,1+(\omega-i\chi)d;1)\Big]. \end{aligned} \tag{67}$$

The on-shell Jost function $f(\chi)$ may be defined as

$$f(\chi) = \frac{D^{(+)}(\chi)}{D(\chi)}. \tag{68}$$

The explicit expression for $D^{(+)}(\chi)$ for Hulthén plus Yamaguchi interaction is obtained from eqs (42), (54) and (58) with $\xi = i\omega$ written as

$$D^{(+)}(\chi) = 1 + \gamma \times \Bigg[d^3 \frac{N\Gamma(N + (\omega + i\chi)d - 1)\Gamma((\omega - i\chi)d)}{(1+M)\Gamma(1+N+(\omega+i\chi)d)\Gamma(2+(\omega-i\chi)d)} \times {}_4F_3(2, M+1, 1 - (\omega + i\chi)d, 1+(\omega - i\chi)d - N; M + 2, 2 + (\omega - i\chi)d, 2 - (\omega + i\chi)d - N; 1) \Bigg]. \tag{69}$$

Therefore, knowledge of eqs (60), (62), (64) and (66)–(69) along with $\lim_{\xi\to\chi} f(\chi,\xi) = f(\chi)$ yields the most compact expression for half-off-shell T-matrix for the potential under consideration. In the following, we shall make some checks on our expressions with respect to their limiting behaviours.

In the proper Coulomb limit, i.e. $d \to \infty$ or $V_0 \to 0$ such that their product $V_0 d = 2\chi\eta$, $\eta \to$the Sommerfeld parameter, one has $M \to i\eta$, $N \to i\eta - 2id\chi$, $K \to -2id\chi$. By the judicious use of eqs (28) and (49) and the following standard relations [60–64,67]

$$ {}_2F_1(a_1, b_1; c_1; z) = (1 - z)^{c_1 - a_1 - b_1} {}_2F_1(c_1 - a_1, c_1 - b_1; c_1; z), \tag{70}$$

$$ c_1(1 - z){}_2F_1(a_1, b_1; c_1; z) - c_1{}_2F_1(a_1 - 1, b_1; c_1; z)(c_1 - b_1)z{}_2F_1(a_1, b_1; c_1 + 1; z) = 0, \tag{71}$$

$$\lim_{x\to\infty} \frac{\Gamma(x + \alpha)}{\Gamma(x + \beta)} = x^{\alpha-\beta}\big[1 + O(x^{-1})\big]; \quad |\arg(x)| < \pi, \tag{72}$$

$$\lim_{\delta\to\infty} \left\{\delta^{\sigma} f_{\sigma}(\delta, \tau; \epsilon; x/\delta)\right\} = \theta_{\sigma}(\tau, \epsilon; x), \tag{73}$$

$$\lim_{d\to\infty} (2d\chi)^{-i\eta} f_H(\chi) = f_C(\chi); \ \chi > 0 \tag{74}$$

and

$$\lim_{d\to\infty} (2d\chi)^{-i\eta} f_H(\chi, s) = f_C(\chi, s); \ \chi > 0, s > 0 \tag{75}$$

in eqs (66), (67) and (69) the associated expressions of the Coulomb field are reproduced [3,8–10]. The quantities for motion in Hulthén plus separable potential also satisfy the same limiting conditions. The Coulomb limits of eqs (15), (19) and (67) take the form [8–11]

$$\lim_{d\to\infty} f_H(\chi, \xi) = f_C(\chi, \xi) = \left[\frac{\xi + \chi}{\xi - \chi}\right]^{i\eta}, \tag{76}$$

$$\lim_{d\to\infty} D(\chi) = D_{CY}(\chi) = 1 - \frac{\gamma}{(1 + i\eta)(\omega - i\chi)} \times \Bigg[(\omega^2 + \chi^2)^{-1}\left(\frac{\omega - i\chi}{\omega + i\chi}\right)^{i\eta} \times {}_2F_1\Big(1, i\eta; 2 + i\eta; \frac{\omega + i\chi}{\omega - i\chi}\Big) - \frac{1}{2\omega(\omega - i\chi)} {}_2F_1\Bigg(1, i\eta; 2 + i\eta; \left(\frac{\omega + i\chi}{\omega - i\chi}\right)^2\Bigg)\Bigg], \tag{77}$$

$$\lim_{d\to\infty} G_H^{(I)}(s, \omega) = G_C^{(I)}(s, \omega) = s e^{i\chi s}\Bigg[\frac{1}{(1 + i\eta)(\omega - i\chi)} \times {}_2F_1\Big(1, i\eta; 2 + i\eta; \frac{\omega + i\chi}{\omega - i\chi}\Big)\Phi(1 + i\eta, 2; -2i\chi s) - \frac{2i\chi\Gamma(1 + i\eta)}{(\omega^2 + \chi^2)}\left(\frac{\omega - i\chi}{\omega + i\chi}\right)^{i\eta} \Psi(1 + i\eta, 2; -2i\chi s) - \frac{1}{2i\chi}\sum_{n=0}^{\infty}\left(\frac{\omega + i\chi}{2i\chi}\right)^n \frac{1}{n!}\theta_{n+1}(1 + i\eta, 2; -2i\chi s)\Bigg], \tag{78}$$

$$\lim_{d\to\infty} G_H^{(I)}(\xi, \omega) = G_C^{(I)}(\xi, \omega) = \frac{1}{(1 + i\eta)(\omega - i\chi)}\Bigg[\frac{1}{(\chi^2 - \xi^2)}\left[\frac{\xi + \chi}{\xi - \chi}\right]^{i\eta} \times {}_2F_1\left(1, i\eta; 2 + i\eta; \frac{\omega + i\chi}{\omega - i\chi}\right) - \frac{e^{i\pi/2}}{(\omega - i\chi)(\xi + \chi)} \times {}_2F_1\left(1, i\eta; 2 + i\eta; \frac{(\omega + i\chi)(\xi - \chi)}{(\omega - i\chi)(\xi + \chi)}\right)\Bigg] \tag{79}$$

and

$$\lim_{d\to\infty} G_H^{(+)}(\xi, \omega) = G_C^{(+)}(\xi, \omega) = -\frac{e^{i\pi/2}}{(1 + i\eta)(\omega - i\chi)(\omega - i\xi)(\xi + \chi)} \times {}_2F_1\left(1, i\eta; 2 + i\eta; \left(\frac{(\omega + i\chi)(\xi - \chi)}{(\omega - i\chi)(\xi + \chi)}\right)\right). \tag{80}$$

Under proper limiting condition, eqs (66) and (76)–(80) reproduce the correct result for Coulomb–Yamaguchi off-shell Jost function [5–11]. It is noticed that our constructed off-shell quantities have correct Coulomb limits. When $\gamma \to 0$ one obtains the pure Hulthén off-shell quantities [12,56,57]. In the following, we shall prove that when Hulthén potential is turned off, the result in eq. (66) reproduces the pure Yamaguchi one [68]. For $V_0 = 0$ the quantities $M = 0$ and $N = -2id\chi$. Thus, substituting these values

$$f_H(\chi, \xi) = f_H(\chi, i\omega) = 1, \tag{81}$$

$$G_H^{(+)}(\xi, \omega) = -\frac{e^{i\pi/2}}{(\omega - i\chi)(\omega - i\xi)(\xi + \chi)}, \tag{82}$$

$$G_H^{(I)}(\xi, \omega) = -\frac{1}{(\omega - i\chi)(\omega - i\xi)(\chi^2 - \xi^2)}, \tag{83}$$

$$D(\chi) = 1 - \frac{\gamma}{2\omega(\omega^2+\chi^2)} \quad (84)$$

and

$$f_Y(\chi,\xi) = 1 + \frac{\gamma(\omega+i\xi)}{D(\chi)(\omega^2+\chi^2)(\omega^2+\xi^2)}. \quad (85)$$

In taking the above limits, the following relations [62]

$$\begin{aligned} f_\sigma(1,b_1;c_1:z) &= \frac{z^\sigma}{\sigma(\sigma+c_1-1)}{}_2F_1(1,\sigma+b_1;\sigma+c_1;z) \\ &= \sigma^{-1}z^{(1-c_1)}(1-z)^{c_1-b_1-1} \\ &\quad\times B_z(\sigma+c_1-1,b_1-c_1+1) \end{aligned} \quad (86)$$

and

$$B_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\mathrm{d}t \quad (87)$$

along with eqs (18) and (28) have been used judiciously, where $B_z(a,b)$ is the incomplete beta function [61,62].

To find the cross-sections for elastic and inelastic processes in terms of the interactions between the reacting systems, one needs the knowledge of the quantum mechanical wave functions. For Coulomb-distorted nuclear scattering, the scattering amplitude is given by

$$f_{nC}(\theta) = f_C(\theta) + f_n(\theta), \quad (88)$$

where

$$f_C(\theta) = -\left\{\frac{\eta}{2\chi\sin^2(\theta/2)}\right\} \times \exp(-i\eta\ln\sin^2(\theta/2) + 2i\sigma_0(\eta)) \quad (89)$$

and

$$f_n(\theta) = \frac{1}{2i\chi}\sum_{\ell=0}^{\infty}(2\ell+1)\exp(2i\sigma_\ell(\eta)) \times P_\ell(\cos(\theta))(S_\ell^n - 1). \quad (90)$$

The negative sign in front of eq. (89) originates from the fact that the Coulomb force between two protons is repulsive. The nuclear S-matrix element $S_\ell^n = \exp(2i\delta_\ell^n)$ with δ_ℓ^n, the Coulomb-distorted nuclear phase shift. The Coulomb-distorted nuclear cross-section $\sigma_{nC}(\theta)$ is written as

$$\sigma_{nC}(\theta) = |f_C(\theta) + f_n(\theta)|^2 = |f_{nC}(\theta)|^2. \quad (91)$$

For identical particle scattering

$$\sigma(\theta) = |f(\theta) + f(\pi-\theta)|^2. \quad (92)$$

The point Coulomb or Rutherford cross-section is

$$\sigma_R(\theta) = \frac{\eta^2}{4\chi^2\sin^4(\theta/2)}. \quad (93)$$

For screened Coulomb potential, the cross-section yields

$$\sigma_{Sc}(\theta) = \left|\frac{V_0 d}{1/d^2 + 4\chi^2\sin^2(\theta/2)}\right|^2. \quad (94)$$

For $d \to \infty$, $V_0 d = 2\chi\eta$ and one gets $\sigma_{Sc}(\theta) = \sigma_R(\theta)$. In the excitation process $a + b \to a + b^*$ the cross-section with two-body initial and final states is [69]

$$\sigma = \frac{2(2\pi)^5\mu^2\xi}{\chi}|T(\chi,\xi,\chi^2)|^2, \quad (95)$$

where μ is the reduced mass and ξ is fixed by energy conservation to satisfy

$$\xi = \left[\chi^2 - 2\mu((E_b)^* - E_b)\right]^{1/2}.$$

4. Results and summary

In this section, half-off-shell T matrix and the scattering phase shifts for nucleon–nucleon and nucleon–nucleus systems are computed by considering $\hbar^2/m_p = 41.47\,\text{MeV fm}^2$. For the Hulthén potential, $V_0 d = 2\chi\eta$, where the screening radius $d = 200$ fm. The quantities $2\chi\eta = 0.0347\,\text{fm}^{-1}$; $0.5255\,\text{fm}^{-1}$ and the parameters for the Yamaguchi potentials [70] are $\gamma = -2.405\,\text{fm}^{-3}$, $\omega = 1.1\,\text{fm}^{-1}$; $\gamma = -1.0855\,\text{fm}^{-3}$, $\omega = 0.39\,\text{fm}^{-1}$, for ($p$–$p$) and ($p$–$O^{16}$) systems respectively. These parameters for (p–O^{16}) system fit the correct binding energy of F^{17}. In figures 1 and 2, it is noticed that the half-shell T-matrices for both the systems under consideration exhibit peaks at the on-shell points $\xi = \chi$. For pure Coulomb or Coulomb-like

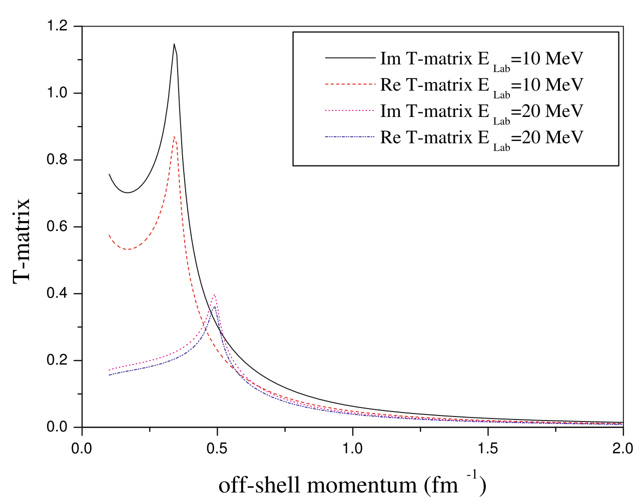

Figure 1. T-matrix $T(\chi,\xi,\chi^2)$ as a function of ξ for the (p–p) system.

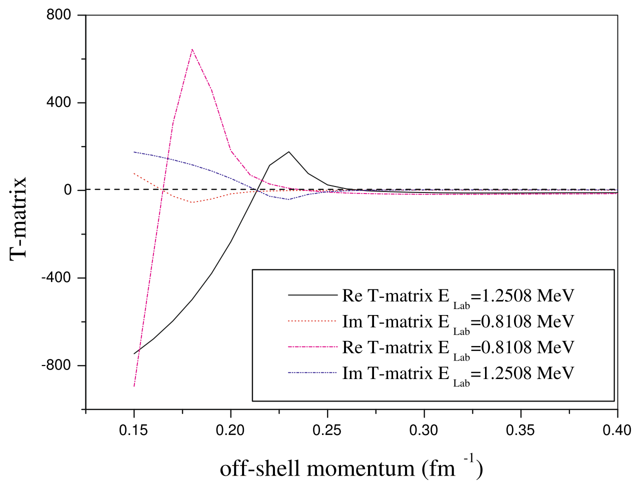

Figure 2. T-matrix $T(\chi, \xi, \chi^2)$ as a function of ξ for the (p–O^{16}) system.

potentials, the off-shell quantities exhibit discontinuities at the on-shell point [1–10]. As the Hulthén-like one is a short-range Coulomb-like interaction, the said half-shell T-matrices show spikes at $\xi = \chi = 0.347$ and 0.49 fm^{-1} and $\xi = \chi = 0.186$ and 0.23 fm^{-1} for (p–p) and (p–O^{16}) systems. For larger values of the off-shell momentum ξ, the half-shell T-matrices decrease rapidly. Essentially, in the semi-classical approximation, the off-shell T-matrix elements illustrate an incomplete collision. In the half-off-shell case, the trajectory extends to infinity at one end (half-complete collision), and in the on-shell case the trajectory extends from infinity to infinity (complete collision). Thus, the regions of larger and smaller matrix elements give an explanation of the oscillatory behaviour of the matrix elements.

It is well known that the half-shell T-matrix reproduces scattering phase shifts. We have computed (p–p) and (p–O^{16}) scattering phase shifts and presented them in tables 1 and 2 respectively. Also our expressions for the Hulthén-modified off-shell quantities reproduce correct expressions under Coulomb limits. In any reaction/inelastic scattering process, some part of energy is always carried away by the other particles present in the system. Therefore, in such physical processes, the half-shell T-matrices play a major role compared to the on-shell one. It is our belief that our newly constructed expression for half-shell T-matrix may be suitable enough for studying the effects of screening of the electromagnetic interaction in physical situations like (p–O^{16}), (p–$2p$) and ($\alpha - 2\alpha$) reactions [71–74].

Table 1. The (p–p) scattering phase shifts (in degree) along with values from ref. [16].

E_{Lab} (MeV)	Phase shift $\delta^+_{1/2}$ (degree) (Present work)	Phase shift $\delta^+_{1/2}$ (degree) Ref. [16]	E_{Lab} (MeV)	Phase shift $\delta^+_{1/2}$ (degree) (Present work)	Phase shift $\delta^+_{1/2}$ (degree) Ref. [16]
1	31.68	32.65	50	35.55	38.84
5	53.02	54.50	75	29.13	31.65
10	52.87	54.57	100	24.56	25.39
15	50.40	52.44	125	21.15	19.75
20	47.75	50.13	150	18.50	14.60
25	45.25	47.95	–	–	–

Table 2. The (p–O^{16}) scattering phase shifts (in degree) along with values from ref. [50].

E_{Lab} (MeV)	Phase shift $\delta^+_{1/2}$ (degree) (Present work)	Phase shift $\delta^+_{1/2}$ (degree) Ref. [50]	E_{Lab} (MeV)	Phase shift $\delta^+_{1/2}$ (degree) (Present work)	Phase shift $\delta^+_{1/2}$ (degree) Ref. [50]
0.3855	179.78	179.59	0.9058	173.92	174.19
0.4871	179.36	179.74	0.979	172.95	172.17
0.6162	178.40	179.65	1.1063	170.45	169.82
0.6631	177.97	178.11	1.2508	167.09	166.43
0.7162	177.35	177.96	1.3704	164.99	163.61
0.759	176.79	176.46	1.5898	160.03	159.42
0.8108	174.66	174.55	1.7903	155.57	155.78
0.8612	174.30	173.37	1.9909	151.20	151.09

Table 3. The (p–O^{16}) scattering phase shifts (in degree) along with values from refs [34] and [37].

E_{Lab} (MeV)	Phase shift $\delta^{+}_{1/2}$ (degree) (Present work)	Phase shift $\delta^{+}_{1/2}$ (degree) Ref. [34]	E_{Lab} (MeV)	Phase shift $\delta^{+}_{1/2}$ (degree) (Present work)	Phase shift $\delta^{+}_{1/2}$ (degree) Ref. [37]
2.768	136.13	135.75	3.593	122.54	121.31
2.857	134.42	133.25	3.896	118.13	116.02
2.945	132.94	130.75	4.063	115.82	111.87
3.031	131.30	130.08	4.313	112.65	108.25
3.119	130.05	128.43	4.572	109.40	102.06
3.211	128.31	126.18	4.824	106.57	99.81
3.299	127.12	125.62	5.033	104.22	95.93

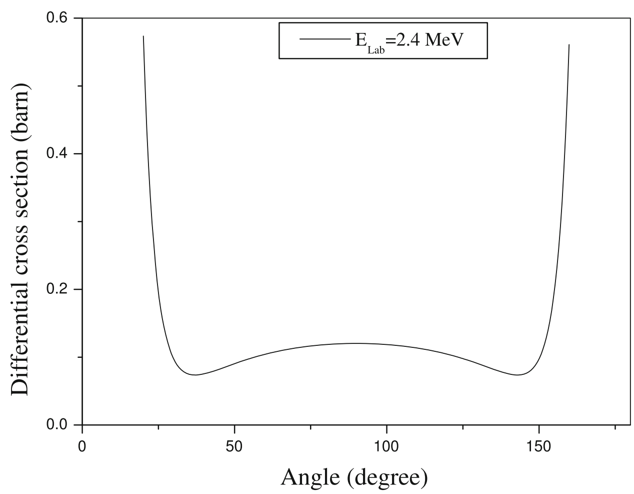

Figure 3. Differential scattering cross-section at $E_{Lab} = 2.4$ MeV.

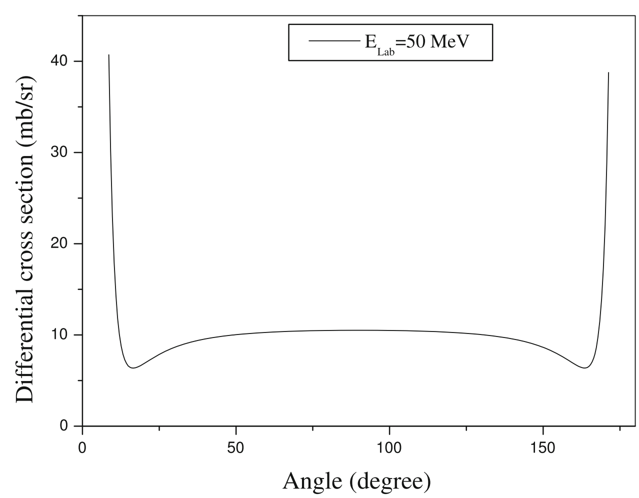

Figure 5. Differential scattering cross-section at $E_{Lab} = 50$ MeV.

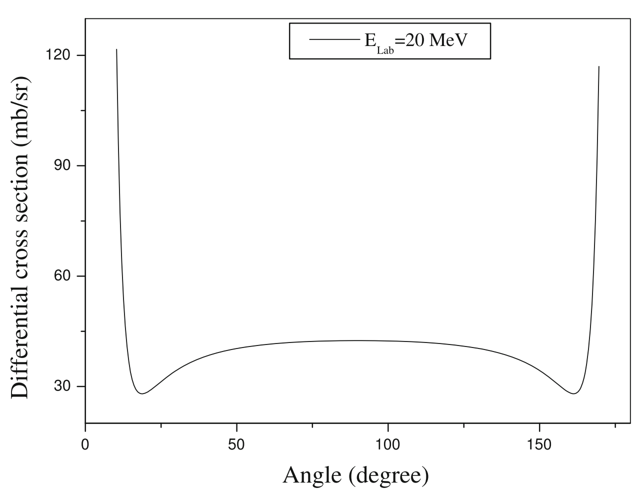

Figure 4. Differential scattering cross-section at $E_{Lab} = 20$ MeV.

The database relating to nucleon–nucleon experimental phase-shift analysis has been modified considerably by a number of groups [13–31]. Although these groups adapted different approaches to the problem, their phase data are more or less identical. Numerous processes of (p–O^{16}) elastic scattering in the energy range of 1.0–3.5 MeV have been advocated by many researchers [42,45–49]. However, phase-shift analysis of the experimental data has not been considered in refs [45–49]. We have also compared our results with the phase-shift analyses of some earlier works done in the 1960s [34,37]. Our phase parameters agree well with those of Arndt *et al* [16], Henry *et al* [34], Harris *et al* [37] and Dobovichenko *et al* [50]. In the recent past, we have treated (p–O^{16}) elastic scattering within the framework of Manning–Rosen-modified separable potential [74,75] to have elastic phase parameters. Our phase shifts in the present text are also in close agreement with our previous works. For (p–p) scattering our phase shifts are slightly lower than those of standard values [16] because we have worked with a two-parameter potential while Arndt *et al* [16] used several parameters for the same. We have checked that our results for (p–p) scattering are also in reasonable agreement with those of refs [14–22,26]. From table 2, it is noticed that our phase parameters are in excellent agreement with ref. [50] while the data in table 3 predicts slight differences from those of refs [34,37] which are within the permissible error.

The elastic scattering of protons by oxygen has been studied intensively by many [34–52] and some of these works have predicted the low-lying states of F^{17} nucleus. Absolute (p–O^{16}) elastic scattering cross-section was measured by Braun and Fried [45] in the energy range of 0.6–2.0 MeV for scattering angles 110°–160° without detecting any resonances in this energy interval. The results of new experimental measurements of (p–O^{16}) elastic scattering in the energy range of 0.6–2.0 MeV at 40°–160° angles have been demonstrated by Dubovichenko *et al* [50] by calculating differential cross-sections and excitation functions. Using the phase parameters given in tables 1–3 for (p–p) and (p–O^{16}) systems we have computed elastic differential cross-section and excitation functions. Our results are presented in figures 3–5 and tables 4–8. For calculating phase shifts, excitation functions and cross-sections, we have used MATLAB software.

It is noticed that the elastic cross-sections for (p–p) scattering at low and intermediate energy regions show symmetric nature about the central region of angles (around 90°) with shallow interference minima on both sides. It is argued that around 90° the scattering is mainly due to the nuclear forces. Our results for differential scattering cross-section at $E_{\text{Lab}} = 2.4$ MeV are in exact agreement with the observation of Jackson and Blatt [32].

Excitation functions of (p–O^{16}) elastic scattering at angles 94°, 160° and 171.5°, in the centre of mass system, presented in tables 4 and 5 are in conformity with those of Dubovichenko *et al* [50] and Chow *et al* [42]. The observed slight differences in numerical values originate due to the nature of potential models used in refs [42,50] and the present work. The excitation functions decrease monotonically with the increment of energy.

Angular distributions/differential cross-sections of (p–O^{16}) elastic scattering between 40° and 160° (c.m.) at E_{Lab} =0.6, 0.8, 1.0, 3.0 and 5.0 MeV are presented in tables 6–8. In tables 6 and 7, our data are in the same order of magnitude with Dubovichenko *et al* [50] and decrease with increase in angles. At larger values of angles, matching of data are satisfactory. For E_{Lab} =3.0 and 5.0 MeV, lower peaks are visualised at angles 72.40° and 62.10°, respectively. Our observations at laboratory energies 3.0 and 5.0 MeV are in qualitative agreement with the experimental findings of Kobayashi [35] at energies 6.87, 7.13 and 7.53 MeV in which lower peaks appear in between $\theta_{\text{c.m.}} = 73.4°–83.5°$. Afterwards, both maintain the same trends as of ref. [35]. Due to the non-availability of any reference data at these energies, we are unable to incorporate them in table 8.

The resulting inelastic cross-sections are plotted in figure 6 against the laboratory energy. The curves A, B and C lead to average excitation of about 6 and 7 MeV in O^{16}. Curve A gives an excitation energy of 6.1 MeV, which is regarded as the average of two discrete energy levels 6.049 and 6.13 MeV, while B and C correspond to 6.92 and 7.11 states in O^{16} which are in agreement with the observation of Ajzenberg-Selove [76]. As obvious, the angle-integrated cross-sections decrease monotonically as incident energy increases [35,77]. Since 1960, several experimental results on total inelastic/reaction cross-sections for (p–O^{16}) system have been published [35–49] which include the effect of all partial waves and resultant forces. At $E_{\text{Lab}} = 12$ MeV, Kobayashi [35] predicted $\sigma = 180$ and 90 mb for Q =6.1 and 7.0 MeV excitations whereas our values are around 20 mb for s-wave case. As we restrict ourselves with s-wave scattering only with central potential model, no such experimental data are available to be incorporated in figure 6.

Chen and Leavitt [33] measured (p–p) inelastic cross-section beyond laboratory energy 410 MeV. Several experimental measurements [26,29–31] have been carried out on (p–p) inelastic cross-section in the TeV region. As the present text restricts itself up to energy 150 MeV, the inelastic (p–p) scattering study goes beyond the scope of this article.

Recently, we have constructed exact analytical expressions for the off-shell solutions and T-matrices for motion in the Hulthén and Manning–Rosen potentials in all partial waves [58,59] through pure quantum mechanical approach to the problems. For complete treatment of the charged hadron systems, one has to consider the additive interaction, namely electromagnetic plus a non-local separable potential with a suitable form of the centrifugal barrier, to have precise expressions for transition matrices. The fully off-shell transition matrix for

Table 4. Excitation functions of (p–O^{16}) elastic scattering at 94° and 160° (c.m.).

E_{Lab}(MeV)	$\frac{d\sigma}{d\Omega}$ (c.m.), (mb/sr) at 160° (c.m.) (Present work)	$\frac{d\sigma}{d\Omega}$ (c.m.), (mb/sr) at 160° (c.m.) Ref. [50]	$\frac{d\sigma}{d\Omega}$ (c.m.), (mb/sr) at 94° (c.m.) (Present work)	$\frac{d\sigma}{d\Omega}$ (c.m.), (mb/sr) at 94° (c.m.) Ref. [50]
0.600	292.748288	318	902.533051	944
0.620	276.111117	298	842.667326	890
0.640	261.155716	276	788.381550	852
0.660	247.677916	258	739.015554	830
0.680	236.113821	254	696.273340	754
0.700	225.034276	228	654.939772	734
0.720	214.984011	226	617.077504	691
0.740	205.849684	219	582.320079	642
0.760	197.532653	205	550.348199	612
0.780	189.946756	202	520.882532	576
0.800	183.365995	192	495.056846	559
0.820	176.995451	177	469.794149	529
0.840	171.158479	188	446.394479	509
0.860	165.802959	170	424.687178	488
0.880	161.131474	173	404.520429	476
0.900	156.356554	165	385.758793	462
0.920	152.399007	163	369.170798	431
0.940	148.538940	153	352.809126	419
0.960	144.975912	147	337.530033	415
0.980	141.683514	145	323.245080	404
1.000	138.638011	145	309.194649	394
1.020	135.961147	141	297.985403	373
1.040	133.336997	134	286.194225	372

Table 5. Excitation functions of (p–O^{16}) elastic scattering at 171.5° (c.m.).

E_{Lab}(MeV)	$\frac{d\sigma}{d\Omega}$ (c.m.), (mb/sr) at 171.5° (c.m.) (Present work)	$\frac{d\sigma}{d\Omega}$ (c.m.), (mb/sr) at 171.5° (c.m.) Ref. [42]
0.3855±0.013	639.077247	638.0
0.4871±0.013	409.446941	398 ± 11
0.6162 ±0.012	270.342639	250 ± 7
0.6631±0.011	234.854790	231 ± 6
0.7162±0.011	207.092999	200± 6
0.759 ±0.011	189.061917	191 ± 5
0.8108±0.015	172.062312	183 ± 5
0.8612±0.015	158.743485	172 ± 5
0.9058±0.010	149.016264	151 ± 3
0.9790±0.010	136.440960	143 ± 3
1.1063±0.015	121.572328	127 ± 3
1.2508±0.015	111.318061	118 ± 2.5
1.3704±0.015	105.896666	113 ± 2.5
1.5898 ±0.025	99.879413	107.5 ± 2.2
1.7903 ±0.025	96.459721	97.0 ± 2.0
1.9909±0.025	93.872379	95.2 ± 1.9

Table 6. Angular distributions of (p–O^{16}) elastic scattering at E_{Lab} = 0.60 MeV and 0.80 MeV.

$\theta°$(c.m.)	$\frac{d\sigma}{d\Omega}$ (c.m.), (mb/sr) $E_{Lab} = 0.60$ MeV (Present work)	$\frac{d\sigma}{d\Omega}$(c.m.), (mb/sr) $E_{Lab} = 0.60$ MeV Ref. [50]	$\frac{d\sigma}{d\Omega}$(c.m.), (mb/sr) $E_{Lab} = 0.80$ MeV (Present work)	$\frac{d\sigma}{d\Omega}$(c.m.), (mb/sr) $E_{Lab} = 0.80$ MeV Ref. [50]
41.3	17051.151462	17068 ± 10%	9459.559193	9271 ± 10%
62.1	3776.122328	3712 ± 10%	2088.069718	1967 ± 10%
72.4	2156.068930	2128 ± 10%	1161.120284	1195 ± 10%
92.6	952.662159	943 ± 5%	520.099211	590 ± 5%
103	702.048696	704 ± 5%	395.157166	461 ± 5%
122	462.394459	475 ± 5%	275.042569	288 ± 5%
141	346.436191	373 ± 5%	213.482614	219 ± 5%
151	313.704143	332 ± 5%	195.205505	203 ± 5%
160	293.278408	312 ± 5%	183.532480	193 ± 5%

Table 7. Angular distributions of (p–O^{16}) elastic scattering at E_{Lab} = 1.0 MeV.

$\theta°$ (c.m.)	$\frac{d\sigma}{d\Omega}$ (c.m.), (mb/sr) $E_{Lab} = 1.0$ MeV (Present work)	$\frac{d\sigma}{d\Omega}$ (c.m.), (mb/sr) $E_{Lab} = 1.0$ MeV Ref. [50]
41.3	6022.856230	5208 ± 10%
62.1	1271.859815	1201 ± 5%
72.4	688.325838	814 ± 5%
92.6	323.597348	381 ± 5%
103	258.416051	293 ± 5%
122	194.240555	208 ± 5%
141	157.993929	161 ± 5%
151	146.452284	146 ± 5%
160	138.861213	139 ± 5%

Table 8. Angular distributions of (p–O^{16}) elastic scattering at E_{Lab} =3.0 MeV and 5.0 MeV.

$\theta°$ (c.m.)	$\frac{d\sigma}{d\Omega}$(c.m.), (mb/sr) $E_{Lab} = 3.0$ MeV (Present work)	$\frac{d\sigma}{d\Omega}$(c.m.), (mb/sr) $E_{Lab} = 5.0$ MeV (Present work)
41.3	605.944672	160.482483
62.1	32.793038	0.958624
72.4	24.324679	8.284527
92.6	52.798511	30.239165
103	64.032408	37.897159
122	76.077587	47.145469
141	80.215713	51.551921
151	80.979694	52.809889
160	81.278974	53.525656

Coulomb-distorted Graz potential exists in the literature [1,8–11]. However, such expression for screened Coulomb-modified Graz potential has not yet been published in the non-relativistic domain of scattering theory. All partial wave dealings of this story involve inordinate mathematical complications. It is in our active

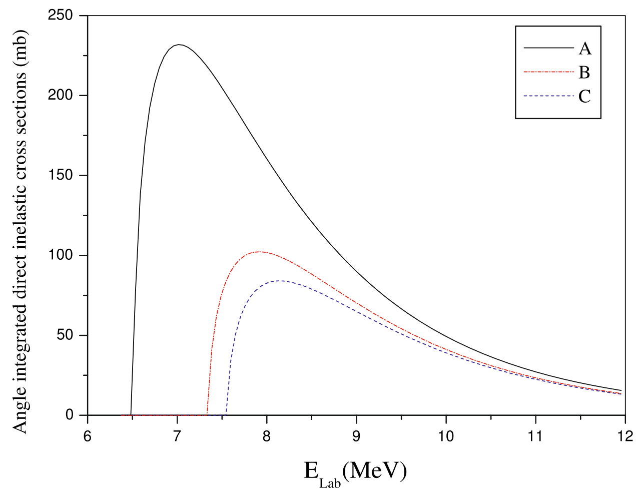

Figure 6. Angle-integrated direct (p–O^{16}) inelastic cross–sections.

consideration and we hope to address it in the near future.

Appendix

From eqs (37) and (58) one has

$$G_H^{(+)}(\xi,\omega) = T_1(\xi,\omega) + T_2(\xi,\omega) = d^2$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(n+1-id(\chi-\xi))\Gamma((\omega-i\chi)d)}{\Gamma(1-id(\chi-\xi))\Gamma(n+3+(\omega-i\chi)d)(\omega+i\chi)}$$
$$\times {}_3F_2(1, M+2+n, N+2+n;$$
$$n+3+(\omega-i\chi)d, n+2; 1)$$
$$+ \frac{d^2 e^{-i\pi/2}\Gamma((\omega-i\chi)d-M-N)}{(\chi+\xi)f_H(\chi)\Gamma((1+(\omega-i\chi)d-M)}$$

$$\times\frac{\Gamma((\omega-i\chi)d)}{(1+(\omega-i\chi)d-N)}{}_3F_2(M,N,-i(\chi+\xi)d;\ K,1-i(\chi+\xi)d;1)\,I_2(\omega,\chi). \quad (96)$$

Utilizing eqs (28) and (39) and the following relation of ${}_3F_2(*)$ function [63]

$$\begin{aligned}{}_3F_2(a,b,c;e;f;1) &= \frac{\Gamma(e)\Gamma(e-a-b)}{\Gamma(e-a)\Gamma(e-b)}\\ &\times {}_3F_2(a,b,f-c;a+b-e+1,f;1)\\ &+\frac{\Gamma(e)\Gamma(f)\Gamma(a+b-e)\Gamma(e+f-a-b-c)}{\Gamma(a)\Gamma(b)\Gamma(f-c)\Gamma(e+f-a-b)}\\ &\times {}_3F_2(e-a,e-b,e+f-a-b-c;\\ &e-a-b+1,e+f-a-b;1)\end{aligned} \quad (97)$$

eq. (96) becomes

$$\begin{aligned}G_H^{(+)}(\xi,\omega) &= -d^3\frac{\Gamma(-i(\xi+\chi)d)}{(1+M)(1+N)\Gamma(1-(\omega+i\chi)d)}\\ &\times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\omega+i\chi)d)(n+1)}{\Gamma(n+2-i(\xi+\chi)d)}\\ &\times {}_3F_2(-n,1,1+i(\xi-\chi)d;\\ &M+2,N+2;1).\end{aligned} \quad (98)$$

Term by term rearrangement and some algebraic manipulation lead to

$$\begin{aligned}G_H^{(+)}(\xi,\omega) &= -\frac{d^2}{(1+M)(1+N)(\omega-i\xi)((\omega-i\chi)d-1)}\\ &\times {}_4F_3(1,2,1-(\omega+i\chi)d,1-i(\xi-\chi)d;\\ &M+2,N+2,2-(\omega-i\xi)d;1).\end{aligned} \quad (99)$$

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