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Periods of periodic travelling wave solutions for an elastic beam equation

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Abstract. The periods of the periodic travelling wave solutions for an elastic beam equation were studied. By the transformation of variables, the elastic beam equation was reduced to a planar differential system. The period function of the planar differential system is examined. It was proved that the period function is a monotonic function. Moreover, the asymptotic behaviour of the period function was also given.

Keywords. Period function; monotonicity; periodic travelling wave; elastic beam equation.

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1. Introduction

In the studies of nonlinear transverse oscillation of elastic beam under tension, Konno *et al* [1] have shown that the nonlinear oscillation can be described by the following equation:

$$u_{tt} - u_{xx} + 2\epsilon\left[\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right]_{xx} = 0, \tag{1}$$

where the dimensionless paremeter ϵ measures the relative size of the bending stiffness over tensile strength along the beam. The travelling wave solutions of eq. (1) and the related equation named Wadati–Konno–Ichikawa–Shimizu (WKIS) equation have been studied by many mathematicians and physicists [1–8]. Most studies concentrated on finding exact physically significant solutions. However, it is also very important to do the qualitative analysis of the solutions [2,3]. There are some interesting problems on the equation as follows. How do the travelling wave solutions depend on the parameters of the system? What is the analytic behaviour of the period functions associated with periodic waves? The first problem has been considered by Li *et al* in [4]. They have studied the travelling wave solutions of eq. (1) by using the bifurcation method of the dynamic systems. For the second problem about WKIS equation, Gorder obtained the expression of the period function in terms of elliptic functions and the monotonicity is given only by numerical simulations [2,3]. In this paper, we plan to consider the analytic behaviour of periodic waves of eq. (1) by using methods of qualitative analysis and prove the monotonicity of the period function in a rigorous mathematical way. Our results will answer the above problems more completely and improve the previous existing results.

The problem on the monotonicity and critical periods of the period functions associated with periodic wave has been studied by many (see, for instance [9–14]). Their proofs are mainly based on Picard–Fuchs equation for algebraic curves or discriminant of a polynomial and Sturm's theorem. These techniques are powerful to deal with the plane dynamical system

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}\xi} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = f(u, y), \end{cases} \tag{2}$$

when $f(u, y)$ is a polynomial with respect to u and y. However, if $f(u, y)$ is not a polynomial, one needs new methods to study the properties of the period function of system (2). In this paper, we focus on the behaviour of period function of system (4) which is reduced from (1). Considering that (4) is not a polynomial dynamical system, the period function of the periodic travelling wave solutions for (1) are investigated mainly by using

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Chicone's monotonicity criterion [15] and asymptotic expansion techniques.

2. Monotonicity of period functions associated with periodic waves

Noticeably, a travelling wave solution of a partial differential equation (PDE) is in the form of $u(x,t) = u(x \pm ct) = u(\xi)$, where c is the wave speed. Usually, $u(\xi)$ satisfies an ordinary differential equation (ODE), i.e. the corresponding travelling wave equation of the PDE. By the existence and uniqueness theorems of ODE theory, every phase orbit of an ODE travelling wave equation gives rise to a travelling wave solution of the corresponding PDE, and different phase orbits correspond to different travelling solutions.

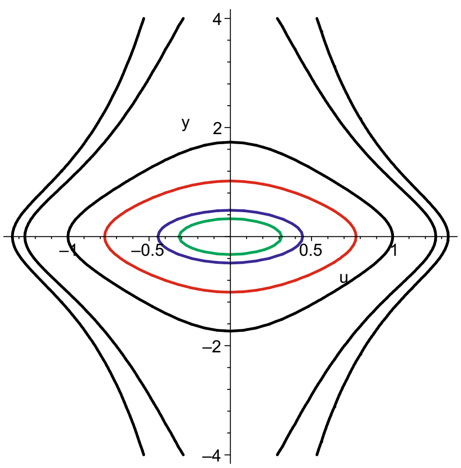

Figure 1. Phase portrait of the planer dynamical system (4)

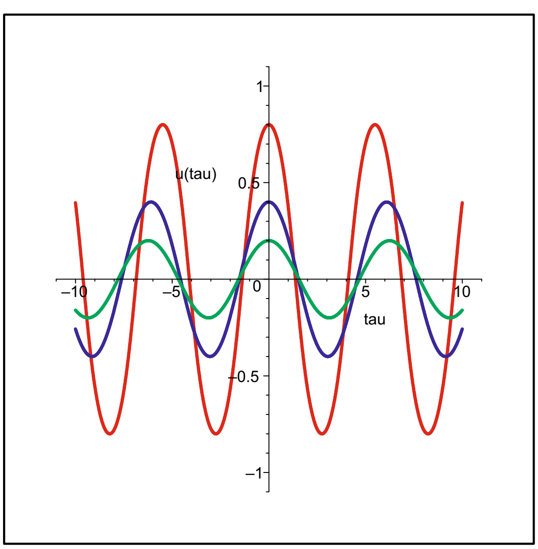

Figure 2. Periodic solutions of the planer dynamical system (4)

By substituting $u(x,t) = u(x+ct) = u(\xi)$ into eq. (1), integrating the resulting equation twice, and then setting the integration constants to 0, one obtains

$$(c^2-1)u + \frac{2\epsilon u''}{\left(1+(u')^2\right)^{\frac{3}{2}}} = 0, \tag{3}$$

which is equivalent to the following planar dynamical system:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}\xi} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = -\frac{1}{2}Au(1+y^2)^{\frac{3}{2}}, \end{cases} \tag{4}$$

where $A = (c^2-1)/\epsilon$. This system has the first integral

$$H(u,y) = \frac{1}{2}u^2 - \frac{2}{A\sqrt{1+y^2}} = h. \tag{5}$$

It is easy to see that system (4) has a unique equilibrium at (0, 0). When $A > 0$, it is a centre, see figure 1.

Moreover, the analytical expressions of periodic solutions of the dynamical system (4) is available and it has been obtained explicitly in [4, eq. (10)]. Figure 2 shows the picture of periodic solutions by numerical simulations.

Now, consider a level curve of the form $H(u,y) = h$. It corresponds to a periodic orbit surrounding the centre (0, 0) if h satisfies $-\frac{2}{A} < h < 0$ (see figure 1). The period of the periodic orbit with energy h is given by

$$T(h) = 2\int_{u_1}^{u_2} \frac{\mathrm{d}u}{y}, \tag{6}$$

where u_1 and u_2 denote the left and right intersections of the curve given by $H(u,y) = h$ with the u-axis. The explicit expression of the period function $T(h)$ is difficult to be obtained by (6) directly. Numerical simulation shows that $T(h)$ is a monotonically decreasing function on $(-\frac{2}{A}, 0)$ (see figure 3).

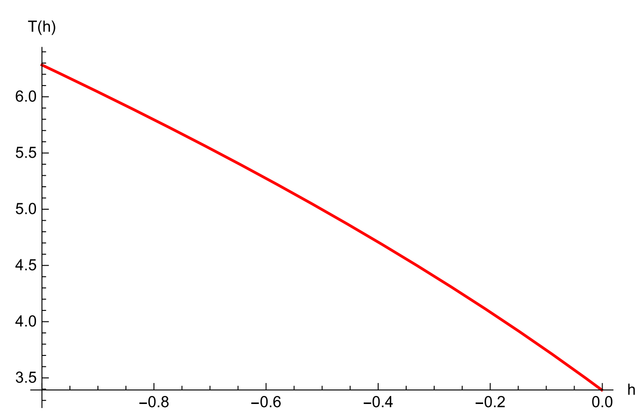

Figure 3. Graph of the period function $T(h)$ when $A = 2$

Next, we prove the monotonicity of the period function $T(h)$ analytically.

Theorem 1. *If $A > 0$, then the period function $T(h)$ is a monotonically decreasing function on $(-\frac{2}{A}, 0)$.*

Proof. Using the transformation

$$p = \frac{y}{\sqrt{1+y^2}}, \quad q = -\frac{1}{2}Au,$$

we can transform system (4) to

$$\begin{cases} \frac{\mathrm{d}p}{\mathrm{d}\xi} = q, \\ \frac{\mathrm{d}q}{\mathrm{d}\xi} = -\frac{Ap}{2\sqrt{1-p^2}}, \end{cases} \tag{7}$$

which is in the Hamiltonian form with Hamiltonian

$$H(p,q) = \frac{1}{2}q^2 + V(p), \tag{8}$$

where $V(p) = -\frac{A}{2}\sqrt{1-p^2}$. To prove that $T'(h) < 0$, we use a monotonicity criterion by Chicone [15] for planar systems with Hamiltonian of the form (8), where V is a smooth function on (p_1, p_2) with a non-degenerate relative minimum at the origin. The period function $T(h)$ is monotonically decreasing if the function

$$W(p) := \frac{V(p)}{(V'(p))^2}$$

is concave in (p_1, p_2). Hence, we only need to prove that $W''(p) < 0$ for every $p \in (p_1, p_2)$. A straightforward computation shows that

$$W''(p) = \frac{6(p^2-2)}{Ap^4\sqrt{1-p^2}} < 0.$$

This completes the proof of Theorem 1. □

3. Asymptotic analysis of period functions

In this section, we investigate the asymptotic behaviour of the period function T as the amplitude of the periodic solution goes to zero first. More precisely, let ρ denote the semi-height of the periodic wave, i.e., $\rho = u_2$, then we have the following theorem.

Theorem 2. *If $A > 0$, when ρ is small enough, the period function T has the following asymptotic behaviour:*

$$T = 2\sqrt{\frac{2}{A}}\pi\left(1 - \frac{3A}{32}\rho^2 + O(\rho^3)\right). \tag{9}$$

Proof. Recall that system (4) has a first integral

$$h = \frac{1}{2}u^2 - \frac{2}{A\sqrt{1+y^2}}, \tag{10}$$

at $u = u_2 = \rho$, $y = 0$, which yields

$$h = \frac{1}{2}\rho^2 - \frac{2}{A}. \tag{11}$$

Noting that $y|_{u_1} = 0$, one can get u_1 from the following equation:

$$\frac{1}{2}\rho^2 - \frac{2}{A} = \frac{1}{2}u_1^2 - \frac{2}{A} \tag{12}$$

and the result is

$$u_1 = -\rho. \tag{13}$$

From (10) it can be deduced that

$$y^2 = \frac{4 - A^2(\frac{1}{2}u^2 - h)^2}{A^2(\frac{1}{2}u^2-h)^2}. \tag{14}$$

Let

$$F(u) = \frac{4 - A^2(\frac{1}{2}u^2 - h)^2}{A^2(\frac{1}{2}u^2-h)^2}. \tag{15}$$

Then the period of the periodic orbit with energy $h = \frac{1}{2}\rho^2 - \frac{2}{A}$ reads as

$$T = 2\int_{-\rho}^{\rho} \frac{\mathrm{d}u}{\sqrt{F(u)}}. \tag{16}$$

To obtain the asymptotic behaviour of T as ρ goes to zero, we introduce a new variable z by

$$u = \rho z. \tag{17}$$

Then, period T becomes

$$T = \int_{-1}^{1} \frac{2\rho\,\mathrm{d}z}{\sqrt{F(z,\rho)}}. \tag{18}$$

For small ρ, combining (15), (11) and (17), one can find the asymptotic behaviour of $\rho/\sqrt{F(z,\rho)}$ as

$$\frac{\rho}{\sqrt{F(z,\rho)}} = \sqrt{\frac{2}{A}}\left(\frac{1}{\sqrt{1-z^2}} - \frac{3A}{16}\sqrt{1-z^2}\rho^2 + O(\rho^3)\right). \tag{19}$$

Therefore, we get

$$T = 2\sqrt{\frac{2}{A}}\pi\left(1 - \frac{3A}{32}\rho^2 + O(\rho^3)\right). \tag{20}$$

This completes the proof of Theorem 2. □

Noting that $\rho \to 0$ as $h \to -\frac{2}{A}$, Theorem 1 yields the following result.

COROLLARY 1

The limit of the period function $T(h)$ as $h \to -2/A$ is

$$\lim_{h\to -2/A} T(h) = 2\sqrt{\frac{2}{A}}\pi. \tag{21}$$

Finally, we study the asymptotic behaviour of $T(h)$ as $h \to 0$.

Theorem 3. *If $A > 0$, when h is small enough, the period function T has the following asymptotic behaviour:*

$$T = 2\left(\frac{4\sqrt{\pi}}{\sqrt{A}}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} - \sqrt{A\pi}\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}h + O(h^2)\right), \tag{22}$$

where $\Gamma(x)$ is the Gamma function.

Proof. By the definition of u_1 and u_2, (5) gives

$$u_1 = -\sqrt{2h + \frac{4}{A}},\ u_2 = \sqrt{2h + \frac{4}{A}}. \tag{23}$$

Then the period function $T(h)$ becomes

$$T(h) = 2\int_{-\sqrt{2h+\frac{4}{A}}}^{\sqrt{2h+\frac{4}{A}}} \frac{\mathrm{d}u}{\sqrt{F(u)}}. \tag{24}$$

Similarly, with the proof of Theorem 2, we introduce a variable x by

$$u = \sqrt{2h + \frac{4}{A}}x. \tag{25}$$

Then we find

$$T(h) = 2\int_{-1}^{1} \frac{\sqrt{2h + \frac{4}{A}}}{\sqrt{F(x,h)}}\mathrm{d}x. \tag{26}$$

For small values of h, one can get

$$\frac{\sqrt{2h + \frac{4}{A}}}{\sqrt{F(x,h)}} = \frac{2x^2}{\sqrt{A}\sqrt{1-x^4}} - \frac{(2-x^2-x^4)\sqrt{A}}{2(1+x^2)\sqrt{1-x^4}}h + O(h^2) \tag{27}$$

which yields

$$T = 2\left(\frac{4\sqrt{\pi}}{\sqrt{A}}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} - \sqrt{A\pi}\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}h + O(h^2)\right). \tag{28}$$

This completes the proof of Theorem 3. □

Taking the limit $h \to 0$ in Theorem 3, we have

COROLLARY 2

$$\lim_{h\to 0} T(h) = \frac{8\sqrt{\pi}}{\sqrt{A}}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}. \tag{29}$$

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