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Studies in scattering using Manning–Rosen-modified separable potential in all partial waves

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Abstract. The aim of this paper is to find the exact solutions of the Schrödinger equation for the Manning–Rosen plus Graz separable potential through two different approaches to the problem. We express the irregular/Jost and physical solutions in terms of the special functions of mathematical physics. Numerical results of the phase shifts are obtained by utilising the properties of the Jost function and Fredholm determinants for nucleon–nucleon and nucleon–nucleus systems. The results obtained are in good agreement with earlier works.

Keywords. Manning–Rosen-modified potential; direct integration; differential equation approach; Jost solution and function; p–p and p–d systems.

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1. Introduction

Finding the exact solutions of the wave equations (non-relativistic and relativistic) for given physical potentials is an important task in quantum mechanics as these solutions contain all relevant information about the quantum systems under investigation. In this regard, considerable progress has been made in the past few decades in developing exact analytical solutions to the Schrödinger equation with interesting potentials. The nucleon–nucleon interaction is fundamental to the understanding of nuclear physics as a whole. The scattering of charged hadron systems is caused by additive interactions. In such a case, scattering occurs due to the combined influence of two potentials, one due to electric charges and the other due to nuclear forces. In a mathematically well-defined sense, a short-range local potential can be represented by a finite rank separable non-local potential. As a result, experiments involving scattering by additive interactions are analysed using a non-local separable model, which makes it easier to develop closed-form analytical expressions for physical observables. Various groups have examined the problems with Coulomb or Hulthén plus rational separable potentials in detail [1–12]. However, the Manning–Rosen plus separable interaction with $\ell > 0$ problem has yet to be addressed in the literature. We use the Manning–Rosen plus Graz potential to build the solutions for wave functions, Jost function and the Fredholm determinants. The Manning–Rosen potential is exactly solvable for s-waves only. Most quantum systems could only be solved by approximation approaches for an arbitrary ℓ state. However, the problem with the Manning–Rosen potential having $\ell > 0$ has been advocated with some approximation techniques and published in a number of publications [13–19]. The Manning–Rosen potential is frequently used as a model for screened and cut-off Coulomb interactions and generally used in molecular dynamics. As the effect of screening should always affect the theory and interpretations of results about charged hadron scattering, it is expected that the analysis in this article is favourable to a broad range of physicists.

In this context, we use the Manning–Rosen potential as a short-range electromagnetic interaction and a non-local Graz separable potential as the nuclear part. We construct the expressions for the scattering states of the wave equation involving two potentials by adapting two different approaches to the problem, one is the direct integration approach and other is the ordinary differential equation approach for various boundary conditions. We investigate the usefulness of our expressions by examining bound and scattering state observables for

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some nuclear systems by exploiting the associated Fredholm determinants.

The contents of this paper is as follows: In §2, we derive exact analytical solutions of the Schrödinger equation related to direct integration and ordinary differential equation approach for associated boundary conditions. In §3 we discuss our results. Finally, a brief conclusion is presented in §4.

2. Solutions of the Schrödinger equation

2.1 *Jost solution*

2.1.1 *Direct integration approach.* The irregular/Jost solution $f_\ell^{MG}(\chi, s)$ for the Manning–Rosen plus Graz separable potential [20–23] satisfy the inhomogeneous differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \chi^2 - V_M(s)\right] f_\ell^{MG}(\chi, s) = \lambda_\ell\, d_\ell(\beta_\ell, \chi)\, g_\ell(\beta_\ell, s), \quad (1)$$

where for all partial waves the Manning–Rosen potential can be written as

$$V_M(s) = b^{-2}\left[\delta(\delta-1)\frac{\exp(-2s/b)}{[1-\exp(-s/b)]^2} - A\frac{\exp(-s/b)}{[1-\exp(-s/b)]}\right], \quad (2)$$

with

$$\delta = \frac{1}{2}\left[1 \pm \sqrt{1 + 4\{\alpha(\alpha-1) + \ell(\ell+1\}}\right] \quad (3)$$

and ℓ takes the values 0, 1, 2, 3, ... Here the centrifugal barrier term is considered as $b^{-2}\frac{\ell(\ell+1)\exp(-2s/b)}{[1-\exp(-s/b)]^2}$, for small values of s which behaves as $\frac{\ell(\ell+1)}{s^2}$. The quantities A, b and α are three adjustable parameters of the Manning–Rosen potential of which A and α are dimensionless and b possesses the dimension of length.

The quantity $d_\ell(\beta_\ell, \chi)$ is expressed as

$$d_\ell(\beta_\ell, \chi) = \int_0^\infty \mathrm{d}s'\, g_\ell(\beta_\ell, s') f_\ell^{MG}(\chi, s'), \quad (4)$$

where $g_\ell(\beta_\ell, s)$ is the form factor of the Graz separable potential with the quantities λ_ℓ and β_ℓ related to the Graz potential denote strength and inverse range parameters and is given by

$$g_\ell(\beta_\ell, s) = 2^{-\ell}(\ell!)^{-1} s^\ell \mathrm{e}^{-\beta_\ell s}. \quad (5)$$

The irregular solution of eq. (1) is expressed as

$$f_\ell^{MG}(\chi, s) = f_\ell^M(\chi, s) + \lambda_\ell d_\ell(\beta_\ell, \chi) \times \int_s^\infty G_\ell^{M(I)}(s, s') g_\ell(\beta_\ell, s')\mathrm{d}s', \quad (6)$$

where $G_\ell^{M(I)}(s, s')$, the irregular Green's function for Manning– Rosen potential [18,24], is as follows:

$$G_\ell^{M(I)}(s, s') = \frac{1}{f_\ell^M(\chi)}\left[\phi_\ell^M(\chi, s') f_\ell^M(\chi, s) - \phi_\ell^M(\chi, s) f_\ell^M(\chi, s')\right]. \quad (7)$$

Here [18]

$$\phi_\ell^M(\chi, s) = b^{\omega+1}\left[1 - \exp(-s/b)\right]^{\omega+1} \exp(\mathrm{i}\chi s) \times {}_2F_1(M', N'; P'; 1 - \exp(-s/b)), \quad (8)$$

$$f_\ell^M(\chi, s) = \left[1 - \exp(-s/b)\right]^{-\omega} \exp(\mathrm{i}\chi s) \times {}_2F_1(M' - 1 - 2\omega, N' - 1 - 2\omega; 1 - 2\mathrm{i}\chi b; \exp(-s/b)) \quad (9)$$

and

$$f_\ell^M(\chi) = b^\omega \frac{\Gamma(2+2\omega)\Gamma(1-2\mathrm{i}\chi b)}{\Gamma(M')\Gamma(N')} \quad (10)$$

with

$$M' = 1 + \omega - \mathrm{i}\chi b + (\omega^2 - \chi^2 b^2 + \omega + A)^{1/2}, \quad (11)$$

$$N' = 1 + \omega - \mathrm{i}\chi b - (\omega^2 - \chi^2 b^2 + \omega + A)^{1/2} \quad (12)$$

and

$$P' = 2\omega + 2. \quad (13)$$

The quantity $\omega = \delta - 1$. Multiplying by $g_\ell(\beta_\ell, s)$ and integrating over the whole range on both sides of eq. (6), $d_\ell(\beta_\ell, \chi)$ is simplified to the form

$$d_\ell(\beta_\ell, \chi) = \frac{1}{D_\ell^{MG}(\chi)} \int_0^\infty \mathrm{d}s\, g_\ell(\beta_\ell, s)\, f_\ell^M(\chi, s), \quad (14)$$

where $D_\ell^{MG}(\chi)$, the Fredholm determinant associated with the regular boundary condition reads as

$$D_\ell^{MG}(\chi) = 1 - \lambda_\ell \int_0^\infty \int_0^s g_\ell(\beta_\ell, s) \times G_\ell^{M(R)}(s, s')\, g_\ell(\beta_\ell, s')\, \mathrm{d}s\, \mathrm{d}s', \quad (15)$$

with [18,24]

$$G_\ell^{M(R)}(s, s') = \frac{1}{f_\ell^M(\chi)}\left[\phi_\ell^M(\chi, s) f_\ell^M(\chi, s') - \phi_\ell^M(\chi, s') f_\ell^M(\chi, s)\right]. \quad (16)$$

Utilizing eq. (14), eq. (6) becomes

$$f_\ell^{MG}(\chi, s) = f_\ell^M(\chi, s) + \lambda_\ell\left(\frac{W_\ell(\beta_\ell, \chi)}{D_\ell^{MG}(\chi)}\right) T_\ell(\beta_\ell, \chi, s), \quad (17)$$

where

$$W_\ell(\beta_\ell,\chi)=\int_0^\infty \mathrm{d}s\, g_\ell(\beta_\ell,s)\, f_\ell^M(\chi,s) \tag{18}$$

and

$$T_\ell(\beta_\ell,\chi,s)=\int_s^\infty \mathrm{d}s'\, g_\ell(\beta_\ell,s')\, G_\ell^{M(I)}(s,s'). \tag{19}$$

To calculate the integral in the above equations, we approximate the form factor of the Graz [25] potential as

$$g_\ell(\beta_\ell,s)=2^{-\ell}(\ell!)^{-1}b^\ell\left[1-\exp(-s/b)\right]^\ell \mathrm{e}^{-\beta_\ell s}. \tag{20}$$

To calculate the definite integral $W_\ell(\beta_\ell,\chi)$ in eq. (18), we substitute $z=\exp(-s/b)$ and one must take recourse to the following standard integral and relations of the Gaussian hyper geometric function [26–30]:

$$\int_0^s z^{\rho-1}(s-z)^{\sigma-1}{}_2F_1(\alpha,\beta;\gamma;cz)\mathrm{d}z = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}s^{\rho+\sigma-1}{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;cs) \tag{21}$$

with Re $\sigma>0$, Re $\rho>0$, $\mathrm{Re}(\gamma+\sigma-\alpha-\beta)>0$. Hence, the value of $W_\ell(\beta_\ell,\chi)$ can be expressed as

$$\begin{aligned}W_\ell(\beta_\ell,\chi)
&=2^{-\ell}(\ell!)^{-1}b^{\ell+1}\frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)}\\
&\quad\times{}_3F_2(M'-1-2\omega,N'-1-2\omega,(\beta_\ell-i\chi)b;\\
&\quad 1-2\mathrm{i}\chi b,\ell-\omega+1+(\beta_\ell-i\chi)b;1).\end{aligned} \tag{22}$$

Equation (19) may be rewritten as

$$T_\ell(\beta_\ell,\chi,s)=I_1(\beta_\ell,\chi,s)-I_2(\beta_\ell,\chi,s) \tag{23}$$

with

$$I_1(\beta_\ell,\chi,s)=\int_0^\infty \mathrm{d}s'\, g_\ell(\beta_\ell,s')\, G_\ell^{M(I)}(s,s') \tag{24}$$

and

$$I_2(\beta_\ell,\chi,s)=\int_0^s \mathrm{d}s'\, g_\ell(\beta_\ell,s')\, G_\ell^{M(I)}(s,s'). \tag{25}$$

Utilising eqs (7), (20) and (21) the value of $I_1(\beta_\ell,\chi,s)$ can be expressed as

$$\begin{aligned}I_1(\beta_\ell,\chi,s)
&=-\frac{2^{-\ell}(\ell!)^{-1}b^{\ell+\omega+2}\mathrm{e}^{\mathrm{i}\chi s}}{f_\ell^M(\chi)}\\
&\quad\times\big[(1-\exp(-s/b))^{\omega+1}\\
&\quad\times{}_2F_1(M',N';P';1-\exp(-s/b))\\
&\quad\times\frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)}\\
&\quad\times{}_3F_2(M'-1-2\omega,N'-1-2\omega,(\beta_\ell-\mathrm{i}\chi)b;\\
&\quad 1-2\mathrm{i}\chi b,\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b;1)\\
&\quad-\left[1-\exp(-s/b)\right]^{-\omega}\\
&\quad\times{}_2F_1(M'-1-2\omega,N'\\
&\quad-1-2\omega;1-2\mathrm{i}\chi b;\exp(-s/b))\\
&\quad\times\frac{\Gamma(\ell+\omega+2)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell+\omega+2+(\beta_\ell-\mathrm{i}\chi)b)}\\
&\quad\times{}_3F_2(M',N',\ell+\omega+2;\\
&\quad P',\ell+\omega+2+(\beta_\ell-\mathrm{i}\chi)b;1)\big].\end{aligned} \tag{26}$$

From eqs (7)–(13), the irregular Green's function for the Manning–Rosen potential becomes

$$\begin{aligned}G_\ell^{M(I)}(s,s')
&=-\frac{1}{f_\ell^M(\chi)}\big[b^{\omega+1}\left[1-\exp(-s/b)\right]^{\omega+1}\exp(\mathrm{i}\chi s)\\
&\quad\times{}_2F_1(M',N';P';1-\exp(-s/b))\\
&\quad\times\left[1-\exp(-s'/b)\right]^{-\omega}\exp(\mathrm{i}\chi s')\\
&\quad\times{}_2F_1(M'-1-2\omega,N'-1-2\omega;\\
&\quad 1-2\mathrm{i}\chi b;\exp(-s'/b))\\
&\quad-b^{\omega+1}\left[1-\exp(-s'/b)\right]^{\omega+1}\exp(\mathrm{i}\chi s')\\
&\quad\times{}_2F_1(M',N';P';1-\exp(-s'/b))\\
&\quad\times\left[1-\exp(-s/b)\right]^{-\omega}\exp(\mathrm{i}\chi s)\\
&\quad\times{}_2F_1(M'-1-2\omega,N'-1-2\omega;\\
&\quad 1-2\mathrm{i}\chi b;\exp(-s/b))\big].\end{aligned} \tag{27}$$

Substituting eqs (20) and (27) in eq. (25), we get

$$\begin{aligned}I_2(\beta_\ell,\chi,s)
&=-\frac{2^{-\ell}(\ell!)^{-1}b^{\ell+\omega+1}\mathrm{e}^{\mathrm{i}\chi s}}{f_\ell^M(\chi)}\\
&\quad\times\big[(1-\exp(-s/b))^{\omega+1}\\
&\quad\times{}_2F_1(M',N';P';1-\exp(-s/b))\\
&\quad\times\int_0^s\left[1-\exp(-s'/b)\right]^{\ell-\omega}\mathrm{e}^{-(\beta_\ell-\mathrm{i}\chi)s'}\\
&\quad\times{}_2F_1(M'-1-2\omega,N'\\
&\quad-1-2\omega;1-2\mathrm{i}\chi b;\mathrm{e}^{-s'/b})\,\mathrm{d}s'\\
&\quad-\left[1-\exp(-s/b)\right]^{-\omega}\exp(\mathrm{i}\chi s)\\
&\quad\times{}_2F_1(M'-1-2\omega,N'\\
&\quad-1-2\omega;1-2\mathrm{i}\chi b;\exp(-s/b))\\
&\quad\times\int_0^s\left[1-\exp(-s'/b)\right]^{\ell+\omega+1}\mathrm{e}^{-(\beta_\ell-\mathrm{i}\chi)s'}\\
&\quad\times{}_2F_1(M',N';P';1-\exp(-s'/b))\,\mathrm{d}s'\big].\end{aligned} \tag{28}$$

Now applying the following analytic continuation of the Gaussian hypergeometric function [26–30] ${}_2F_1(M'-1-2\omega,N'-1-2\omega;1-2\mathrm{i}\chi b;\mathrm{e}^{-s'/b})$ present in the above equation

$$
{}_2F_1(P_1, Q_1; R_1; Z) = \frac{\Gamma(R_1)\Gamma(R_1-P_1-Q_1)}{\Gamma(R_1-P_1)\Gamma(R_1-Q_1)} \times {}_2F_1(P_1, Q_1; P_1+Q_1-R_1+1; 1-Z) + (1-Z)^{R_1-P_1-Q_1}\frac{\Gamma(R_1)\Gamma(P_1+Q_1-R_1)}{\Gamma(P_1)\Gamma(Q_1)} \times {}_2F_1(R_1-P_1, R_1 - Q_1; R_1-P_1-Q_1+1; 1-Z). \quad (29)
$$

Equation (28) simplifies to

$$
I_2(\beta_\ell, \chi, s) = -\frac{2^{-\ell}(\ell!)^{-1}b^{\ell+\omega+2}\mathrm{e}^{\mathrm{i}\chi s}}{f_\ell^M(\chi)}\frac{\Gamma(1+2\omega)\Gamma(1-2\mathrm{i}\chi b)}{\Gamma(M')\Gamma(N')} \times (1-\exp(-s/b))^{\omega+1}\Big[{}_2F_1(M', N'; P'; 1-\mathrm{e}^{-s/b}) \times \int_0^s \left[1-\exp(-s'/b)\right]^{\ell-\omega}\mathrm{e}^{-(\beta_\ell-\mathrm{i}\chi)s'}{}_2F_1(M'-1 -2\omega, N'-1-2\omega; 1-2\mathrm{i}\chi b; \mathrm{e}^{-s'/b})\,\mathrm{d}s' - \left[1-\exp(-s/b)\right]^{-2\omega-1}{}_2F_1(M'-1-2\omega, N'-1 -2\omega; 1-2\mathrm{i}\chi b; \mathrm{e}^{-s/b}) \times \int_0^s \left[1-\exp(-s'/b)\right]^{\ell+\omega+1}\mathrm{e}^{-(\beta_\ell-\mathrm{i}\chi)s'} \times {}_2F_1(M', N'; P'; 1-\exp(-s'/b))\,\mathrm{d}s'\Big]. \quad (30)
$$

The above equation can be evaluated by using the following standard integral of the non-homogeneous Gaussian hypergeometric function [29]:

$$
f_\sigma(d, e; f; z) = \frac{1}{f-1}\Big[{}_2F_1(d, e; f; z)\int_0^z s^{\sigma-1} \times (1-s)^{d+e-f}{}_2F_1(d-f+1, e-f +1; 2-f; s)\mathrm{d}s - z^{1-f}{}_2F_1(d-e+1, e-f+1; 2-f; z) \times \int_0^z s^{\sigma+f-2}(1-s)^{d+e-f}{}_2F_1(d, e; f; s)\mathrm{d}s\Big]. \quad (31)
$$

Hence, eq. (30) becomes

$$
I_2(\beta_\ell, \chi, s) = -2^{-\ell}(\ell!)^{-1}b^{\ell+2}\mathrm{e}^{\mathrm{i}\chi s} \times (1-\exp(-s/b))^{\omega+1} \times \sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!} \times f_{n+1+\ell-\omega}(M', N'; P'; 1-\mathrm{e}^{-s/b}). \quad (32)
$$

Equation (15) in conjunction with eq. (24) and use the following standard integral relation [29]

$$
\int_0^s z^{\delta-1}(s-z)^{\mu-1}f_\sigma(d, e; f; az)\mathrm{d}z = \frac{\Gamma(\delta+\sigma)\Gamma(\mu)}{\sigma(\sigma+f-1)\Gamma(\delta+\sigma+\mu)}a^\sigma s^{\delta+\sigma+\mu-1} \times {}_4F_3(1, \sigma+d, \sigma+e, \delta+\sigma; \sigma+1, \sigma+f, \delta+\sigma+\mu; as) \quad (33)
$$

with $\mathrm{Re}\,\sigma > 0$, $\mathrm{Re}\,\sigma + a > 1$, $\mathrm{Re}\,\mu > 0$, $\mathrm{Re}\,\delta > 0$, $|as| < 1$, the value of $D_\ell^{MG}(\chi)$ can be expressed as

$$
D_\ell^{MG}(\chi) = 1 - \lambda_\ell 2^{-2\ell}(\ell!)^{-2}b^{2\ell+3}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!} \times \frac{\Gamma(n+2\ell+3)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{(n+\ell-\omega+1)(n+\ell+\omega+2)\Gamma(n+2\ell+3+(\beta_\ell-\mathrm{i}\chi)b)}{}_4F_3(1, n+\ell-\omega+1+M', n+\ell -\omega+1+N', n+2\ell+3; n+\ell-\omega+2, n+\ell+\omega+3, n+2\ell+3+(\beta_\ell-\mathrm{i}\chi)b; 1). \quad (34)
$$

Hence, substituting eqs (22), (23), (26), (32) and (34) in eq. (17), one obtains the complete expression for the required irregular solution of eq. (1) as

$$
f_\ell^{MG}(\chi, s) = f_\ell^M(\chi, s) + \lambda_\ell 2^{-\ell}(\ell!)^{-1}b^{\ell+1} \times \frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{D_\ell^{MG}(\chi)\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)} \times {}_3F_2(M'-1-2\omega, N'-1-2\omega, (\beta_\ell-\mathrm{i}\chi)b; 1-2\mathrm{i}\chi b, \ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b; 1) \times \Big\{-\frac{2^{-\ell}(\ell!)^{-1}b^{\ell+\omega+2}\mathrm{e}^{\mathrm{i}\chi s}}{f_\ell^M(\chi)}\Big[(1-\exp(-s/b))^{\omega+1} \times {}_2F_1(M', N'; P'; 1-\exp(-s/b)) \times \frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)} \times {}_3F_2(M'-1-2\omega, N' -1-2\omega, (\beta_\ell-\mathrm{i}\chi)b; 1-2\mathrm{i}\chi b, \ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b; 1) - \left[1-\exp(-s/b)\right]^{-\omega} \times {}_2F_1(M'-1-2\omega, N'-1-2\omega;
$$

$$1-2\mathrm{i}\chi b;\exp(-s/b)) \times\frac{\Gamma(\ell+\omega+2)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell+\omega+2+(\beta_\ell-\mathrm{i}\chi)b)} \times{}_3F_2(M',N',\ell+\omega+2; P',\ell+\omega+2+(\beta_\ell-\mathrm{i}\chi)b;1)\Big] +2^{-\ell}(\ell!)^{-1}b^{\ell+2}\mathrm{e}^{\mathrm{i}\chi s}(1-\exp(-s/b))^{\omega+1} \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!} \times f_{n+1+\ell-\omega}(M',N';P';1-\mathrm{e}^{-s/b})\Big\}. \quad (35)$$

It is well known that near the origin, the behaviour of the Jost/irregular solution $f_\ell^{MG}(\chi,s)$ represents the Jost function $f_\ell^{MG}(\chi)$ [18] as

$$f_\ell^{MG}(\chi)=(2\omega+1)b^\omega \lim_{s\to0}\left[1-\exp(-s/b)\right]^\omega f_\ell^{MG}(\chi,s). \quad (36)$$

Thus, from eq. (35) the Jost function reads as

$$f_\ell^{MG}(\chi) = f_\ell^M(\chi)+\lambda_\ell 2^{-2\ell}(\ell!)^{-2}b^{2\ell+3+\omega} \times\frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{D_\ell^{MG}(\chi)\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)} \times{}_3F_2(M'-1-2\omega,N'-1-2\omega,(\beta_\ell-\mathrm{i}\chi)b; 1-2\mathrm{i}\chi b,\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b;1) \times\frac{\Gamma(\ell+\omega+2)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell+\omega+2+(\beta_\ell-\mathrm{i}\chi)b)} \times{}_3F_2(M',N',\ell+\omega+2; P',\ell+\omega+2+(\beta_\ell-\mathrm{i}\chi)b;1). \quad (37)$$

2.1.2 *Ordinary differential equation approach.* Applying the following transformation for the irregular solution

$$f_\ell^{MG}(\chi,s) = b^{\omega+1}\left[1-\exp(-s/b)\right]^{\omega+1}\mathrm{e}^{\mathrm{i}\chi s}\eta_\ell(\chi,s) \quad (38)$$

in eq. (1) we get

$$\exp(s/b)b^2(1-\exp(-s/b))\frac{\mathrm{d}^2\eta_\ell}{\mathrm{d}s^2} +\left[2(\omega+1)+2\mathrm{i}\chi b^2\exp(s/b)(1-\exp(-s/b))\right] \times\frac{\mathrm{d}\eta_\ell}{\mathrm{d}s}+\big(2\mathrm{i}\chi b(\omega+1)-(\omega+1)+A\big)\eta_\ell(\chi,s) =\lambda_\ell 2^{-\ell}(\ell!)^{-1}b^{\ell-\omega+1}\exp(s/b) \times\left[1-\exp(-s/b)\right]^{\ell-\omega}\mathrm{e}^{-(\beta_\ell+\mathrm{i}\chi)s}d_\ell(\beta_\ell,\chi). \quad (39)$$

The transformation in eq. (38) is identified by considering the behaviour of eq. (1) as $s\to0$ and $s\to\infty$. In the limit $s\to0$ the solution behaves as $s^{\omega+1}$ and asymptotically it goes as $\mathrm{e}^{\mathrm{i}\chi s}$. The quantity $s^{\omega+1}$ is approximated by $b^{\omega+1}\left[1-\exp(-s/b)\right]^{\omega+1}$ for simplicity of calculation and to achieve an analytical solution. Change of variable $z=(1-\exp(-s/b))$ in eq. (39) yields

$$z(1-z)\frac{\mathrm{d}^2\eta_\ell}{\mathrm{d}z^2}+\left[2(\omega+1)-1(1+2(\omega+1)-2\mathrm{i}\chi b)z\right] \times\frac{\mathrm{d}\eta_\ell}{\mathrm{d}z}-\big(-2\mathrm{i}\chi b(\omega+1)+(\omega+1)-A\big)\eta_\ell(\chi,z) =\lambda_\ell 2^{-\ell}(\ell!)^{-1}b^{\ell-\omega+1}z^{\ell-\omega}(1-z)^{(\beta_\ell+\mathrm{i}\chi)b-1}d_\ell(\beta_\ell,\chi). \quad (40)$$

Comparing eq. (40) with the non-homogeneous Gaussian hypergeometric equation [29,30]

$$z(1-z)\frac{\mathrm{d}^2\eta_\ell}{\mathrm{d}z^2}+\left\{P'-(1+M'+N')z\right\}\frac{\mathrm{d}\eta_\ell}{\mathrm{d}z} -M'N'\eta_\ell=z^{\sigma-1}(1-\rho z)^{\tau-1}, \quad (41)$$

where M', N', P', ρ, τ and σ are constants. Equation (41) resembles the inhomogeneous Gaussian hypergeometric equation [29] and its complementary functions, namely $u_1(z)$ and $u_2(z)$, read as

$$u_1(z)={}_2F_1(M',N';P';z);\ P'>0 \quad (42)$$

and

$$u_2(z)={}_2F_1(M',N';M'+N'-P'+1;1-z); M'+N'-P'+1\neq0,-1,\ldots. \quad (43)$$

The particular solution [29] of eq. (41) is written as

$$F_P(z)=\lambda_\ell d_\ell(\beta_\ell,\chi)2^{-\ell}(\ell!)^{-1}b^{1-\omega+\ell} \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!} \times f_{n+1+\ell-\omega}(M',N';P';z) \quad (44)$$

with

$$f_n(a,b;c;z) = z^n \sum_{j=0}^{\infty} \frac{\Gamma(n+a+j)\Gamma(n+b+j)\Gamma(n)\Gamma(n+c-1)}{\Gamma(n+a)\Gamma(n+b)\Gamma(n+j+1)\Gamma(n+c+j)} z^j$$
$$= \frac{z^n}{n(n+c-1)} {}_3F_2(1, n+a, n+b; n+1, n+c; z). \quad (45)$$

Equation (38) together with eqs (42)–(44) yields

$$\begin{aligned} & f_\ell^{MG}(\chi, s) \\ & = b^{\omega+1}\left[1-\exp(-s/b)\right]^{\omega+1}\exp(\mathrm{i}\chi s) \\ & \times\Bigg[D_1\, {}_2F_1(M', N'; P'; 1-\exp(-s/b)) \\ & +D_2\, (1-\exp(-s/b))^{-2\omega-1} \\ & \times {}_2F_1(M'-1-2\omega, \\ & N'-1-2\omega; 1-2\mathrm{i}\chi b; \exp(-s/b)) \\ & +\lambda_\ell d_\ell(\beta_\ell,\chi)2^{-\ell}(\ell!)^{-1}b^{1-\omega+\ell} \\ & \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!} \\ & \times f_{n+1+\ell-\omega}(M', N'; P'; 1-\exp(-s/b))\Bigg]. \end{aligned} \quad (46)$$

The two unknown constants D_1 and D_2 will be calculated by applying the boundary conditions. As $s \to 0$, the solution $f_\ell^{MG}(\chi, s)$ produces the Jost function $f_\ell^{MG}(\chi)$ for the Manning–Rosen-modified Graz potential. For $s = 0$, eqs (10), (36) and (46) give

$$D_2 = \frac{f_\ell^{MG}(\chi)}{b^{\omega+1} f_\ell^M(\chi)}. \quad (47)$$

To calculate the limit $s \to \infty$ in eq. (46) we apply the following identity. The term involving infinite sum on the RHS of eq. (46) can be expressed in terms of the regular Manning–Rosen Green's function $G_\ell^{M(R)}(s, s')$ as

$$\begin{aligned} & \lambda_\ell d_\ell(\beta_\ell,\chi)\int_0^s \mathrm{d}s'\, g_\ell(\beta_\ell, s')\, G_\ell^{M(R)}(s,s') \\ & = 2^{-\ell}(\ell!)^{-1}b^{\ell+2}\lambda_\ell d_\ell(\beta_\ell,\chi)\mathrm{e}^{\mathrm{i}\chi s}\,(1-\exp(-s/b))^{\omega+1} \\ & \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!} \\ & \times f_{n+1+\ell-\omega}(M', N'; P'; 1-\mathrm{e}^{-s/b}). \end{aligned} \quad (48)$$

Under the limit $s \to \infty$, eq. (46) yields

$$\begin{aligned} D_1 = & -\frac{\lambda_\ell d_\ell(\beta_\ell,\chi)2^{-\ell}(\ell!)^{-1}b^{\ell+1}}{f_\ell^M(\chi)} \\ & \times\frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)} \\ & \times {}_3F_2(M'-1-2\omega, N'-1-2\omega, (\beta_\ell-\mathrm{i}\chi)b; \\ & 1-2\mathrm{i}\chi b, \ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b; 1). \end{aligned} \quad (49)$$

Equation (46) along with eqs (47) and (49) provides the Jost solution for Manning–Rosen plus Graz potential,

$$\begin{aligned} & f_\ell^{MG}(\chi, s) = b^{\omega+1}\left[1-\exp(-s/b)\right]^{\omega+1}\exp(\mathrm{i}\chi s) \\ & \times\Bigg[-\frac{\lambda_\ell d_\ell(\beta_\ell,\chi)2^{-\ell}(\ell!)^{-1}b^{\ell+1}}{f_\ell^M(\chi)} \\ & \times\frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)} \\ & \times {}_3F_2(M'-1-2\omega, N'-1-2\omega, (\beta_\ell-\mathrm{i}\chi)b; \\ & 1-2\mathrm{i}\chi b, \ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b; 1) \\ & \times {}_2F_1(M', N'; P'; 1-\exp(-s/b)) \\ & +\frac{f_\ell^{MG}(\chi)}{b^{\omega+1}f_\ell^M(\chi)}\,(1-\exp(-s/b))^{-2\omega-1} \\ & \times {}_2F_1(M'-1-2\omega, N'-1-2\omega; 1-2\mathrm{i}\chi b; \mathrm{e}^{-s/b}) \\ & +\lambda_\ell d_\ell(\beta_\ell,\chi)2^{-\ell}(\ell!)^{-1}b^{1-\omega+\ell} \\ & \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!} \\ & \times f_{n+1+\ell-\omega}(M', N'; P'; 1-\exp(-s/b))\Bigg]. \end{aligned} \quad (50)$$

2.2 *Physical solution*

2.2.1 *Direct integration approach.* The physical solution for Manning–Rosen plus Graz potential satisfies the Schrödinger like differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\chi^2-V_M(s)\right]\psi_\ell^{(+)MG}(\chi,s)$$
$$= \lambda_\ell\, \mathrm{d}_\ell^{(+)}(\beta_\ell,\chi)\, g_\ell(\beta_\ell, s). \quad (51)$$

The physical solution of eq. (51) is expressed as

$$\psi_\ell^{(+)MG}(\chi,s) = \psi_\ell^{(+)M}(\chi,s) + \lambda_\ell d_\ell^{(+)}(\beta_\ell,\chi) \times\int_0^{\infty} G_\ell^{(+)M}(s,s')g_\ell(\beta_\ell,s')\mathrm{d}s'. \quad (52)$$

The all partial wave physical solution [18] of the Manning–Rosen potential is given by

$$\psi_\ell^{(+)M}(\chi,s) = \frac{\chi}{f_\ell^M(\chi)} b^{\omega+1}\left[1-\exp(-s/b)\right]^{\omega+1} \times \exp(\mathrm{i}\chi s)\, {}_2F_1(M', N'; P'; 1-\exp(-s/b)). \quad (53)$$

The all partial wave physical Green's function is defined as [18,24]

$$G_\ell^{(+)M}(s,s') = -\chi^{-1}\psi_\ell^{(+)M}(\chi, s_<) f_\ell^M(\chi, s_>), \quad (54)$$

where $s_<$ symbol stands for either s or s' whichever is smaller and $s_>$ symbol stands for either s or s' whichever is larger.

Substituting eq. (54) in eq. (52) and rearranging the integral involved, we arrive at

$$\begin{aligned}\psi_\ell^{(+)MG}(\chi,s) &= \psi_\ell^{(+)M}(\chi,s) + \frac{\lambda_\ell}{\chi} d_\ell^{(+)}(\beta_\ell,\chi) \\ &\times\Bigg[f_\ell^M(\chi,s)\int_0^s \psi_\ell^{(+)M}(\chi,s') g_\ell(\beta_\ell,s')\mathrm{d}s' \\ &- \psi_\ell^{(+)M}(\chi,s)\int_0^s f_\ell^M(\chi,s') g_\ell(\beta_\ell,s')\mathrm{d}s' \\ &+ \psi_\ell^{(+)M}(\chi,s)\int_0^\infty f_\ell^M(\chi,s') g_\ell(\beta_\ell,s')\mathrm{d}s'\Bigg]. \end{aligned} \quad (55)$$

Multiplying by $g_\ell(\beta_\ell, s)$ and integrating over the whole range on both sides of eq. (52), $d_\ell^{(+)}(\beta_\ell,\chi)$ can be simplified to

$$d_\ell^{(+)}(\beta_\ell,\chi) = \frac{1}{D_\ell^{(+)MG}(\chi)}\int_0^\infty \psi_\ell^{(+)M}(\chi,s) g_\ell(\beta_\ell,s)\mathrm{d}s. \quad (56)$$

$D_\ell^{(+)MG}(\chi)$ stands for the Fredholm determinant associated with the physical boundary condition and is written as

$$D_\ell^{(+)MG}(\chi) = 1 - \lambda_\ell \int_0^\infty\int_0^\infty g_\ell(\beta_\ell,s) \times G_\ell^{(+)M}(s,s')\, g_\ell(\beta_\ell,s')\,\mathrm{d}s\,\mathrm{d}s'. \quad (57)$$

The integrals involved in eq. (55) can be evaluated using eqs (18) and (28), and hence the physical solution is expressed as

$$\begin{aligned}\psi_\ell^{(+)MG}(\chi,s) &= \psi_\ell^{(+)M}(\chi,s) + \frac{\lambda_\ell}{\chi} d_\ell^{(+)}(\beta_\ell,\chi) \\ &\times\Bigg[-2^{-\ell}(\ell!)^{-1} b^{2+\ell}\left[1-\exp(-s/b)\right]^{\omega+1} \\ &\times \exp(\mathrm{i}\chi s)\sum_{n=0}^\infty \frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!} \\ &\times f_{n+1+\ell-\omega}(M', N'; P'; 1-\exp(-s/b)) \\ &+ \frac{\chi}{f_\ell^M(\chi)} b^{\omega+1}\left[1-\exp(-s/b)\right]^{\omega+1} \\ &\times \exp(\mathrm{i}\chi s) {}_2F_1(M', N'; P'; 1-\exp(-s/b)) \\ &\times 2^{-\ell}(\ell!)^{-1} b^{\ell+1} \times \frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)} \\ &\times {}_3F_2(M'-1-2\omega, N'-1-2\omega, (\beta_\ell-\mathrm{i}\chi)b; \\ &1-2\mathrm{i}\chi b, \ell-\omega+1+(\beta_\ell-0\mathrm{i}\chi)b; 1)\Bigg]. \end{aligned} \quad (58)$$

The integral used in eq. (56) can be solved using eq. (21) and can be expressed as follows:

$$\begin{aligned}\int_0^\infty \psi_\ell^{(+)M}(\chi,s) g_\ell(\beta_\ell,s)\mathrm{d}s &= \frac{2^{-\ell}(\ell!)^{-1}b^{2+\ell+\omega}\chi}{f_\ell^M(\chi)} \frac{\Gamma(\ell+\omega+2)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell+\omega+2+(\beta_\ell-\mathrm{i}\chi)b)} \\ &\times {}_3F_2(M', N', \ell+\omega+2; \\ &P', \ell+\omega+2+(\beta_\ell-\mathrm{i}\chi)b; 1). \end{aligned} \quad (59)$$

Equation (57) involves the double Laplace transform of Manning–Rosen's physical Green's function with form factor of Graz potential

$$g_\ell(\beta_\ell,s) = 2^{-\ell}(\ell!)^{-1} b^\ell \left[1-\exp(-s/b)\right]^\ell \mathrm{e}^{-\beta_\ell s}.$$

The double Laplace transform of $G_\ell^{(+)M}(s,s')$ can be calculated by using eqs (21), (32) and (33) to get

$$\bar{G}_\ell^{(+)M}(\beta_\ell,\chi)=\int_0^\infty\int_0^\infty g_\ell(\beta_\ell,s)\,G_\ell^{(+)M}(s,s')\,g_\ell(\beta_\ell,s')\,\mathrm{d}s\,\mathrm{d}s'$$
$$=b^{2\ell+3}\times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!}\frac{\Gamma(n+2\ell+3)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{(n+\ell-\omega+1)(n+\ell+\omega+2)\Gamma(n+2\ell+3+(\beta_\ell-\mathrm{i}\chi)b)}$$
$$\times{}_4F_3(1,n+\ell-\omega+1+M',n+\ell-\omega+1+N',n+2\ell+3;n+\ell-\omega+2,n+\ell+\omega$$
$$+3,n+2\ell+3+(\beta_\ell-\mathrm{i}\chi)b;1)-b^{2\ell+3}\frac{\Gamma(M')\Gamma(N')}{\Gamma(2+2\omega)\Gamma(1-2\mathrm{i}\chi b)}\frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)}$$
$${}_3F_2(M'-1-2\omega,N'-1-2\omega,(\beta_\ell-\mathrm{i}\chi)b;1-2\mathrm{i}\chi b,\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b;1)$$
$$\times\frac{\Gamma(\ell+\omega+2)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell+\omega+2+(\beta_\ell-vi\chi)b)}{}_3F_2(M',N',\ell+\omega+2;P',\ell+\omega+2+(\beta_\ell-\mathrm{i}\chi)b;1).\quad(60)$$

Combining eqs (57) and (60), one gets the expression for $D_\ell^{(+)MG}(\chi)$ as

$$D_\ell^{(+)MG}(\chi)=1-\lambda_\ell 2^{-2\ell}(\ell!)^{-2}\bar{G}_\ell^{(+)M}(\beta_\ell,\chi).\quad(61)$$

2.2.2 *Ordinary differential equation approach.* The physical solution of eq. (51) can be expressed in terms of Laplace transform of Manning–Rosen's physical Green's function with the form factor of Graz potential,

$$\psi_\ell^{(+)MG}(\chi,s)=\psi_\ell^{(+)M}(\chi,s)+\lambda_\ell d_\ell^{(+)}(\beta_\ell,\chi)\bar{G}_\ell^{(+)M}(s,\beta_\ell).\quad(62)$$

We adapt the differential equation approach to evaluate the single Laplace transform of the pure Manning–Rosen Green's function $\bar{G}_\ell^{(+)M}(s,\beta_\ell)$. The Green's function $G_\ell^{(+)M}(s,s')$ satisfies [9,10,24]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\chi^2-V_M(s)\right]G_\ell^{(+)M}(s,s')=\delta(s-s').\quad(63)$$

Taking the Laplace transform of the above equation and using the following transformation

$$\bar{G}_\ell^{(+)M}(s,\beta_\ell)=b^{\omega+1}\left[1-\exp(-s/b)\right]^{\omega+1}\mathrm{e}^{\mathrm{i}\chi s}\bar{F}_\ell(s,\beta_\ell)\quad(64)$$

in eq. (63) gives

$$\exp(s/b)b^2(1-\exp(-s/b))\frac{\mathrm{d}^2\bar{F}_\ell}{\mathrm{d}s^2}+\left[2(\omega+1)+2\mathrm{i}\chi b^2\exp(s/b)\times(1-\exp(-s/b))\right]\times\frac{\mathrm{d}\bar{F}_\ell}{\mathrm{d}s}+\left(2\mathrm{i}\chi b(\omega+1)-(\omega+1)+A\right)\bar{F}_\ell$$
$$=2^{-\ell}(\ell!)^{-1}b^{\ell-\omega+1}\exp(s/b)\times\left[1-\exp(-s/b)\right]^{\ell-\omega}\mathrm{e}^{-(\beta_\ell+\mathrm{i}\chi)s}.\quad(65)$$

Making change of variable $z=1-\exp(-s/b)$ in the above equation one has

$$z(1-z)\frac{\mathrm{d}^2\bar{F}_\ell}{\mathrm{d}z^2}+\left[2(\omega+1)+(1+2(\omega+1)-2\mathrm{i}\chi b)z\right]\frac{\mathrm{d}\bar{F}_\ell}{\mathrm{d}z}-\left(-2\mathrm{i}\chi b(\omega+1)+(\omega+1)-A\right)\bar{F}_\ell$$
$$=2^{-\ell}(\ell!)^{-1}b^{\ell-\omega+1}z^{\ell-\omega}(1-z)^{(\beta_\ell+\mathrm{i}\chi)b-1}.\quad(66)$$

Combining eqs (41), (64) and (66) leads to

$$\bar{G}_\ell^{(+)M}(s,\beta_\ell)=\int_0^\infty G_\ell^{(+)M}(s,s')g_\ell(\beta_\ell,s')\mathrm{d}s'$$
$$=b^{\omega+1}\left[1-\exp(-s/b)\right]^{\omega+1}\exp(\mathrm{i}\chi s)\times\Bigg[R_1\,{}_2F_1(M',N';P';1-\exp(-s/b))+R_2(1-\exp(-s/b))^{-2\omega-1}\times{}_2F_1(M'-1-2\omega,N'-1-2\omega;1-2\mathrm{i}\chi b;\mathrm{e}^{-s/b})+2^{-\ell}(\ell!)^{-1}b^{1-\omega+\ell}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!}\times f_{n+1+\ell-\omega}(M',N';P';1-\exp(-s/b))\Bigg].\quad(67)$$

In eq. (67) the constants R_1 and R_2 will be found from the boundary conditions at $s=0$ and $s=\infty$. At $s=0$, $\bar{G}_\ell^{(+)M}(s,\beta_\ell)=0$ and we obtain $R_2=0$.

Under the limit $s\to\infty$, eq. (67) along with eq. (48) yields

$$R_1=-\frac{2^{-\ell}(\ell!)^{-1}b^{\ell+1}}{f_\ell^M(\chi)}\frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)}\times{}_3F_2(M'-1-2\omega,N'-1-2\omega,(\beta_\ell-\mathrm{i}\chi)b;1-2\mathrm{i}\chi b,\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b;1).\quad(68)$$

Hence, from eqs (67) and (68) along with $R_2 = 0$ we have

$$\bar{G}_\ell^{(+)M}(s, \beta_\ell)$$
$$= b^{\omega+1}\left[1 - \exp(-s/b)\right]^{\omega+1} \exp(\mathrm{i}\chi s)$$
$$\times\Bigg[-\frac{2^{-\ell}(\ell!)^{-1}b^{\ell+1}\chi}{f_\ell^M(\chi)}\frac{\Gamma(\ell-\omega+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b)}$$
$$\times{}_3F_2(M'-1-2\omega, N'-1-2\omega, (\beta_\ell-\mathrm{i}\chi)b;$$
$$1-2\mathrm{i}\chi b, \ell-\omega+1+(\beta_\ell-\mathrm{i}\chi)b; 1)$$
$$\times{}_2F_1(M', N'; P'; 1-\exp(-s/b))$$
$$+2^{-\ell}(\ell!)^{-1}b^{1-\omega+\ell}\chi\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!}$$
$$\times f_{n+1+\ell-\omega}(M', N'; P'; 1-\exp(-s/b))\Bigg]. \quad (69)$$

Substituting the results of eq. (69) in eq. (62), we shall get the required physical solution same as in eq. (58). We have performed various checks on our expressions in terms of their limiting behaviours. Appendix I addresses these limiting behaviours. The Jost function $f_\ell^{MG}(\chi)$ is usually correlated with the Fredholm determinants [31,32] $D_\ell^{(+)MG}(\chi)$ and $D_\ell^{MG}(\chi)$ as $f_\ell^{MG}(\chi) = D_\ell^{(+)MG}(\chi)/D_\ell^{MG}(\chi)$. For a local plus a non-local potential, $D_\ell^{MG}(\chi)$ is always a real quantity, whereas $D_\ell^{(+)MG}(\chi)$ is a complex quantity. The significance of the Jost function $f_\ell^{MG}(\chi)$ in scattering theory may be viewed on two levels. On the one hand, it has been useful in establishing a relativistic theory of the S-matrix by allowing the conjectured analyticity properties of the S-matrix to be tested in potential scattering; there the Schrödinger theory provides an alternate reliable standard against which the plausibility of the arguments of the new theory is to be measured. On the other hand, it has become an invaluable tool strictly in the domain of potential scattering where it has been instrumental in the study of bound and resonant states, and in the general analysis of low-energy scattering data.

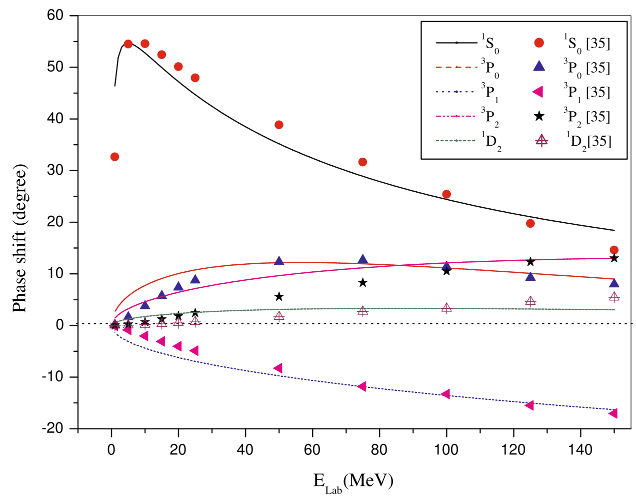

Figure 1. The p–p phase shifts as a function of E_{Lab}. The continuous and the dashed lines represent the present work while the different symbols are for experimental results of Arndt *et al* [35]. The zero line is represented by the black dotted line.

In particular, the Jost solution is holomorphic in the upper complex χ-plane, where Im $\chi > 0$. For our approach, the most important quantity is the Jost function and the roots of the Jost function in the upper complex χ-plane correspond to the bound-state energies of the related system [24,33]. As the phase of the Jost function is negative of the scattering phase shift, the same is true for $D_\ell^{(+)MG}(\chi)$. The acceptability of a potential model strictly depends on the reproduction of low-energy scattering parameters including binding energies. Thus, we focus particularly on the computation of the scattering phase shifts up to $\ell = 2$ for

Table 1. Parameters for the p–p system

States	α	β_ℓ (fm^{-1})	λ_ℓ (fm^{-3})	b (fm)	A
$^1S_0$	−0.005	1.1	−2.405	0.997	−0.0346
$^3P_0$	−0.5	1.55	−11.52	1.62	−0.0563
$^3P_1$	0.005	1.45	−25.39	0.35	−0.0122
$^3P_2$	0.005	2.5	−325.41	1.08	−0.0375
$^1D_2$	0.5	1.63	−552.54	1.099	−0.0382

Table 2. Parameters for the p–$^2H_1$ system

States	α	β_ℓ (fm^{-1})	λ_ℓ (fm^{-3})	b (fm)	A
$^2S_{1/2}$	0.05	0.1855	−1.0228	0.256	−0.0119
$^2P_{1/2}$	0.005	0.862	−829.84	0.196	−0.0091
$^2P_{3/2}$	0.005	0.761	−515.23	0.217	−0.010
$^2D_{3/2}$	0.05	1.02	−513.75	0.715	−0.0331

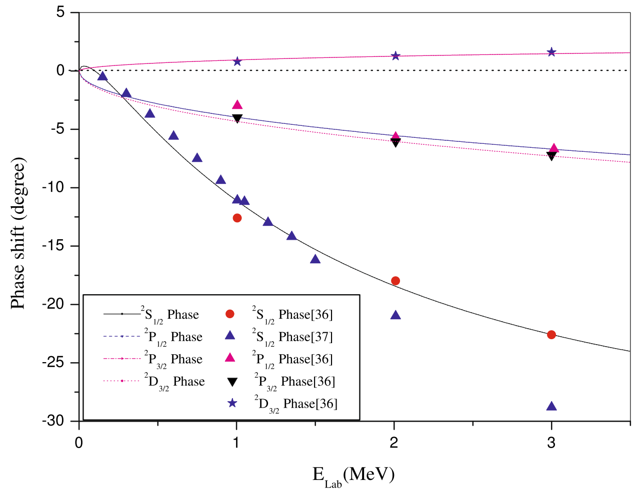

Figure 2. The p–$^2H_1$ phase shifts as a function of E_{Lab}. The continuous and the dashed lines represent the present work while the different symbols are for experimental results of Huttel *et al* [36] and Chen *et al* [37]. The zero line is represented by the black dotted line.

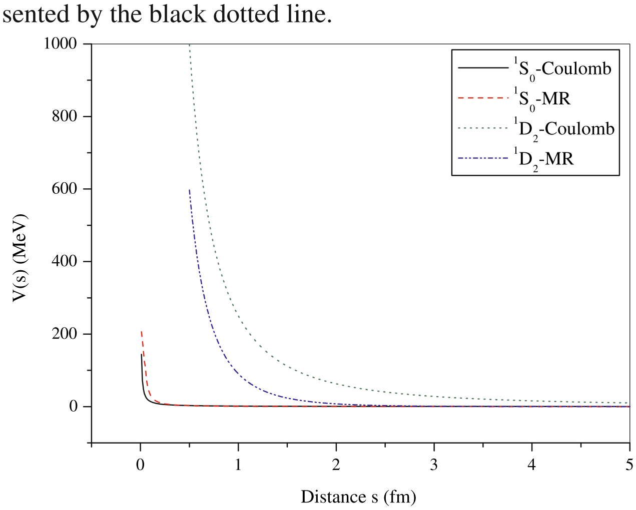

Figure 3. Pure Coulomb and Manning–Rosen potentials for p–p $^1S_0$ and $^1D_2$ states as a function of distance.

the p–p and p–$^2H_1$ systems by exploiting the expression for $D_\ell^{(+)MG}(\chi)$. We have made some checks on our expression for $D_\ell^{(+)MG}(\chi)$ with respect to its limiting behaviours and found in order with the Hulthén potential. We present it in Appendix I.

3. Results and discussion

Based on the above-described methods, we have applied our formalism to compute the scattering phase shifts for the p–p and p–$^2H_1$ systems. In our calculations, we have used the exact values of $\hbar^2/2\mu$ =41.47 MeV fm^2 and $\hbar^2/2\mu$ = 31.1025 MeV fm^2 respectively for the systems under consideration. The parameters of the nuclear

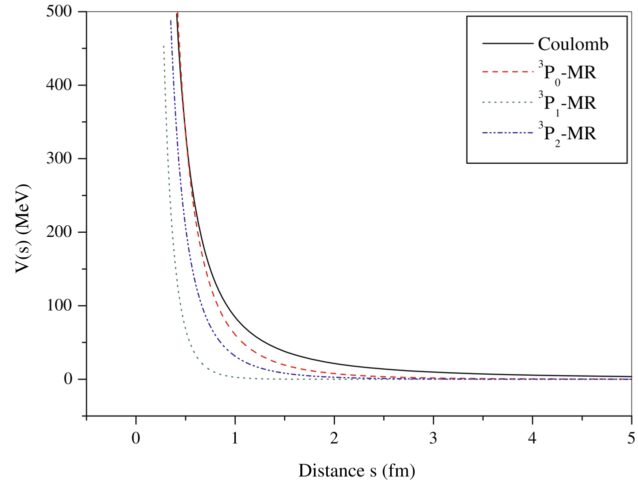

Figure 4. Pure Coulomb and Manning–Rosen potentials for p–p $^3P_0$, $^3P_1$ and $^3P_2$ states as a function of distance.

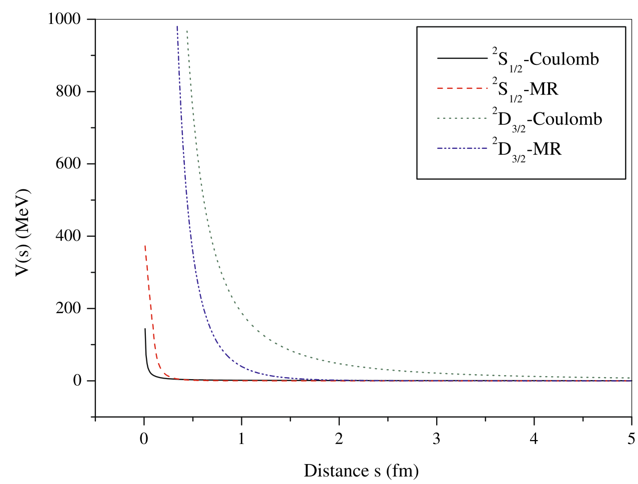

Figure 5. Pure Coulomb and Manning–Rosen potentials for p–d $^2S_{1/2}$ and $^2D_{3/2}$ states as a function of distance.

part of the potential for the p–$^2H_1$ system are fixed by fitting its doublet state binding energy and considered to be E_B = 7.718 MeV for the He^3 nucleus. The higher excited state energies of the He^3 nucleus are fixed from the shell model calculations [34]. On the other hand, as the p–p system is unbound, we give free running to our parameters in the numerical program to have proper values of the phase shifts. The best fitted parameters for the Manning–Rosen and Graz potentials are listed in tables 1 and 2.

Results of our phase analysis for the p–p and p–$^2H_1$ systems are shown in figures 1 and 2 respectively. In figure 1, we portrayed the p–p elastic scattering phase shifts for the partial wave states *s*, *p* and *d* up to E_{Lab} = 150 MeV.

From figure 1, it can be seen that our $^1S_0$ state phase shift (continuous line) produces correct nature and the peak value of the phase shifts matches exactly but in the

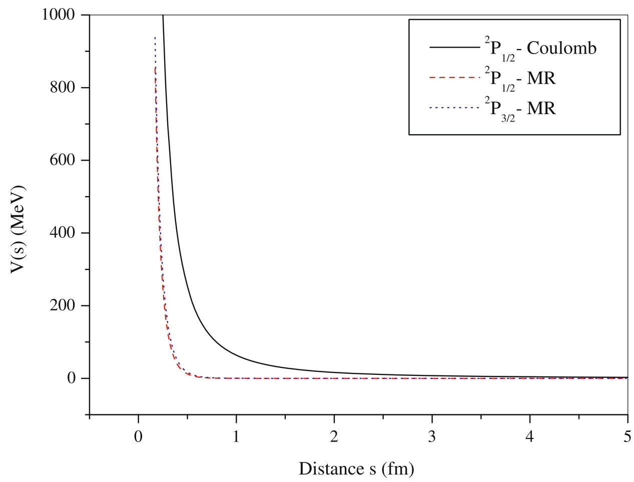

Figure 6. Pure Coulomb and Manning–Rosen potentials for p–d $^2P_{1/2}$ and $^2P_{3/2}$ states as a function of distance.

intermediate energy range they differ slightly by 2 to 3%, from those of Arndt *et al* [35]. For the $^3P_0$ state (dashed line), it is observed that our results match exactly and shows the peak value at $E_{\mathrm{Lab}} = 75$ MeV but differs slightly in the low-energy regions. On the other hand, for the $^3P_1$ state (dotted line), our results are in close agreement with the standard data [35] and show little higher values in the low-energy ranges. The phase shift values for the $^3P_2$ state (dash–dot–dot line) differ at low and intermediate energies but reproduce better results beyond $E_{\mathrm{Lab}} = 100$ MeV. The phase shift values for the $^1D_2$ state (short-dash line) differ slightly in their numerical value at low and high energies but reproduce better results in intermediate energy regions.

The p–d scattering phase shifts for different states under consideration plotted in figure 2 are in good agreement with those of Huttel *et al* [36] and Chen *et al* [37]. However, our results for the $^2S_{1/2}$ state (continuous line) gradually increase with energy from the standard data of Chen *et al* [37] beyond $E_{\mathrm{Lab}} = 1.5$ MeV but match well with Huttel *et al* [36]. The $^2P_{1/2}$ (dashed line), $^2P_{3/2}$ (dotted line) and $^2D_{3/2}$ (dash–dot–dot line) phase parameters are in excellent agreement with Huttel *et al* [36].

As $s \to 0$ the Manning–Rosen potential in eq. (2) behaves as

$$V_M(s) = \left[\delta(\delta-1) - As/b\right]/s^2.$$

In this case, question may naturally arise regarding the electromagnetic character of our potential model at short distances. The extra factor $\delta(\delta-1)/s^2$ becomes the source of confusion. To remove the confusion, we portray both the Manning–Rosen and the pure Coulomb potentials in figures 3–6 for the p–p and p–$^2H_1$ systems. From the figures it is noticed that both behave in the same way and decrease with almost equal gradients with distance. As obvious, the Manning–Rosen ones die out much faster than the pure Coulomb potentials with distance due to their short-range character. At very short distances, the order of their magnitudes is more or less the same as shown in the figures with their closed view.

4. Conclusions

Separable interactions are commonly used in different areas of physics such as particle, nuclear and atomic physics because of the simplicity involved in analytical calculation. In reality, for a correct description of the nucleon–nucleon interaction in terms of the separable potential, one requires at least two terms in the potential with the strength parameter having opposite signs. Since the low-energy scattering experiments sample out only the outer region of the potential, one term separable potential may be of importance for this energy range. For intermediate and high energy ranges, one has to consider higher rank potential because of the sensitivity of scattering data to the choice of the inner core. Our model does not pose any problem irrespective of whether the separable potential is symmetric or non-symmetric. It is worthwhile to mention that for $\alpha = 0$ the present case reduces to the Hulthén potential. The Hulthén potential behaves like the Coulomb potential near the origin, i.e., when $s \to 0$, but decreases exponentially in the asymptotic region and its capacity for bound states is smaller than the Coulomb potential. Obviously, the behaviours of the Manning–Rosen potentials are in good agreement with those of Coulomb one for low values of α and ℓ. We have demonstrated that the Manning–Rosen potential has the ability to reproduce the correct nature of the electromagnetic force essential for such model potentials in figures 3–6. Recently, physicists show much interest in searching for exponential-type potentials as they play an important role in plasma, solid-state, atomic and molecular physics. The charged hadronic systems are generally represented by the screened/cut-off Coulomb interaction as pure Coulomb potential does not exist in reality. The overall quality of the consistency between the theory and the experiment, in the low-energy region, is noteworthy. Therefore, the present approach may turn out to be interesting to theoretical and experimental physicists.

Appendix I

When $\alpha = 0$, the Manning–Rosen potential is converted to Hulthén potential with $-Ab^{-2} = V_0$, where V_0 denotes the strength of the atomic Hulthén potential. With $\alpha = 0$, eq. (60) reads as

$$\bar{G}_\ell^{(+)}(\beta_\ell,\chi) = b^{2\ell+3}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(1-(\beta_\ell+\mathrm{i}\chi)b)n!} \times\frac{\Gamma(n+2\ell+3)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{(n+1)(n+2\ell+2)\Gamma(n+2\ell+3+(\beta_\ell-\mathrm{i}\chi)b)} \times{}_3F_2(1,n+1+M',n+1+N';n+2, n+2\ell+3+(\beta_\ell-\mathrm{i}\chi)b;1) -b^{2\ell+2}\frac{\Gamma(M')\Gamma(N')}{\Gamma(2+2\ell)\Gamma(1-2\mathrm{i}\chi b)(\beta_\ell-\mathrm{i}\chi)} \times{}_3F_2(M'-1-2\ell,N'-1-2\ell,(\beta_\ell-\mathrm{i}\chi)b; 1-2\mathrm{i}\chi b,1+(\beta_\ell-\mathrm{i}\chi)b;1) \times\frac{\Gamma(2\ell++2)\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(2\ell+2+(\beta_\ell-\mathrm{i}\chi)b)} \times{}_2F_1(M',N';2\ell+2+(\beta_\ell-\mathrm{i}\chi)b;1). \quad (70)$$

To get the exact limiting value, it is necessary to remove the infinite summation series present in eq. (70). For this, one can proceed by applying the following analytic continuation [38]

$$ {}_3F_2(a,b,c;e;f;1) = \frac{\Gamma(e)\Gamma(e-a-b)}{\Gamma(e-a)\Gamma(e-b)} \times{}_3F_2(a,b,f-c;a+b-e+1,f;1) +\frac{\Gamma(e)\Gamma(f)\Gamma(a+b-e)\Gamma(e+f-a-b-c)}{\Gamma(a)\Gamma(b)\Gamma(f-c)\Gamma(e+f-a-b)} \times{}_3F_2(e-a,e-b,e+f-a-b-c; e-a-b+1,e+f-a-b;1) \quad (71)$$

to the hypergeometric series present in the first term of eq. (70) along with the transformation relation of generalised Gaussian hypergeometric function [29,38]

$$ {}_3F_2(a,b,c;e;f;z) = \frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(b)\Gamma(c)} \times\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+n)\Gamma(f+n)}\frac{z^n}{n!} \quad (72)$$

and eq. (70) simplifies to

$$\bar{G}_\ell^{(+)}(\beta_\ell,\chi) = -b^{2\ell+3}\sum_{n=0}^{\infty}\frac{\Gamma(n+2\ell+2)\Gamma(n+1-(\beta_\ell+\mathrm{i}\chi)b)}{(A'+\ell+1)\Gamma(n+1)\Gamma((1-\beta_\ell+\mathrm{i}\chi)b)} \times\frac{\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(n+2\ell+3+(\beta_\ell-\mathrm{i}\chi)b)}{}_3F_2(1,n+2+A'+\ell,\ell+1+(\beta_\ell-\mathrm{i}\chi)b-B'; 2+A'+\ell,n+2\ell+3+(\beta_\ell-\mathrm{i}\chi)b;1) + b^{2\ell+3}\frac{\Gamma(A'+\ell+1)\Gamma(B'+\ell+1)\Gamma(2\ell+2)}{\Gamma(A'+\ell+2)\Gamma(2-2\mathrm{i}\chi b-A'+\ell)} \times\frac{\Gamma((\beta_\ell-\mathrm{i}\chi)b)\Gamma((\beta_\ell+\mathrm{i}\chi)b)}{\Gamma((\beta_\ell-\mathrm{i}\chi)b-A'+\ell+1)\Gamma((\beta_\ell-\mathrm{i}\chi)b-B'+\ell+1)} \times{}_3F_2(1,2\ell+2,1-(\beta_\ell+\mathrm{i}\chi)b; A'+\ell+2,2-2\mathrm{i}\chi b-A'+\ell;1) - b^{2\ell+2}\frac{\Gamma(A'+\ell+1)\Gamma(B'+\ell+1)}{\Gamma(C')(\beta_\ell-\mathrm{i}\chi)} \frac{\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(\ell+1+(\beta_\ell-\mathrm{i}\chi)b-A')}\frac{\Gamma((\beta_\ell+\mathrm{i}\chi)b)}{\Gamma(\ell+1+(\beta_\ell-i\chi)b-B')} \times{}_3F_2(A'-\ell,B'-\ell,(\beta_\ell-\mathrm{i}\chi)b;C',1+(\beta_\ell+\mathrm{i}\chi)b;1), \quad (73)$$

where $M'=\ell+1+A'$, $N'=\ell+1+B'$, $A'=-\mathrm{i}b\chi+\mathrm{i}b\sqrt{\chi^2+V_0}$, $B'=-\mathrm{i}b\chi-\mathrm{i}b\sqrt{\chi^2+V_0}$ and $C'=2\ell+2$. Again, applying the following formula [38] to the second term of the above equation

$$ {}_3F_2(a_1,b_1,c_1;e_1,f_1;1) = \frac{\Gamma(S)\Gamma(f_1)}{\Gamma(S+a_1)\Gamma(f-a_1)} \times{}_3F_2(a_1,e_1-b_1,e_1-c_1;S+a_1,e_1;1); \quad S=e_1+f_1-a_1-b_1-c_1. \quad (74)$$

$\bar{G}_\ell^{(+)}(\beta_\ell,\chi)$ becomes

$$\bar{G}_\ell^{(+)}(\beta_\ell,\chi) = -b^{2\ell+3} \times\sum_{n=0}^{\infty}\frac{\Gamma(n+2\ell+2)\Gamma((n+1-\beta_\ell+\mathrm{i}\chi)b)}{(A'+\ell+1)\Gamma(n+1)\Gamma((1-\beta_\ell+\mathrm{i}\chi)b)} \times\frac{\Gamma((\beta_\ell-\mathrm{i}\chi)b)}{\Gamma(n+2\ell+3+(\beta_\ell-\mathrm{i}\chi)b)} \times{}_3F_2(1,n+2+A'+\ell,\ell+1+(\beta_\ell-\mathrm{i}\chi)b-B';$$

$$2 + A' + \ell, n + 2\ell + 3 + (\beta_\ell - i\chi)b; 1). \quad (75)$$

Then proceeding with the hypergeometric transformation [38]

$$\begin{aligned}&{}_3F_2(a_1, a_2, a_3; b_1, b_2; 1)\\ &= \frac{\Gamma(b_2)\Gamma(b_1 + b_2 - a_1 - a_2 - a_3)}{\Gamma(b_2 - a_3)\Gamma(b_1 + b_2 - a_1 - a_2)}\, {}_3F_2(b_1 - a_1,\\ &\quad b_1 - a_2, a_3; b_1, b_1 + b_2 - a_1 - a_2; 1)\end{aligned} \quad (76)$$

eq. (75) converts to

$$\begin{aligned}&\bar{G}_\ell^{(+)}(\beta_\ell, \chi) = -b^{2\ell+3}\\ &\times \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\ell + 2)\Gamma((n + 1 - \beta_\ell + i\chi)b)}{(A' + \ell + 1)\Gamma(n + 1)\Gamma((1 - \beta_\ell + i\chi)b)}\\ &\times \frac{\Gamma((\beta_\ell - i\chi)b)}{\Gamma(n + 2\ell + 2 + (\beta_\ell - i\chi)b)}\, {}_3F_2(-n, 1 - (\beta_\ell\\ &+ i\chi)b, 1; 2 + A' + \ell, 2 + B' + \ell; 1).\end{aligned} \quad (77)$$

Expanding eq. (77) and rearranging the terms suitably along with [26–28]

$$\begin{aligned}&{}_2F_1(a, b; c; z)\\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)} \frac{z^n}{n!}\end{aligned} \quad (78)$$

and

$$\begin{aligned}&{}_4F_3(a, b, c, d; e, f, g; z)\\ &= \frac{\Gamma(e)\Gamma(f)\Gamma(g)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\\ &\times \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)\Gamma(c + n)\Gamma(d + n)}{\Gamma(e + n)\Gamma(f + n)\Gamma(g + n)} \frac{z^n}{n!}\end{aligned} \quad (79)$$

and eqs (72) and (77) yield

$$\begin{aligned}&\bar{G}_\ell^{(+)}(\beta_\ell, \chi) = -b^{2\ell+3}\\ &\times \frac{\Gamma(2\ell + 2)\Gamma(2\beta_\ell b - 1)}{(A'+\ell+1)(B'+\ell+1)\Gamma(n+1)\Gamma(2\beta_\ell b+2\ell+1)}\\ &\times {}_4F_3(1, 2\ell + 2, 1 - (\beta_\ell + i\chi)b, 1 - (\beta_\ell + i\chi)b;\\ &2 + A' + \ell, 2 + B' + \ell, 2 - 2\beta_\ell b; 1).\end{aligned} \quad (80)$$

The above result is in agreement with Behera *et al* [25] for the partial waves $\ell = 0$, 1 and 2. For the s-wave, it results in the double Laplace transform of the Hulthén Green's function [12,39]. For $\ell = 0$ the Graz potential coincides with Yamaguchi [40] potential. In this limit, eq. (60) reproduces the result for Manning–Rosen plus Yamaguchi potential [41,42].

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