



Infinite-dimensional symmetry group, Kac–Moody–Virasoro algebras and integrability of Kac–Wakimoto equation

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MS received 10 January 2021; revised 2 May 2022; accepted 11 June 2022

Abstract. An eighth-order equation in $(3 + 1)$ dimension is studied for its integrability. Its symmetry group is shown to be infinite-dimensional and is checked for Virasoro-like structure. The equation is shown to have no Painlevé property. One- and two-dimensional classifications of infinite-dimensional symmetry algebra are also given.

Keywords. Lie symmetries; Kac–Wakimoto equation; Virasoro-like algebra; integrability.

PACS Nos 02.20.Sv; 02.30.Jr; 02.20.Tw; 02.30.Ik

1. Introduction

Over the last three decades, there have been various approaches for solving nonlinear partial differential equations, including computational and analytical methods. The Lie group method continues to prove its competence compared to all the other available methodologies. The Lie group approach not only helps to solve nonlinear equations for solutions that are physically important, but also helps to detect the inherent geometric properties of the equation [1–10]. The symmetries of nonlinear partial differential equations, particularly infinite symmetries, play a vital role in studying the integrability of the equation, especially when the equation admits Virasoro-like algebra. It has been seen that many integrable equations, such as Nizhnik–Novikov–Veselov equation [11], nonlinear Schrödinger equation [12], sine-Gordon equation, $(2 + 1)$ -dimensional long dispersive wave equation [13], admit Virasoro algebra and other non-integrable equations such as Infeld–Rowlands equation [14], dispersive long-wave equation [15] do not admit Virasoro algebra. This coincidence of Virasoro-like algebra may be helpful in investigating whether the equation is integrable or not, though we need to investigate additional integrability parameters (like Painlevé property) to support the claim. So, in this work, we plan to investigate the integrability of the following eighth-order $(3 + 1)$ -dimensional Kac–Wakimoto equation:

$$u_{8x} + 28 u_x u_{6x} + 28 u_{xx} u_{5x} + 70 u_{xxx} u_{4x}$$

$$\begin{aligned} &+ 210 u_x^2 u_{4x} + 420 u_x u_{xx} u_{xxx} \\ &+ 420 u_x^3 u_{xx} + a (u_{xxx} u_y + 3 u_y u_{xx} + 3 u_x u_{xy}) \\ &+ b u_{zz} + c u_{xt} = 0, \end{aligned} \quad (1)$$

where $a = -280\sqrt{6}$, $b = 210$, $c = -240\sqrt{2}$. This equation associated with affine Lie algebra $\mathfrak{e}_6^{(1)}$ was once derived by Kac and Wakimoto [16] in Hirota’s bilinear form. Dodd [17] has obtained its one- and two-soliton solutions which subsequently were corrected by Pekcan [18] and the author proved that the Kac–Wakimoto equation is not integrable in Hirota’s sense. The non-integrability of the Kac–Wakimoto equation is further reinforced by Sakovich [19], who has shown that the equation does not possess the Painlevé property. Later, Wang *et al* [20] have constructed the rational, kink-type breather and degenerate three-solitary wave solutions for eq. (1). In this work, we mainly focus on establishing non-integrability of Kac–Wakimoto equation on the basis of non-existence of Virasoro-like Lie algebra, and along with this we shall provide exclusive classification of infinite-dimensional Lie algebra of eq. (1) in one and two dimensions.

2. Lie group analysis

In this section, we start with brief and relevant discussion on Lie group analysis. Consider a partial differential equation $F(x, y, z, t, u, \partial u) = 0$ and one-parameter Lie

group of point transformations in the following the form:

$$\begin{aligned} x^* &= T_1(x, y, z, t; \epsilon), & y^* &= T_2(x, y, z, t; \epsilon), \\ z^* &= T_3(x, y, z, t; \epsilon), & t^* &= T_4(x, y, z, t; \epsilon), \\ u^* &= U(x, y, z, t; \epsilon). \end{aligned}$$

The differential equation F shall be invariant under these one-parameter transformations if and only if

$$F(x^*, y^*, z^*, t^*, u^*, \partial u^*) = F(x, y, z, t, u, \partial u)$$

or more precisely

$$\lim_{\epsilon \rightarrow 0} \frac{F(x^*, y^*, z^*, t^*, u^*, \partial u) - F(x, y, z, t, u, \partial u)}{\epsilon} = 0.$$

The above equation is nothing but Lie derivative of differential equation $F(x, y, z, t, u, \partial u) = 0$ along the following direction:

$$\begin{aligned} X &= \left(\frac{\partial x^*}{\partial \epsilon} \right)_{\epsilon=0} \frac{\partial}{\partial x} + \left(\frac{\partial y^*}{\partial \epsilon} \right)_{\epsilon=0} \frac{\partial}{\partial y} + \left(\frac{\partial z^*}{\partial \epsilon} \right)_{\epsilon=0} \frac{\partial}{\partial z} \\ &+ \left(\frac{\partial t^*}{\partial \epsilon} \right)_{\epsilon=0} \frac{\partial}{\partial t} + \left(\frac{\partial u^*}{\partial \epsilon} \right)_{\epsilon=0} \frac{\partial}{\partial u}, \end{aligned}$$

or equivalently

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial z} + \xi_4 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}.$$

The eighth-order prolongation $X^{(8)}$ of the above vector field when acting on eq. (1) shall provide the following set of infinitesimals:

$$\begin{aligned} \eta &= -\frac{1}{6} y z f_1''(t) + \frac{1}{3} y f_2'(t) + z f_3(t) + f_4(t), \\ \xi_1 &= -\frac{1}{2} z f_1'(t) + f_2(t), \quad \xi_2 = c_2, \quad \xi_3 = f_1(t), \quad \xi_4 = c_1 \end{aligned} \tag{2}$$

and consequently, infinite-dimensional Lie algebra \mathfrak{g} is obtained:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= -\frac{1}{2} z f_1'(t) \frac{\partial}{\partial x} + f_1(t) \frac{\partial}{\partial z} - \frac{1}{6} y z f_1''(t) \frac{\partial}{\partial u}, \\ X_4 &= f_2(t) \frac{\partial}{\partial x} + \frac{1}{3} y f_2'(t) \frac{\partial}{\partial u}, & X_5 &= z f_3(t) \frac{\partial}{\partial u}, \\ X_6 &= f_4(t) \frac{\partial}{\partial u}. \end{aligned} \tag{3}$$

The Lie algebra \mathfrak{g} is closed under Lie bracket $[X_i, X_j] = X_i X_j - X_j X_i$, which is the commutator of two generators X_i and X_j . The Lie algebra (3) might not appear closed under Jacobi identity, but the closeness under Jacobi's identity can be achieved by the change of basis. The detailed results of all the Lie commutators are listed in table 1.

Remark 2.1. The Virasoro algebra has a basis consisting of generators X_m , $m \in \mathbb{Z}$ satisfying Lie commutation in following the manner:

$$[X_m, X_n] = (m - n) X_{m+n} + \frac{c}{12} (m^2 - m) \delta_{m+n,0},$$

where c is a central element commuting with all the generators. In the classical sense, $c = 0$, but it plays a crucial role in quantum mechanics (for more details, see refs [21–23]). The presence of Virasoro algebra is a good predictor of integrability that can be seen in the typically integrable equations in (2 + 1) dimension [14,15,24,25]. On the basis of the above argument for Virasoro algebra, we can infer from the Lie commutations written in table 1 that the Lie algebra \mathfrak{g} does not contain Virasoro algebra. This supports the argument presented in [18] that Kac–Wakimoto equation is not integrable, whereas the investigation of Zakharov–Strachan equation [26] reveals that it does not admit Virasoro algebra-like structure but it is still integrable. However, the integrability of an equation can be judged from the number of arguments in favour of or against it.

3. Classification of Lie algebra under adjoint transformation

Each infinitesimal generator in the Lie algebra (3) is capable of generating Lie group of point transformations [3,27] through exponentiation

$$\begin{aligned} \tilde{\psi} &= \exp(a_i X_i) \psi, \quad i = 1, \dots, 6, \\ &\text{for } \psi = \psi(x, y, z, t, u). \end{aligned} \tag{4}$$

Such transformation $\tilde{\psi}$ sometimes is also called one-parameter group of infinitesimal transformations. The 6-parameter version of Lie group of point transformation may be expressed as

$$\tilde{\psi} = \exp \left(\sum_{i=1}^6 a_i X_i \right) \psi. \tag{5}$$

We know that a group-invariant solution Ψ corresponding to a sub-group of the widest invariance group can be transformed into another group-invariant solution using relation (5). The two solutions which can be connected through transformation (5) shall be essentially identical. So it becomes necessary to find out only those solutions which cannot be connected through this relation. The problem reduces to finding a minimal list of generators from (3) which guarantees that the two group-invariant solutions Ψ_i and Ψ_j are not connected by relation (5), that is, the problem reduces to finding an optimal list of group generators.

Table 1. Commutation relations for Lie algebra (3).

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	$X_3(f'_1)$	$X_4(f'_2)$	$X_5(f'_3)$	$X_6(f'_4)$
X_2	0	0	$-X_5(\frac{1}{6}f''_1)$	$X_6(\frac{1}{3}f'_2)$	0	0
X_3	$-X_3(f'_1)$	$X_5(\frac{1}{6}f''_1)$	0	0	$X_6(f_1 f_3)$	0
X_4	$-X_4(f'_2)$	$-X_6(\frac{1}{3}f'_2)$	0	0	0	0
X_5	$-X_5(f'_3)$	0	$-X_6(f_1 f_3)$	0	0	0
X_6	$-X_6(f'_4)$	0	0	0	0	0

Here comes the importance of adjoint transformations which will divide all the sub-groups of (3) into equivalence classes, and hence, invariant solutions from such sub-groups would be essentially different. The following theorem emphasise the importance of adjoint transformations.

Theorem 1. *The two group-invariant solutions are essentially the same if the underlying sub-groups are adjoint or conjugate subgroups.*

Proof. Let Ψ be a group-invariant solution under the sub-group $H \subset G$, where $G = \{\exp(\epsilon X) | X \in \mathfrak{g}, \epsilon \in \mathbb{R}\}$ is the Lie group and H is its sub-group under Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Suppose $h\Psi$ is transformation (5) such that h belongs to exponentiated sub-group H with generator from Lie sub-algebra \mathfrak{h} . Since Ψ is invariant under sub-group H , we must have

$$\Psi = h\Psi \quad \text{for all } h \in H.$$

Consider now a transformed solution $\tilde{\Psi} = g\Psi$ with $g \in G$. We may ask a simple query, that, under what sub-group K the solution $\tilde{\Psi}$ would be invariant? That is, what type of sub-group K would be, such that $\tilde{\Psi} = k\tilde{\Psi}$ for all $k \in K$? The following brief calculations answer the query:

$$\begin{aligned} \tilde{\Psi} &= g\Psi = gh\Psi = ghg^{-1}g\Psi = ghg^{-1}\tilde{\Psi} \\ \implies k &= ghg^{-1}. \end{aligned}$$

This shows that, when $h\Psi$ is an H -invariant solution and $g\Psi$ is a K -invariant solution, then $K = K_g(H) = \{ghg^{-1}, g \in G, h \in H\}$. Therefore, sub-group K is the adjoint or conjugate sub-group of group H under G . \square

The above theorem establishes the importance of adjoint actions or adjoint transformations in partitioning the Lie algebra into equivalence classes or we can say optimal list of sub-algebras. We define adjoint transformation

$$\text{ad}_{\exp(\epsilon X_i)}(X_j) = e^{-\epsilon X_i} X_j e^{\epsilon X_i} = \tilde{X}_j(\epsilon). \tag{6}$$

This actually is equivalent to the operator $K_g(H)$. The adjoint transformation (6) can be written through Lie

brackets using Campbell–Hausdorff formula as

$$\begin{aligned} \text{ad}_{\exp(\epsilon X_i)}(X_j) &= X_j - \epsilon[X_i, X_j] \\ &\quad + \frac{\epsilon^2}{2}[X_i, [X_i, X_j]] - \dots, \tag{7} \end{aligned}$$

where $[\cdot, \cdot]$ is the Lie bracket defined in table 1. Relation (7) helps to compile a table of the adjoint actions among each element in (3). All such adjoint actions are listed in table 2.

3.1 Construction of invariants of full adjoint action

A real function ϕ defined on the Lie algebra \mathfrak{g} is called an invariant function if $\phi(\text{ad}_g X) = \phi(X)$ for all $X \in \mathfrak{g}$ and $g \in G$. For adjoint transformation $\text{ad}(X) : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by $\text{ad}(X)Y = [X, Y]$ for all $Y \in \mathfrak{g}$, the bilinear form function $K(X, Y) = \text{trace}(\text{ad}(X), \text{ad}(Y))$ is an invariant function (for more details, see ref. [28]). The special invariant function is also called Killing function and is the key to classify Lie algebra into optimal list. Let $X = \sum_{i=1}^6 a_i X_i$. Then

$$\text{ad}(X) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 0 & -a_1 & 0 & 0 & 0 \\ a_4 & 0 & 0 & -a_1 & 0 & 0 \\ a_5 & -a_3 & a_2 & 0 & -a_1 & 0 \\ a_6 & a_4 & a_5 & -a_2 & -a_3 & -a_1 \end{bmatrix}. \tag{8}$$

The above adjoint transformation matrix quickly gives Killing form $K(X, X) = \text{trace}(\text{ad}(X), \text{ad}(X)) = 4a_1^2$. Beside the Killing form as invariant of full adjoint action, more general invariant function ϕ can also be calculated on the basis of the procedure described in [29]. The general invariant function ϕ satisfies the following system of partial differential equations:

$$\begin{aligned} \frac{\partial \phi}{\partial a_6} = 0, \quad a_1 \frac{\partial \phi}{\partial a_4} + a_2 \frac{\partial \phi}{\partial a_6} = 0, \quad a_1 \frac{\partial \phi}{\partial a_5} + a_3 \frac{\partial \phi}{\partial a_6} = 0, \\ a_3 \frac{\partial \phi}{\partial a_5} - a_4 \frac{\partial \phi}{\partial a_6} = 0, \quad a_1 \frac{\partial \phi}{\partial a_3} - a_2 \frac{\partial \phi}{\partial a_5} - a_5 \frac{\partial \phi}{\partial a_6} = 0, \\ a_3 \frac{\partial \phi}{\partial a_3} + a_4 \frac{\partial \phi}{\partial a_4} + a_5 \frac{\partial \phi}{\partial a_5} + a_6 \frac{\partial \phi}{\partial a_6} = 0. \tag{9} \end{aligned}$$

Table 2. Table of adjoint actions defined at (7).

$\text{Ad}_{\exp(\epsilon X_i)}(X_j)$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	X_2	$e^{-\epsilon} X_3$	$e^{-\epsilon} X_4$	$e^{-\epsilon} X_5$	$e^{-\epsilon} X_6$
X_2	X_1	X_2	$X_3 + \epsilon X_5$	$X_4 - \epsilon X_6$	X_5	X_6
X_3	$X_1 + \epsilon X_3$	$X_2 - \epsilon X_5 + \frac{1}{2} \epsilon^2 X_6$	X_3	X_4	$X_5 - \epsilon X_6$	X_6
X_4	$X_1 + \epsilon X_4$	$X_2 + \epsilon X_6$	X_3	X_4	X_5	X_6
X_5	$X_1 + \epsilon X_5$	X_2	$X_3 + \epsilon X_6$	X_4	X_5	X_6
X_6	$X_1 + \epsilon X_6$	X_2	X_3	X_4	X_5	X_6

On solving the above system of equation, the general invariant function $\phi = f(a_1, a_2)$ is obtained, and as we can see the Killing form is also included in this general invariant function. The actual format of this general invariant function can be predicted from the full adjoint action.

For $X = \sum_{i=1}^6 a_i X_i$, the repeated application of formula (7) gives the following result:

$$\text{ad}_{\exp(\epsilon_5 X_5)} \text{ad}_{\exp(\epsilon_6 X_6)} \text{ad}_{\exp(\epsilon_1 X_1)} \text{ad}_{\exp(\epsilon_3 X_3)} \text{ad}_{\exp(\epsilon_4 X_4)} \text{ad}_{\exp(\epsilon_2 X_2)}(X) = \sum_{i=1}^6 \tilde{a}_i X_i, \tag{10}$$

where the coefficient \tilde{a}_i are given as follows:

$$\begin{aligned} \tilde{a}_1 &= a_1, \quad \tilde{a}_2 = a_2, \quad \tilde{a}_3 = (a_1 \epsilon_3 + a_3) e^{-\epsilon_1}, \\ \tilde{a}_4 &= (a_1 \epsilon_4 + a_4) e^{-\epsilon_1}, \\ \tilde{a}_5 &= a_1 \epsilon_5 - e^{-\epsilon_1} a_2 \epsilon_3 + e^{-\epsilon_1} a_3 \epsilon_2 + e^{-\epsilon_1} a_5, \\ \tilde{a}_6 &= a_1 \epsilon_3 \epsilon_5 e^{-\epsilon_1} + a_1 \epsilon_6 + \frac{1}{2} a_2 \epsilon_3^2 e^{-\epsilon_1} - a_3 \epsilon_2 \epsilon_3 e^{-\epsilon_1} \\ &\quad + a_3 \epsilon_5 e^{-\epsilon_1} + a_2 \epsilon_4 e^{-\epsilon_1} \\ &\quad - a_4 \epsilon_2 e^{-\epsilon_1} - a_5 \epsilon_3 e^{-\epsilon_1} + a_6 e^{-\epsilon_1}. \end{aligned} \tag{11}$$

The first two equations in (11) agree with the general invariant function $\phi = f(a_1, a_2)$, that is, a_1 and a_2 are invariants of full adjoint action (10). Once relations (11) are written, then it becomes quite easy to construct optimal list of generators. For example, \tilde{a}_3 and \tilde{a}_4 can be made zero by taking $\epsilon_3 = -a_3/a_1$ and $\epsilon_4 = -a_4/a_1$, respectively in (11), and this process can be repeated for further simplification of the general element $X = \sum_{i=1}^6 a_i X_i$ in (3).

3.2 Construction of the one-dimensional optimal system

To construct optimal systems, we need to stick to the invariants a_1 and a_2 . The coefficients $\tilde{a}_i, i = 1, \dots, 6$ in (11) can be annihilated by choosing appropriate values for $\epsilon_i, i = 1, \dots, 6$. To begin with the classification process, we have the following four cases depending on the values of invariants a_1 and a_2 .

Case 3.2.1. When $a_1 \neq 0, a_2 \neq 0$, we set $\epsilon_1 = \epsilon_2 = 0$ (this means in (10), the adjoint actions $\text{ad}_{\exp(\epsilon_1 X_1)}, \text{ad}_{\exp(\epsilon_2 X_2)}$ are being inactivated), and on setting $\epsilon_3 = -a_3/a_1$ and $\epsilon_4 = -a_4/a_1$, the coefficients \tilde{a}_3, \tilde{a}_4 are annihilated. Further, on setting

$$\begin{aligned} \epsilon_5 &= -\frac{a_5 a_1 + a_2 a_3}{a_1^2}, \\ \epsilon_6 &= -\frac{2 a_6 a_1^2 - 2 a_2 a_4 a_1 + 2 a_5 a_3 a_1 + a_2 a_3^2}{2 a_1^3}, \end{aligned}$$

the coefficients \tilde{a}_5 and \tilde{a}_6 are also annihilated. The general element X finally reduces to $a_1 X_1 + a_2 X_2$ or $X_1 + \alpha X_2$ for $\alpha = a_2/a_1$.

Case 3.2.2. When $a_1 = 0, a_2 \neq 0$, we may take $a_2 = 1$. Setting $\epsilon_3 = \epsilon_4 = 0$ in (11), and the selections $\epsilon_2 = -a_5/a_3$ and $\epsilon_5 = -(a_3 a_6 + a_4 a_5)/a_3^2$ will annihilate the coefficients \tilde{a}_5 and \tilde{a}_6 , respectively, and the general element X reduces to $X_2 + a_3 e^{-\epsilon_1} X_3 + a_4 e^{-\epsilon_1} X_4$. The coefficient of X_3 can be scaled to ± 1 by taking $\epsilon_1 = \log |a_3|$. Final simplification shall be

$$X = X_2 \pm X_3 + \beta X_4, \quad \beta = \frac{a_4}{|a_3|}.$$

Subcase 3.2.2.1. When $a_1 = 0, a_2 \neq 0, a_3 = 0$, we may take $a_2 = 1$ in (11). We may set $\epsilon_4 = 0$, and then on taking $\epsilon_3 = a_5, \epsilon_2 = -(a_5^2 - 2 a_6)/2 a_4$, the coefficients \tilde{a}_5 and \tilde{a}_6 can be annihilated, so that X reduces to $X_2 + a_4 e^{-\epsilon_1} X_4$. The coefficient of X_4 can be scaled to ± 1 by taking $\epsilon_1 = \log |a_4|$. The final simplification shall be

$$X = X_2 \pm X_4.$$

Subcase 3.2.2.2. When $a_1 = 0, a_2 \neq 0, a_4 = 0$, we may take $a_2 = 1$ as usual. We may start by setting $\epsilon_3 = \epsilon_4 = 0$ in (11), and on taking $\epsilon_2 = -a_5/a_3$ and $\epsilon_5 = -a_6/a_3$, the coefficients \tilde{a}_5 and \tilde{a}_6 can be annihilated, so that X reduces to $X_2 + a_3 e^{-\epsilon_1} X_3$. The coefficient of X_3 can be scaled to ± 1 by taking $\epsilon_1 = \log |a_3|$. The final simplification shall be

$$X = X_2 \pm X_3.$$

Case 3.2.3. When $a_1 \neq 0, a_2 = 0$, we may take $a_1 = 1$. The coefficients \tilde{a}_3 and \tilde{a}_4 can be annihilated by taking $\epsilon_3 = -a_3$ and $\epsilon_4 = -a_4$, such that X reduce to

$$X_1 + (\epsilon_5 + a_3\epsilon_2e^{-\epsilon_1} + a_5e^{-\epsilon_1}) X_5 + (\epsilon_6 + a_3^2\epsilon_2e^{-\epsilon_1} - a_4\epsilon_2e^{-\epsilon_1} + a_5a_3e^{-\epsilon_1} + a_6e^{-\epsilon_1}) X_6.$$

Further, we may set $\epsilon_1 = \epsilon_2 = 0$, and on taking $\epsilon_5 = -a_5$ and $\epsilon_6 = -a_5a_3 - a_6$, the coefficients of X_5 and X_6 can be annihilated. The final simplification shall be $X = X_1$.

Case 3.2.4. When $a_1 = a_2 = 0$. The coefficients \tilde{a}_5 and \tilde{a}_6 can be annihilated by taking $\epsilon_2 = -a_5/a_3$ and $\epsilon_5 = -(a_3a_6 + a_4a_5)/a_3^2$. The final simplification shall be $X = a_3X_3 + a_4X_4 = X_3 + \gamma X_4$ for $\gamma = a_4/a_3$.

Subcase 3.2.4.1. When $a_1 = a_2 = a_3 = 0$. On taking $\epsilon_2 = 0$ and $\epsilon_3 = a_6/a_5$, the final simplification shall be $X = X_4 + \delta X_5$ for $\delta = a_5/a_4$.

Subcase 3.2.4.2. When $a_1 = a_2 = a_4 = 0$. On taking $\epsilon_2 = -a_5/a_3$ and $\epsilon_5 = -a_6/a_3$, the final simplification shall be $X = X_3$.

The result is a one-dimensional optimal system Θ_1 as follows:

$$\begin{aligned} &X_1 + \alpha X_2, X_2 \pm X_3 + \beta X_4, \\ &X_2 \pm X_4, X_2 \pm X_3, X_1, \\ &X_3 + \gamma X_4, X_4 + \delta X_5, X_3. \end{aligned} \tag{12}$$

3.3 Construction of two-dimensional optimal system

To construct two-dimensional optimal system Θ_2 , we follow the standard procedure given in [1]. To prepare the list of sub-algebras for Θ_2 , we need to find sub-algebra of the type $\mathcal{H}(x_i, x_j)$, where the element x_i is taken from the list of sub-algebras Θ_1 obtained at (12), and x_j belongs to the normaliser sub-algebra $\text{Nor}_{\mathfrak{g}}(x_i) = \{x \in \mathfrak{g} | [x, x_i] \in x_i\}$, that is, $[x_i, x_j] = \lambda x_i$. While selecting x_j , one can avoid the occurrence of x_i in x_j by selecting x_j from factor algebra $\text{Nor}_{\mathfrak{g}}(x_i)/x_i$, where the normaliser $\text{Nor}_{\mathfrak{g}}(x_i)$ can be obtained by setting

$$\left[x_i, \sum_{j=1}^6 a_j X_j \right] = \lambda x_i, \tag{13}$$

where $[\cdot, \cdot]$ is the usual Lie bracket and λ is an arbitrary constant. In relation (13), the coefficients of X_i can be equated to find out all possible non-zero a_i 's for the construction of $\text{Nor}_{\mathfrak{g}}(x_i)$ and hence $\text{Nor}_{\mathfrak{g}}(x_i)/x_i$. So the following list of two-dimensional sub-algebras is obtained:

$$\mathcal{H}_1(X_2 \pm X_3 + \beta X_4, a X_5 + b X_6),$$

$$\begin{aligned} &\mathcal{H}_2(X_2 \pm X_4, a X_5 + b X_6), \\ &\mathcal{H}_3(X_2 \pm X_3, -a X_4 + a X_5 + b X_6), \\ &\mathcal{H}_4(X_1, X_2), \mathcal{H}_5(X_3 + \gamma X_4, X_6), \\ &\mathcal{H}_6(X_4 + \delta X_5, -a\delta X_2 + a X_3 + b X_6), \\ &\mathcal{H}_7(X_3, a X_1 + b X_4 + c X_6). \end{aligned} \tag{14}$$

In the following, we shall try to simplify each pair in the list (14) with the adjoint action (7) as much as possible. The two elements $\{x_1, x_2\}$ and $\{x'_1, x'_2\}$ are equivalent under the adjoint action if

$$\begin{aligned} x'_1 &= k_1 \text{ad}_{\exp(\epsilon X)}(x_1) + k_2 \text{ad}_{\exp(\epsilon X)}(x_2), \\ x'_2 &= k_3 \text{ad}_{\exp(\epsilon X)}(x_1) + k_4 \text{ad}_{\exp(\epsilon X)}(x_2), \end{aligned} \tag{15}$$

where constants k_i are such that at least one of the pair (k_1, k_4) or (k_2, k_3) is non-zero and X is the general element of Lie algebra \mathfrak{g} given in (3). The following inverse version of (15) is more appropriate for the classification of the two-dimensional subalgebra.

$$\begin{aligned} \text{ad}_{\exp(\epsilon X)}(x_1) &= k_1 x'_1 + k_2 x'_2, \\ \text{ad}_{\exp(\epsilon X)}(x_2) &= k_3 x'_1 + k_4 x'_2. \end{aligned} \tag{16}$$

As an example, we consider $\mathcal{H}_7(X_3, aX_1 + bX_4 + cX_6)$ for simplification under adjoint actions. We must consider different possible values of the triplet (a, b, c) . For example, (a, b, c) : all constants are non-zero constants, $(a, b, 0)$, $(a, 0, c)$, $(0, b, c)$: one constant is zero, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$: two constants are zero.

For (a, b, c) : all constants are non-zero constants. Taking full adjoint action (10) on X_3 and $aX_1 + bX_4 + cX_6$, eqs (16) can be written as follows:

$$\begin{aligned} &e^{-\epsilon_1} (-X_6\epsilon_2\epsilon_3 + \epsilon_2X_5 + X_6\epsilon_5 + X_3) \\ &= k_1 X_3 + k_2 (a'X_1 + b'X_4 + c'X_6), \end{aligned} \tag{17a}$$

$$\begin{aligned} &aX_1 + a\epsilon_3e^{-\epsilon_1} X_3 + (a\epsilon_4e^{-\epsilon_1} + be^{-\epsilon_1}) X_4 \\ &+ a\epsilon_5X_5 + (a\epsilon_3\epsilon_5e^{-\epsilon_1} + a\epsilon_6 - b\epsilon_2e^{-\epsilon_1} + ce^{-\epsilon_1}) X_6 \\ &= k_3 X_3 + k_4 (a'X_1 + b'X_4 + c'X_6). \end{aligned} \tag{17b}$$

The coefficients of X_1, X_4 give $a' = 0$ and $b' = 0$. So the final simplification of \mathcal{H}_7 shall be $\mathcal{H}_7(X_3, X_6)$, and the simplification is the same for other cases too. Repeating this procedure for the other two-dimensional sub-algebra in (14), the final simplification is given as follows:

$$\begin{aligned} &\mathcal{G}_1(X_2 \pm X_3 + \beta X_4, X_5), \mathcal{G}_2(X_2 \pm X_3 + \beta X_4, X_6), \\ &\mathcal{G}_3(X_2 \pm X_4, X_5), \mathcal{G}_4(X_2 \pm X_4, X_6), \mathcal{G}_5(X_2 \pm X_3, X_6), \\ &\mathcal{G}_6(X_1, X_2), \mathcal{G}_7(X_3 + \gamma X_4, X_6), \\ &\mathcal{H}_9(X_4 + \delta X_5, X_6), \mathcal{H}_8(X_3, X_6). \end{aligned} \tag{18}$$

In a similar fashion, a three-dimensional optimal system can also be constructed. We have avoided writing that here due to space constraints. In the following section,

we show reductions with respect to optimal systems (14) and (18).

Remark 3.1. We may have continued for symmetry reductions of eq. (1) but that would not be fruitful as the equation is of eighth order. Its order can be reduced by one or two using one-dimensional sub-algebra (12) and two-dimensional sub-algebra (14) respectively, and even then the equation shall remain of a higher order. And unfortunately, the symmetry reductions of (1) cannot be accomplished with one- and two-dimensional Lie algebras (12) and (18) respectively, until the arbitrary functions are set to polynomials.

4. Painlevé analysis

In Remark 1, the non-existence of Virasoro algebra suggests that the Kac–Wakimoto equation may not be integrable, and with the Painlevé analysis we try to strengthen this claim. The Painlevé analysis is a well-established tool for investigating complete integrability of nonlinear partial differential equations. Looking at the singularity structure of the equation, one can predict the complete integrability (for details, see refs [30–32]). In this section, we investigate the singularity structure of eq. (1) based on the algorithm suggested by Weiss–Tabor–Carnevale. The Laurent expansion can be taken in the following form:

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j, \quad \text{where } \phi = \phi(x, y, z, t). \quad (19)$$

When (19) is substituted into (1), the following conditions need to be satisfied for the Painlevé property to be valid.

1. α should be an integer.
2. ϕ should be an analytic function of (x, y, z, t) .
3. The equations for u_j should have self-consistent solutions.

The quick survey of leading-order analysis gives

First branch: $\alpha = -1, u_0 = 2\phi_x,$

Second branch: $\alpha = -1, u_0 = 4\phi_x,$

Third branch: $\alpha = -1, u_0 = 6\phi_x.$

The resonant points can be determined by substituting

$$u = u_0 \phi^{-1} + u_j \phi^{j-1}, \quad (20)$$

into (1) and on retaining the most singular part, the resonance points at each branch are obtained as follows:

First branch: $j = -1, 1, 2, 3, 4, 5, 8, 14,$

Second branch: $j = -2, -1, 1, 2, 3, 8,$

$$\frac{25 + \sqrt{65}}{2}, \frac{25 - \sqrt{65}}{2},$$

Third branch: $j = -3, -2, -1, 1, 8, 10,$

$$\frac{23 + \sqrt{193}}{2}, \frac{23 - \sqrt{193}}{2}.$$

The non-integral resonance points at the second and third branches suggest that the Kac–Wakimoto equation fails to pass the Painlevé property, and therefore cannot be integrable.

5. Conclusion

The eighth-order $(3 + 1)$ -dimensional Kac–Wakimoto equation is studied for its integrability. This equation is already shown to be non-integrable using Hirota’s bilinear method as it does not have three-soliton solutions [18]. In this paper, the non-integrability of the equation is checked from different angles, such as the infinite-dimensional Lie algebra for Kac–Wakimoto equation does not have Virasoro-like structure, which in fact is the common characteristic of most of the integrable equations, and absence of Virasoro-like structure implies that the equation may be non-integrable. In order to further strengthen the argument for the non-integrability of equation, the Painlevé property is also checked, which is also turned out to be negative.

It is important to note that there are some integrable systems that are invariant under finite- or infinite-dimensional Lie algebra without having Virasoro-like structure (see ref. [26]), but all non-integrable systems that are invariant finite- or infinite-dimensional Lie algebra do not have Virasoro-like structure. In this way, the presence of Virasoro-like structure can be a weak predictor of integrability and it can be used along with other strong predictors of integrability.

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