Pramana – J. Phys. (2022) 96:199
https://doi.org/10.1007/s12043-022-02436-6

Non-local symmetries, exact solutions and conservation laws for the coupled Lakshmanan–Porsezian–Daniel equations

FENG ZHANG, YURU HU, XIANGPENG XIN* and HANZE LIU

School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, People's Republic of China
*Corresponding author. E-mail: xinxiangpeng2020@163.com

MS received 11 March 2022; revised 19 May 2022; accepted 7 June 2022

Abstract. The non-local symmetries of the coupled Lakshmanan–Porsezian–Daniel (LPD) equations are obtained with the help of the known Lax pair. By introducing an auxiliary variable, the coupled LPD equations are extended to a closed prolonged system and the non-local symmetries are localised to the Lie point symmetries of the prolonged system. Furthermore, based on the Lie point symmetries of the prolonged system, the exact solutions and non-local conservation laws of the coupled LPD equations are derived.

Keywords. Non-local symmetry; Lie point symmetry; exact solutions; non-local conservation laws.

PACS Nos 05.45.Yv; 02.30.Jr

1. Introduction

The discipline of nonlinear equations has gained considerable attention and importance over time, because nonlinear equations have many applications in many different fields, such as science and engineering [1–3]. In addition, the study of nonlinear evolution equations has received extensive attention because of its ability to accurately describe many nonlinear phenomena [4–6]. There have been many classical methods to obtain exact solutions of nonlinear evolution equations in the past decades, such as inverse scattering method [7], Darboux transformation method [8,9], bilinear method [10,11], etc.

The symmetry provides an efficient, powerful and systematic method for solving exact solutions of nonlinear partial differential equations (PDEs). Lie symmetry is one of the more classic of the various symmetry methods. Lie symmetry is a method that only involves point transformations of differential equations, i.e., the infinitesimal functions of the symmetry contain only independent and dependent variables and this symmetry is usually called Lie point symmetry which belongs to the local symmetry [12–15]. Furthermore, when the infinitesimal functions contain non-local variables, i.e., the relevant variables cannot be expressed as functions of the dependent variables and the finite-order derivatives of the dependent variables, this symmetry is called non-local symmetry [16,17]. Compared with the Lie point symmetry, the acquisition and application of non-local symmetry are more difficult and complicated. However, it often leads to some novel exact solutions that are not available through Lie point symmetry in general [18]. Therefore, the non-local symmetry is gradually attracting more and more attention.

The non-local potential symmetry is a type of non-local symmetry that can be constructed based on the Lax pair of nonlinear differential system. This symmetry consists of the potential functions of the Lax pair, i.e., the infinitesimal generators depend on the potential functions. In addition, the non-local symmetry is localised by introducing necessary auxiliary variables and finding the closed prolonged system of nonlinear differential system, i.e., the non-local symmetry in the low-dimensional space is equated to the Lie point symmetry in the high-dimensional space. Due to the arbitrariness of λ in the Lax pair, the principle of linear superposition can be further used to construct infinite number of non-local symmetries of nonlinear differential system and then extend the single symmetry to the N symmetry transformations. Based on the finite symmetry transformation, more different types of exact solutions for nonlinear differential system can be constructed. The above approach has been further developed in [19–21] and many important nonlinear mathematical–physical models have been studied, which lay the foundation for better application of non-local symmetry.

Published online: 08 October 2022

In this paper, we study the coupled Lakshmanan–Porsezian–Daniel (LPD) [22] equations that are

$$\begin{aligned} & iq_t-\alpha\left(\frac{1}{2}q_{xx}-q^2r\right) \\ & \quad -\beta(q_{xxxx}-8qq_{xx}r-2q^2r_{xx}-4qq_xr_x \\ & \quad -6q_x^2r+6q^3r^2)=0, \\ & ir_t+\alpha\left(\frac{1}{2}r_{xx}-r^2q\right) \\ & \quad +\beta(r_{xxxx}-8rr_{xx}q-2r^2q_{xx}-4rr_xq_x \\ & \quad -6r_x^2q+6r^3q^2)=0, \end{aligned} \tag{1.1}$$

where $q=q(x,t)$ and $r=r(x,t)$ are complex fields. In optical fibres, x and t denote the propagation distance and the retarded time coordinate, respectively, q, r are the slow varying electronic-field envelopes and parameter β denotes the strength of the higher-order nonlinear effect [23]. When $r=-q^*$, eqs (1.1) transform into the famous LPD equation

$$\begin{aligned} & iq_t+\alpha\left(\frac{1}{2}q_{xx}+|q|^2q\right) \\ & \quad +\beta(q_{xxxx}+8|q|^2q_{xx}+2q^2q^*_{xx}+4q|q_x|^2 \\ & \quad +6q_x^2q^*+6q|q|^4)=0, \end{aligned} \tag{1.2}$$

where q^* denotes the complex conjugate of q. The LPD equation was originally identified in connection with the dynamics of the limit of the Heisenberg ferromagnetic spin chain. The effect of biquadratic interactions on the integrable properties of Heisenberg bilinear spin chains under the classical limit is a hot topic of significant research. Lakshmanan, Porsezian and Daniel studied the integrable properties of the classical one-dimensional isotropic biquadratic Heisenberg spin chain in its continuum limit using the geometric method and discussed the integrable properties of the LPD equation [24,25]. The LPD equation is a nonlinear Schrödinger-type equation by adding higher-order nonlinear terms to the nonlinear Schrödinger equation, which has spatio-temporal dispersion, full nonlinearity and higher-order dispersion. The LPD equation, which is a complex nonlinear Schrödinger model, plays an important role in nonlinear optics and photonics [26,27]. The study of the exact solutions of the LPD equation contributes to the development of nonlinear fibre optics and optical communications. In the study of symmetry of the LPD equation, Bansal *et al* [28] studied the Lie point symmetry and group invariant solutions of a single uncoupled LPD equation. In addition, the correlated non-local potential system can be constructed through the conservation law form of a single uncoupled LPD equation and then the Lie point symmetry of the whole system can be studied to eventually find its non-local symmetry. After reviewing the literature, the authors found that no one has studied the non-local symmetry of the coupled LPD equations.

The outline of this article is as follows: In §2, the non-local symmetries of the coupled LPD equations are derived on the basis of the Lax pair. In §3, an auxiliary variable is introduced to extend the coupled LPD equations into a prolonged system while transforming the non-local symmetries into Lie point symmetries. Solving the initial value problem yields the finite symmetry transformation. In §4, some exact solutions of the coupled LPD equations are obtained by using Lie point symmetries and finite symmetry transformation of the prolonged system. In §5, the non-local conservation laws of the coupled LPD equations are derived. Finally, some conclusions are given in §6.

2. Non-local symmetries of the coupled LPD equations

In this section, we are going to explore the non-local symmetries of the coupled LPD equations based on their Lax pair. The Lax pair of eqs (1.1) has been derived in [24,29] as

$$\Phi_x=U\Phi,\quad \Phi_t=V\Phi, \tag{2.1}$$

where $\Phi=(\varphi(x,t),\psi(x,t))^T$ and U, V are the matrices determined by q, r with spectral parameter λ,

$$U=\begin{pmatrix} i\lambda & q \\ r & -i\lambda \end{pmatrix},\quad V=\begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{2.2}$$

with

$$\begin{aligned} A= & -8i\beta\lambda^4+i(\alpha-4\beta qr)\lambda^2 \\ & +2\beta(qr_x-rq_x)\lambda+i\beta(qr_{xx}+rq_{xx} \\ & -r_xq_x-3q^2r^2)+\frac{1}{2}i\alpha qr, \\ B= & -8\beta q\lambda^3+4i\beta q_x\lambda^2 \\ & +(\alpha q+2\beta q_{xx}-4\beta rq^2)\lambda+i\beta(6qrq_x-q_{xxx}) \\ & -\frac{1}{2}i\alpha q_x, \\ C= & -8\beta r\lambda^3-4i\beta r_x\lambda^2 \\ & +(\alpha r+2\beta r_{xx}-4\beta r^2q)\lambda+i\beta(r_{xxx}-6rr_xq) \\ & +\frac{1}{2}i\alpha r_x. \end{aligned} \tag{2.3}$$

The coupled LPD equations (1.1) can be obtained by the zero-curvature equation $U_t-V_x+[U,V]=0$. We shall use a direct algebraic approach to derive the non-local symmetries of eqs (1.1) in the following. First, we define the symmetries of q, r to be σ_1, σ_2 and consider the invariant property of coupled LPD equations under the infinitesimal transformations

$$q \to q + \varepsilon\sigma_1, \quad r \to r + \varepsilon\sigma_2, \tag{2.4}$$

where ε is an infinitesimal parameter. Therefore, σ_1, σ_2 should be the solutions of the linearised equations

$$\begin{aligned}
&\sigma_{1,t} + i\alpha\left(\frac{1}{2}\sigma_{1,xx} - 2qr\sigma_1 - q^2\sigma_2\right)\\
&\quad +i\beta\big(\sigma_{1,xxxx} - 2(q^2\sigma_{2,xx} + 2qr_{xx}\sigma_1)\\
&\quad -6(q_x^2\sigma_2 + 2q_xr\sigma_{1,x}) - 4\big(r_xq_x\sigma_1\\
&\quad +qr_x\sigma_{1,x} + q\sigma_{2,x}q_x\big)\\
&\quad -8\left(rq_{xx}\sigma_1 + qr\sigma_{1,xx} + qq_{xx}\sigma_2\right)\\
&\quad +6q^2(3r^2\sigma_1 + 2qr\sigma_2)\big) = 0,\\
&\sigma_{2,t} - i\alpha\left(\frac{1}{2}\sigma_{2,xx} - 2qr\sigma_2 - r^2\sigma_1\right)\\
&\quad -i\beta\big(\sigma_{2,xxxx} - 2(r^2\sigma_{1,xx} + 2rq_{xx}\sigma_2)\\
&\quad -6(r_x^2\sigma_1 + 2r_xq\sigma_{2,x})\\
&\quad -4\left(r_xq_x\sigma_2 + rr_x\sigma_{1,x} + rq_x\sigma_{2,x}\right)\\
&\quad -8\left(rr_{xx}\sigma_1 + rq\sigma_{2,xx} + r_{xx}q\sigma_2\right)\\
&\quad +6r^2(3q^2\sigma_2 + 2qr\sigma_1)\big) = 0
\end{aligned} \tag{2.5}$$

and the form of symmetries σ_1, σ_2 can be defined as

$$\begin{aligned}
\sigma_1 &= T(t,x,q,r,\varphi,\psi)\,q_t + X(t,x,q,r,\varphi,\psi)\,q_x\\
&\quad - Q(t,x,q,r,\varphi,\psi),\\
\sigma_2 &= T(t,x,q,r,\varphi,\psi)\,r_t + X(t,x,q,r,\varphi,\psi)\,r_x\\
&\quad - R(t,x,q,r,\varphi,\psi).
\end{aligned} \tag{2.6}$$

Unlike the local symmetries, T, X, Q, R are determined functions to be defined by $(t, x, q, r, \varphi, \psi)$. We substitute eqs (2.6) into the linearised equations (2.5), using eqs (1.1) and its Lax pair (2.1) to eliminate the terms are $q_t, r_t, \varphi_x, \varphi_t, \psi_x, \psi_t$ and subsequently collect the coefficients of the equivalent derivative terms of q, r to obtain a system of determining equations for T, X, Q, R. Solving this system yields

$$\begin{aligned}
&X = C_3, \quad T = C_4, \quad R = C_2\psi^2 - C_1r,\\
&Q = C_2\varphi^2 + C_1q,
\end{aligned} \tag{2.7}$$

where C_i $(i = 1, 2, 3, 4)$ are arbitrary constants. Observing (2.7), we can find that σ_1, σ_2 are non-local symmetries of the coupled LPD equations when $C_2 \neq 0$.

3. Localisation of the non-local symmetries

In our perception, Lie point symmetries can be used directly to reduce the dimensionality of PDEs and then obtain the group invariant solutions. However, it is hard for non-local symmetries [30]. Because of this fact, we have to transform the non-local symmetries into local ones, especially the Lie point symmetries. According to this idea, we extend the coupled LPD equations to a closed prolonged system by introducing an auxiliary variable and localising the non-local symmetries σ_1, σ_2 of the coupled LPD equations to the Lie point symmetries of the prolonged system [31,32].

To alleviate the complexity of the calculation, let $C_2 = 1, C_1 = 0, C_3 = 0, C_4 = 0$ to simplify the non-local symmetries to

$$\sigma_1 = -\varphi^2, \quad \sigma_2 = -\psi^2, \tag{3.1}$$

and in order to achieve the localisation of the simplified non-local symmetries (3.1), we introduce the infinitesimal transformations

$$\varphi \to \varphi + \varepsilon\sigma_3, \quad \psi \to \psi + \varepsilon\sigma_4, \quad f \to f + \varepsilon\sigma_5. \tag{3.2}$$

It is considered that the invariant property of the Lax pair of eqs (1.1) under the infinitesimal transformations (3.2), σ_1 and σ_2 satisfying the linearised equations

$$\begin{aligned}
&\sigma_{3,x} - i\lambda\sigma_3 - \sigma_1\psi - q\sigma_4 = 0,\\
&\sigma_{4,x} + i\lambda\sigma_4 - \sigma_2\varphi - r\sigma_3 = 0,
\end{aligned} \tag{3.3}$$

where $\sigma_1 = -\varphi^2, \sigma_2 = -\psi^2$ and σ_3, σ_4 have the solution of the following form:

$$\sigma_3 = \varphi f, \quad \sigma_4 = \psi f, \tag{3.4}$$

with f determined by

$$\begin{aligned}
f_x &= -\varphi\psi,\\
f_t &= i(\varphi^2 r - \psi^2 q)(2\beta qr + (12\beta\lambda^2 - \frac{1}{2}\alpha))\\
&\quad + 2i\beta\psi\varphi\,(r_xq - rq_x)\\
&\quad - 4\beta\lambda(r_x\varphi^2 + \psi^2q_x) + 8\lambda\beta\psi\varphi(rq + 4\lambda^2)\\
&\quad + i\beta(\psi^2q_{xx} - r_{xx}\varphi^2) - 2\alpha\lambda\varphi\psi,
\end{aligned} \tag{3.5}$$

and it is not difficult to verify that the symmetry associated with f is

$$\sigma_5 = \sigma_f = f^2. \tag{3.6}$$

The above processes show that we have successfully localised the simplified non-local symmetries (3.1) in the original space $\{t, x, q, r\}$ to the Lie point symmetries on the prolonged space $\{t, x, q, r, \varphi, \psi, f\}$ with the help of the auxiliary function f. The Lie point symmetries of the prolonged systems (1.1), (2.1) and (3.5) are

$$\begin{aligned}
&\sigma_1 = -\varphi^2, \quad \sigma_2 = -\psi^2, \quad \sigma_3 = \varphi f,\\
&\sigma_4 = \psi f, \quad \sigma_5 = f^2
\end{aligned} \tag{3.7}$$

and the corresponding vector form is

$$V_1 = -\varphi^2\frac{\partial}{\partial q} - \psi^2\frac{\partial}{\partial r} + \varphi f\frac{\partial}{\partial\varphi} + \psi f\frac{\partial}{\partial\psi} + f^2\frac{\partial}{\partial f}. \tag{3.8}$$

In the following, we use the theory related to Lie point symmetry analysis to solve the group invariant solutions of the prolonged systems (1.1), (2.1) and (3.5) to obtain exact solutions of the coupled LPD equations. Solving the initial value problem

$$\frac{\mathrm{d}\bar{q}}{\mathrm{d}\varepsilon}=-\varphi^2,\ \bar{q}\,|_{\varepsilon=0}=q,\quad \frac{\mathrm{d}\bar{r}}{\mathrm{d}\varepsilon}=-\psi^2,\ \bar{r}\,|_{\varepsilon=0}=r,$$
$$\frac{\mathrm{d}\bar{\varphi}}{\mathrm{d}\varepsilon}=\varphi f,\ \bar{\varphi}\,|_{\varepsilon=0}=\varphi,\quad \frac{\mathrm{d}\bar{\psi}}{\mathrm{d}\varepsilon}=\psi f,\ \bar{\psi}\,|_{\varepsilon=0}=\psi,$$
$$\frac{\mathrm{d}\bar{f}}{\mathrm{d}\varepsilon}=f^2,\ \bar{f}\,|_{\varepsilon=0}=f, \tag{3.9}$$

the finite symmetry transformation is constructed as

$$\bar{q}=\frac{\varepsilon f q+\varepsilon\varphi^2-q}{\varepsilon f-1},\qquad \bar{r}=\frac{\varepsilon f r+\varepsilon\psi^2-r}{\varepsilon f-1},$$
$$\bar{\varphi}=-\frac{\varphi}{\varepsilon f-1},\ \ \bar{\psi}=-\frac{\psi}{\varepsilon f-1},\ \ \bar{f}=-\frac{f}{\varepsilon f-1}, \tag{3.10}$$

where ε is an infinitesimal parameter. For a known solution (q, r, φ, ψ, f), a new solution $(\bar{q}, \bar{r}, \bar{\varphi}, \bar{\psi}, \bar{f})$ can be obtained by the finite symmetry transformation (3.10). Based on the classical Lie symmetry method, the vector of the symmetry of the prolonged systems (1.1), (2.1) and (3.5) has the form is

$$\bar{V}=T\frac{\partial}{\partial t}+X\frac{\partial}{\partial x}+Q\frac{\partial}{\partial q}+R\frac{\partial}{\partial r}+A\frac{\partial}{\partial\varphi}$$
$$+B\frac{\partial}{\partial\psi}+F\frac{\partial}{\partial f}, \tag{3.11}$$

where T, X, Q, R, A, B, F are determined by $t, x, q, r, \varphi, \psi, f$ implying that prolonged systems (1.1), (2.1) and (3.5) are invariant under infinitesimal transformations

$$(t, x, q, r, \varphi, \psi, f)$$
$$\rightarrow(t+\varepsilon T, x+\varepsilon X, q+\varepsilon Q, r+\varepsilon R, \varphi$$
$$+\varepsilon A, \psi+\varepsilon B, f+\varepsilon F), \tag{3.12}$$

with

$$\sigma_1=Tq_t+Xq_x-Q,\quad \sigma_2=Tr_t+Xr_x-R,$$
$$\sigma_3=T\varphi_t+X\varphi_x-A,\quad \sigma_4=T\psi_t+X\psi_x-B,$$
$$\sigma_5=Tf_t+Xf_x-F \tag{3.13}$$

and σ_i $(i=1,\ldots,5)$ are the solutions of the linearised equations (2.5), (3.3) and

$$\sigma_{5,x}+2\varphi f\psi=0,$$
$$\sigma_{5,t}+if(24\beta\lambda^2-\alpha+4\beta rq)(\psi^2 q-r\varphi^2)$$
$$+8\beta\lambda\psi\varphi(r\varphi^2+\psi^2 q-\psi\varphi_x-\psi_x\varphi)$$
$$+2i\beta(\varphi^3 r_x\psi-\psi^3\varphi q_x+\varphi_x^2\psi^2-\psi_x^2\varphi^2+\varphi^4 r^2$$
$$-\psi^4 q^2+\varphi\psi\,(\varphi_{xx}\psi-\psi_{xx}\varphi))$$
$$+4i\beta\varphi\psi\,(\psi\psi_x q-r\varphi\varphi_x+f\,(rq_x-r_x q))$$
$$+4\varphi\psi\lambda f(\alpha-16\beta\lambda^2-4\beta rq)$$
$$+2i\beta f(r_{xx}\varphi^2-\psi^2 q_{xx})$$
$$+8\beta\lambda f(r_x\varphi^2+\psi^2 q_x)=0. \tag{3.14}$$

Substituting eqs (3.13) into the linearised equations (2.5), (3.3), (3.14), using the prolonged systems (1.1), (2.1), (3.5) to eliminate the terms $q_t, r_t, \varphi_x, \varphi_t, \psi_x, \psi_t, f_x, f_t$ and collecting the coefficients of the equivalent derivative terms of q, r, φ, ψ, f, we are able to obtain overdetermined equations with respect to T, X, Q, R, A, B, F. By solving this system, we get

$$T=k_3,\quad X=k_4,\quad Q=k_2\varphi^2-k_1 q,$$
$$R=k_2\psi^2+k_1 r,$$
$$A=-k_2 f\varphi+\frac{1}{2}\varphi\,(k_5-k_1)\,,$$
$$B=-k_2 f\psi+\frac{1}{2}\psi\,(k_1+k_5)\,,$$
$$F=-k_2 f^2+k_5 f+k_6, \tag{3.15}$$

where k_i $(i=1,\ldots,6)$ are constants. When we take $k_2=1, k_j=0\,(j=1,3,4,5,6)$, symmetries (3.13) are equivalent to symmetries (3.7).

4. Similarity reductions and exact solutions of the coupled LPD equations

In this section, our task is to solve the group invariant solutions of the coupled LPD equations. First, we consider the whole prolonged system and use similarity reduction [33,34] to obtain the group invariant solutions of the prolonged systems (1.1), (2.1) and (3.5). Obviously, the group invariant solutions of the prolonged systems contain the solutions of the coupled LPD equations. For computational simplicity, we just take $k_1=k_2=1, k_4=\gamma, k_3=k_5=k_6=0$, as an example and other cases can use the same method to derive the solutions of the coupled LPD equations.

In this case, the corresponding characteristic equation is

$$\frac{\mathrm{d}x}{\gamma}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}q}{\varphi^2-q}=\frac{\mathrm{d}r}{\psi^2+r}=\frac{\mathrm{d}\varphi}{-\,(2f+1)\,\varphi/2}$$
$$=\frac{\mathrm{d}\psi}{-\,(2f-1)\,\psi/2}=\frac{\mathrm{d}f}{-f^2}, \tag{4.1}$$

and the similarity variables obtained by solving this equation are

$$q=\left(-\frac{G_2(t)^2}{\gamma\,(\gamma G_1(t)+x)}+G_4(t)\right)\mathrm{e}^{-x/\gamma},$$
$$r=\left(-\frac{G_3(t)^2}{\gamma\,(\gamma G_1(t)+x)}+G_5(t)\right)\mathrm{e}^{x/\gamma},$$

$$\varphi = \frac{G_2(t)}{\gamma G_1(t) + x} \mathrm{e}^{-x/2\gamma}, \quad \psi = \frac{G_3(t)}{\gamma G_1(t) + x} \mathrm{e}^{x/2\gamma},$$
$$f = \frac{\gamma}{\gamma G_1(t) + x}, \tag{4.2}$$

where γ is an arbitrary constant, $G_j(t)\,(j = 1, \ldots, 5)$ are any functions with respect to t. Substituting eqs (4.2) into the prolonged systems (1.1), (2.1) and (3.5), we obtain

$$G_1(t) = \frac{5i\beta t}{2\gamma^4} - \frac{3\beta\lambda t}{\gamma^3} + (\alpha - 12\beta\lambda^2) \times \left(\frac{i}{2\gamma^2} + \frac{\lambda}{\gamma}\right) t + l_2,$$
$$G_2(t) = l_1 \mathrm{e}^{(1/2)\Delta}, \quad G_3(t) = \frac{\gamma}{l_1} \mathrm{e}^{-(1/2)\Delta},$$
$$G_4(t) = -\frac{l_1^2(2i\gamma\lambda + 1)}{2\gamma^2} \mathrm{e}^{\Delta},$$
$$G_5(t) = \frac{2i\gamma\lambda + 1}{2l_1^2} \mathrm{e}^{-\Delta}, \tag{4.3}$$

where

$$\Delta = i\lambda^2(\alpha - 6\beta\lambda^2)t + \frac{\lambda(\alpha - 12\beta\lambda^2)t}{\gamma} + \frac{3i(26\beta\lambda^2 - \alpha)t}{4\gamma^2} + \frac{15\beta\lambda t}{\gamma^3} - \frac{35i\beta t}{8\gamma^4} \tag{4.4}$$

and $l_j\,(j = 1, 2)$ are any constants. Substituting eqs (4.3) into the similarity variables (4.2), the group invariant solutions of the prolonged systems are

$$q = -\frac{1}{2}\frac{l_1^2\Psi}{\gamma^2\Theta} \mathrm{e}^{\Delta - (x/\gamma)}, \quad r = \frac{1}{2}\frac{\Psi}{l_1^2\Theta} \mathrm{e}^{(x/\gamma) - \Delta},$$
$$\varphi = \frac{2l_1\gamma^3}{\Theta} \mathrm{e}^{\frac{\Delta}{2} - \frac{x}{2\gamma}}, \quad \psi = \frac{2\gamma^4}{l_1\Theta} \mathrm{e}^{\frac{x}{2\gamma} - \frac{\Delta}{2}}, \quad f = \frac{2\gamma^4}{\Theta}, \tag{4.5}$$

with

$$\Theta = 2l_2\gamma^4 + \gamma^2(-12i\beta\lambda^2 t + i\alpha t) + \gamma^3(-24\beta\lambda^3 t + 2\alpha\lambda t + 2x) - 6\gamma\beta\lambda t + 5i\beta t,$$
$$\Psi = 2\gamma^3 x + \gamma^4(-48i\beta\lambda^4 t + 4i\alpha\lambda^2 t + 4i\lambda x + 2l_2 + 4) + \gamma^2(-24i\beta\lambda^2 t + i\alpha t) - 16\gamma\beta\lambda t + 5i\beta t + 4i\gamma^5 l_2\lambda. \tag{4.6}$$

In §3, we derived the finite symmetry transformation (3.10) of the prolonged system. Now, we take solutions (4.5) as seed solutions and through finite symmetry transformations (3.10), we get the completely new solutions of the coupled LPD equations as

$$q = \frac{1}{2}\frac{l_1^2(8\varepsilon\gamma^8 - 2\Psi\varepsilon\gamma^4 + \Theta\Psi)}{\Theta\gamma^2\left(2\varepsilon\gamma^4 - \Theta\right)} \mathrm{e}^{\frac{\Delta\gamma - x}{\gamma}}, \tag{4.7}$$

$$r = \frac{1}{2}\frac{(8\varepsilon\gamma^8 + 2\Psi\varepsilon\gamma^4 - \Psi\Theta)}{\Theta l_1^2\left(2\varepsilon\gamma^4 - \Theta\right)} \mathrm{e}^{-\frac{\Delta\gamma - x}{\gamma}}, \tag{4.8}$$

where ε is an infinitesimal parameter. We can also obtain a series of new exact solutions for the coupled LPD equations in this way. The exact solutions of the single uncoupled LPD equation (1.2) can be obtained by doing the transformation $r = -q^*$ on the exact solutions obtained above. The acquisition of new exact solutions of the coupled LPD equations is significant for studying optical solitons and propagating solitons in optical fibres.

5. Non-local conservation laws of the coupled LPD equations

The conservation laws of all physical systems are extremely significant in describing their dynamics. These conservation laws can provide us with deeper insight into the physical meaning of the systems [35–37]. In this section, we derive the non-local conservation laws for the coupled LPD equations. First, we recall the relevant definitions and theorems. We consider a system that consists of n partial differential equations

$$F_a\left(x, q, q_{(1)}, \ldots, q_{(s)}\right) = 0, \; a = 1, \ldots, n, \tag{5.1}$$

with $x = (x^1, \ldots, x^m)$ are m independent variables and $q_{(i)} = D_i(q)$ is the ith-order partial derivative.

Definition 1 [38]

The adjoint system of system (5.1) is defined by

$$F_a^*\left(x, q, r, q_{(1)}, r_{(1)}, \ldots, q_{(s)}, r_{(s)}\right) = \frac{\delta L}{\delta q^a} = 0,$$
$$a = 1, \ldots, n, \tag{5.2}$$

where L is the formal Lagrangian given as

$$L = \sum_{b=1}^{n} r^b F_b\left(x, q, q_{(1)}, \ldots, q_{(s)}\right) \tag{5.3}$$

and $\delta/\delta q^a$ is the variational derivative given as

$$\frac{\delta}{\delta q^a} = \frac{\partial}{\partial q^a} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial q^a_{i_1 \cdots i_s}}. \tag{5.4}$$

Theorem 1 [39]. *For any constant λ_a^b, if adjoint system (5.2) of system (5.1) satisfies the condition*

$$F_a^*\left(x, q, r, q_{(1)}, r_{(1)}, \ldots, q_{(s)}, r_{(s)}\right) = \lambda_a^b F_b\left(x, q, \ldots, q_{(s)}\right), \; a = 1, \ldots, n, \tag{5.5}$$

then system (5.1) is nonlinearly self-adjoint.

Theorem 2 [38]. *For all infinitesimal symmetries*

$$V = \xi_i \left(x, q, q_{(1)}, \dots\right) \frac{\partial}{\partial x^i} + \eta_a \left(x, q, q_{(1)}, \dots\right) \frac{\partial}{\partial q^a}, \tag{5.6}$$

of system (5.1) *provides us a conservation law* $D_i(C^i) = 0$, *which is constructed by*

$$\begin{aligned} C^i = {} & \xi_i L + W^a \left(\frac{\partial}{\partial q_i^a} - D_j \left(\frac{\partial L}{\partial q_{ij}^a} \right) + D_j D_k \left(\frac{\partial L}{\partial q_{ijk}^a} \right) - \cdots \right) \\ & + D_j \left(W^a\right) \left(\frac{\partial L}{\partial q_{ij}^a} - D_k \left(\frac{\partial L}{\partial q_{ijk}^a} \right) + \cdots \right) \\ & + D_j D_k \left(W^a\right) \left(\frac{\partial L}{\partial q_{ijk}^a} - \cdots \right), \end{aligned} \tag{5.7}$$

where $W^a = \eta_a - \xi_j q_j^a$ *and* L *is the formal Lagrangian* (5.3).

Based on the above definitions and theorems, we derive the non-local conservation laws of the coupled LPD equations. First, we consider the whole prolonged systems (1.1), (2.1) and (3.5). According to eq. (5.6), eqs (3.15) can be rewritten as

$$\begin{aligned} & \xi^1 = k_3, \quad \xi^2 = k_4, \quad \eta^1 = k_2 \varphi^2 - k_1 q, \\ & \eta^2 = k_2 \psi^2 + k_1 r, \\ & \eta^3 = -k_2 f \varphi + \frac{1}{2} \varphi \left(k_5 - k_1\right), \\ & \eta^4 = -k_2 f \psi + \frac{1}{2} \psi \left(k_1 + k_5\right), \\ & \eta^5 = -k_2 f^2 + k_5 f + k_6, \end{aligned} \tag{5.8}$$

and the formal Lagrangian (5.3) is

$$\begin{aligned} L = {} & \tilde{q}\Big(q_t - \alpha \left(i q^2 r - \frac{1}{2} i q_{xx} \right) - \beta(-i q_{xxxx} \\ & + 8 i q q_{xx} r + 2 i q^2 r_{xx} + 4 i q q_x r_x \\ & + 6 i q_x^2 r - 6 i q^3 r^2)\Big) \\ & + \tilde{r}\Big(r_t - \alpha \left(\frac{1}{2} i r_{xx} - i r^2 q \right) + \beta(i r_{xxxx} \\ & - 8 i r r_{xx} q - 2 i r^2 q_{xx} - 4 i r r_x q_x \\ & - 6 i r_x^2 q + 6 i r^3 q^2)\Big) \\ & + \tilde{\varphi}_1 \left(\varphi_x - i \lambda \varphi - q \psi\right) + \tilde{\varphi}_2 \left(\varphi_t - A \varphi - B \psi\right) \\ & + \tilde{\psi}_1 \left(\psi_x - r \varphi + i \lambda \psi\right) + \tilde{\psi}_2 \left(\psi_t - C \varphi + A \psi\right) \\ & + \tilde{f}_1 \left(f_x + \varphi \psi\right) + \tilde{f}_2 \Big(f_t - i \left(\varphi^2 r - \psi^2 q\right) \\ & \times \left(2 \beta q r + \left(12 \beta \lambda^2 - \frac{1}{2} \alpha \right) \right) - 2 i \beta \psi \varphi \left(r_x q - r q_x\right) \\ & + 4 \beta \lambda (r_x \varphi^2 + \psi^2 q_x) - 16 \lambda \beta \psi \varphi \left(\frac{1}{2} r q + 2 \lambda^2 \right) \\ & - i \beta (\psi^2 q_{xx} - r_{xx} \varphi^2) + 2 \alpha \lambda \varphi \psi \Big), \end{aligned} \tag{5.9}$$

where A, B, C can be found in Lax pair (2.1) and $\tilde{q}, \tilde{r}, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\psi}_1, \tilde{\psi}_2, \tilde{f}_1, \tilde{f}_2$ are new independent variables determined by $(t, x, q, r, \varphi, \psi, f)$. Based on Theorem 1, the values of $\tilde{q}, \tilde{r}, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\psi}_1, \tilde{\psi}_2, \tilde{f}_1, \tilde{f}_2$ are

$$\begin{aligned} & \tilde{q} = m r + h \psi^2, \quad \tilde{r} = m q - h \varphi^2, \quad \tilde{\varphi}_1 = -\frac{\partial h}{\partial t} \psi, \\ & \tilde{\varphi}_2 = \frac{\partial h}{\partial x} \psi, \quad \tilde{\psi}_1 = \frac{\partial h}{\partial t} \varphi, \quad \tilde{\psi}_2 = -\frac{\partial h}{\partial x} \varphi, \\ & \tilde{f}_1 = 0, \quad \tilde{f}_2 = 0, \end{aligned} \tag{5.10}$$

where m is any constant and $h = h\,(x, t)$ is an arbitrary function about x and t. For computational simplicity, let $m = 1, h = 1$. Thus, the formal Lagrangian (5.9) can be rewritten as

$$\begin{aligned} L = {} & (r + \psi^2) \Big(q_t - \alpha \left(i q^2 r - \frac{1}{2} i q_{xx} \right) \\ & - \beta(-i q_{xxxx} + 8 i q q_{xx} r + 2 i q^2 r_{xx} \\ & + 4 i q q_x r_x + 6 i q_x^2 r - 6 i q^3 r^2) \Big) \\ & + (q - \varphi^2) \Big(r_t - \alpha \left(\frac{1}{2} i r_{xx} - i r^2 q \right) \\ & + \beta(i r_{xxxx} - 8 i r r_{xx} q - 2 i r^2 q_{xx} \\ & - 4 i r r_x q_x - 6 i r_x^2 q + 6 i r^3 q^2) \Big). \end{aligned} \tag{5.11}$$

By applying Theorem 2, we can obtain the conservation laws for the coupled LPD equations as

$$\begin{aligned} C^t = {} & 6 i k_3 \beta (\varphi^2 q (r^3 q - r_x^2) + \psi^2 r (r q^3 - q_x^2) \\ & + q^2 (r r_{xx} + r_x^2) - r^2 (q q_{xx} + q_x^2)) \\ & - 8 i k_3 \beta q r (\psi^2 q_{xx} + \varphi^2 r_{xx}) \\ & + (k_2 - k_1 - 4 i k_3 \beta q_x r_x - i \alpha k_3 q r)(\psi^2 q + \varphi^2 r) \\ & + i k_3 \left(\frac{1}{2} \alpha r_{xx} + \beta r_{xxxx} \right) (\varphi^2 - q) \\ & + i k_3 \left(\frac{1}{2} \alpha q_{xx} + \beta q_{xxxx} \right) (r + \psi^2) \\ & + k_4 (r_x \varphi^2 - \psi^2 q_x - q_x r - q r_x) \\ & - 2 i k_3 \beta (\varphi^2 r^2 q_{xx} + \psi^2 r_{xx} q^2), \end{aligned} \tag{5.12}$$

$$
\begin{aligned}
C^x = {} & ik_3\beta(6(\varphi_{xx}\varphi_x r_t + \psi_{xx}\psi_x q_t) \\
& +6(r^2 q_x q_t - q^2 r_x r_t) \\
& -4(\varphi_x\varphi r^2 q_t + \psi_x\psi q^2 r_t)) \\
& +\frac{1}{2}i\alpha(k_1 - k_2)(\varphi^2 r_x - \psi^2 q_x) \\
& +2ik_3\beta(r_{xt}q^2\psi^2 + q_{xt}r^2\varphi^2 + \varphi_{xxx}\varphi r_t \\
& +\psi_{xxx}\psi q_t + q_{xxt}\psi_x\psi + r_{xxt}\varphi_x\varphi - q_{xt}\psi_x^2 \\
& -r_{xt}\varphi_x^2 - r_{xt}\varphi_{xx}\varphi - q_{xt}\psi_{xx}\psi) \\
& +2ik_4\beta(q_{xxx}\psi_x\psi + r_{xxx}\varphi_x\varphi \\
& +\varphi_{xxx}\varphi r_x + \psi_{xxx}\psi q_x - \varphi_{xx}\varphi r_{xx}. \\
& -\psi_{xx}\psi q_{xx} - \psi^2 q_x^2 r - \varphi^2 r_x^2 q - \psi_x^2 q_{xx} \\
& -\varphi_x^2 r_{xx}) + i(\alpha - 16\beta qr)(k_4(\psi_x q_x\psi + \varphi_x r_x\varphi) \\
& +k_3(\psi_x\psi q_t + \varphi_x\varphi r_t)) - ik_4\alpha(\psi^2 rq^2 + \varphi^2 qr^2) \\
& +k_4(rq_t + qr_t) + ik_3\beta(r_{xxxt}q - r_{xxxt}\varphi^2 \\
& -q_{xxxt}r - q_{xxxt}\psi^2 + r_{xxx}q_t - q_{xxx}r_t + r_{xt}q_{xx} \\
& -q_{xt}r_{xx} + q_{xxt}r_x - r_{xxt}q_x) \\
& +(4ik_3\beta(qr_x - q_x r) - k_4)(r_t\varphi^2 - \psi^2 q_t) \\
& +i(k_1 - k_2)(\beta(r_{xxx}\varphi^2 - q_{xxx}\psi^2) \\
& +\alpha(\psi_x q\psi - \varphi_x r\varphi)) \\
& +\frac{1}{2}ik_3((24\beta qr - \alpha)(q_x r_t - r_x q_t) \\
& +(16\beta qr - \alpha)(r_{xt}\varphi^2 + q_{xt}\psi^2) \\
& +(12\beta qr - \alpha)(q_{xt}r - r_{xt}q)) \\
& +6i\beta(k_2 - k_1)(r_x q^2\psi^2 - q_x r^2\varphi^2 \\
& +\varphi_{xx}\varphi_x r - \psi_{xx}\psi_x q \\
& +2qr(\psi_x\psi q - \varphi_x\varphi r - \psi^2 q_x + \varphi^2 r_x)) \\
& +ik_4\beta(6(\varphi_{xx}\varphi_x r_x + \psi_{xx}\psi_x q_x) \\
& -4(\psi_x\psi r_x q^2 + \varphi_x\varphi q_x r^2) \\
& +(6q^2r^2 - 8r_x q_x)(\psi^2 q + \varphi^2 r)) \\
& +2i\beta(k_1 - k_2)(\varphi_x^2 r_x - \psi_x^2 q_x \\
& +\psi_x\psi q_{xx} - \varphi_x\varphi r_{xx} + \psi_{xxx}\psi q \\
& -\varphi_{xxx}\varphi r + \varphi_{xx}\varphi r_x - \psi_{xx}\psi q_x).
\end{aligned} \tag{5.13}
$$

For (5.12) and (5.13), using Maple software we have verified that $D_t\left(C^t\right) + D_x\left(C^x\right) = 0$ holds. This also shows that we obtain the non-local conservation laws for the coupled LPD equations.

6. Conclusions

In conclusion, the non-local symmetries of the coupled LPD equations were obtained using Lax pair. With the help of an auxiliary variable, the original system was extended to a prolonged system, while the non-local symmetries were localised to the Lie point symmetries of the prolonged system and thus some new exact solutions and non-local conservation laws of the coupled LPD equations were obtained. Another different exact solution was obtained by finite symmetry transformation. In future research, we will consider the construction of the relevant non-local potential system through the conservation law form of the single uncoupled LPD equation and then investigate the Lie point symmetry of the whole system to eventually derive its non-local symmetry.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (No. 11505090), Research Award Foundation for Outstanding Young Scientists of Shandong Province (No. BS2015SF009), the doctoral foundation of Liaocheng University under Grant No. 318051413 and Liaocheng University level science and technology research fund No. 318012018.

References

[1] B Q Li, Y L Ma, L P Mo and Y Y Fu, *Comput. Math. Appl.* **74**, 504 (2017)

[2] Y Zhou, S Manukure and W X Ma, *Commun. Nonlinear Sci* **68**, 56 (2019)

[3] B Kaur and R K Gupta, *Pramana – J. Phys.* **93**, 59 (2019)

[4] S Kumar, D Kumar and H Kharbanda, *Pramana – J. Phys.* **95**, 33 (2021)

[5] Y Y Li, H X Jia and D W Zuo, *Optik* **241**, 167019 (2021)

[6] S T Demiray, Y Pandir and H Bulut, *Ocean Eng.* **103**, 153 (2015)

[7] M J Ablowitz, M A Ablowitz and P A Clarkson, *Solitons, nonlinear evolution equations and inverse scattering* (Cambridge University Press, Cambridge, 1991)

[8] C C Ding, Y T Gao, G F Deng and D Wang, *Chaos Solitons Fractals* **133**, 109580 (2020)

[9] C D Zhu and L H Wu, *Pramana – J. Phys.* **94**, 138 (2020)

[10] R Hirota, *Phys. Rev. Lett.* **27**, 1192 (1971)

[11] Y Shen, B Tian, X Zhao, W R Shan and Y Jiang, *Pramana – J. Phys.* **95**, 137 (2021)

[12] P J Olver, *Applications of Lie groups to differential equations* (Springer, Berlin, 1986)

[13] L V Ovsiannikov, *Group analysis of differential equations* (Academic, New York, 1982)

[14] N H Ibragimov, *Transformation groups applied to mathematical physics* (Reidel, Boston, MA, 1985)

[15] M Lakshmanan and P Kaliappan, *J. Math. Phys.* **24**, 795 (1983)

[16] F Galas, *J. Phys. A: Math. Gen.* **25**, 981 (1992)

[17] G W Bluman and Z Z Yang, *J. Math. Phys.* **54**, 093504 (2013)

[18] X P Xin, Q Miao and Y Chen, *Chin. Phys. B* **23**, 010203 (2014)
[19] S Y Lou, X R Hu and Y Chen, *J. Phys. A: Math. Theor.* **45**, 155209 (2012)
[20] X R Hu, S Y Lou and Y Chen, *Phys. Rev. E* **85**, 056607 (2012)
[21] X P Xin, Y T Liu and X Q Liu, *Appl. Math. Lett.* **55**, 63 (2016)
[22] M Wang, B Tian and T Y Zhou, *Chaos Solitons Fractals* **152**, 111411 (2021)
[23] W Liu, D Q Qiu, Z W Wu and J S He, *Commun. Theor. Phys.* **65**, 671 (2016)
[24] M Lakshmanan, K Porsezian and M Daniel, *Phys. Lett. A* **133**, 483 (1988)
[25] K Porsezian, M Daniel and M Lakshmanan, *J. Math. Phys.* **33**, 1807 (1992)
[26] P Veeresha, D G Prakasha, H M Baskonus and G Yel, *Math. Meth. Appl. Sci.* **43**, 4316 (2020)
[27] W Liu, D Q Qiu, Z W Wu and J S He, *Commun. Theor. Phys.* **65**, 671 (2016)
[28] A Bansal, A Biswas, H Triki, Q Zhou, S P Moshokoa and M Belic, *Optik* **160**, 86 (2018)
[29] A Biswas, *Optik* **160**, 14 (2018)
[30] L L Huang and Y Chen, *Appl. Math. Lett.* **64**, 177 (2017)
[31] X P Xin, Y Chen, *Chin. Phys. Lett.* **30**, 100202 (2013)
[32] Y R Xia, R X Yao and X P Xin, *J. Nonlinear Math. Phys.* **27**, 581 (2020)
[33] N Benoudina, Y Zhang and C M Khalique, *Commun. Nonlinear Sci*. **94**, 105560 (2021)
[34] R Cherniha, M Serov and Y Prystavka, *Commun. Nonlinear Sci.* **92**, 105466 (2021)
[35] Z F Liang, X Y Tang and W Ding, *Commun. Theor. Phys.* **73**, 055003 (2021)
[36] D Baleanu, M Inc, A Yusuf and A I Aliyu, *Commun. Nonlinear Sci*, **59**, 222 (2018)
[37] S Singh, R Sakthivel, M Inc, A Yusuf and K Murugesan, *Pramana – J. Phys.* **95**, 43 (2021)
[38] N H Ibragimov, *J. Phys. A: Math. Theor.* **44**, 432002 (2011)
[39] N H Ibragimov, *J. Math. Anal. Appl.* **333**, 311 (2007)