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Invariant subspace method for time-fractional nonlinear evolution equations of the third order

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Abstract. In this article, we consider a general time-fractional differential equation, $(\partial^\alpha v/\partial t^\alpha) = F[x, v, v_x, v_{xx}, v_{xxx}]$, where F is a third-order polynomial operator with quadratic nonlinearities. We describe the operator F which admits two-dimensional invariant subspaces which are solutions of second-order ordinary differential equation with constant coefficients. Using the invariant subspace method, we find explicit solution for fractional differential equations for various cases of operator F. We show that explicit solutions to many of the known equations like time-fractional Rosenau–Hyman equation, Korteweg–de Vries equation, Benjamin equation etc. can be derived using this method.

Keywords. Invariant subspace method; Mittag–Leffler function; fractional differential equation; Riemann–Liouville fractional derivative.

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1. Introduction

One of the significant problems in the study of partial differential equations (PDEs) is to find exact solutions or closed form solutions to a given nonlinear equation. Exact solutions are very useful in understanding the dynamics of the physical system. They play a crucial role in developing existence and uniqueness theory and also in studying the asymptotic behaviour of the dynamical system. Some of the methods used to find exact solutions to nonlinear PDEs, in particular, evolutionary equations, are inverse scattering method, Lax pair, symmetry analysis, Painleve approach, Hirota bilinear method etc. [1–3]. Another powerful method to find explicit solutions to nonlinear PDEs is the invariant subspace method developed by Galaktionov and Svirshchevski [4]. This method was illustrated and used effectively by Galaktionov and Svirshchevskii to find explicit solutions to PDEs in gas dynamics models, reaction–diffusion absorption and combustion models, thin film equations, nonlinear dispersion PDEs with compactons, higher-order Rosenau–Hyman equation, modifications of Camassa–Holm equations etc. Also they have extensively studied the problem of finding invariant subspaces of maximal dimension for first- and second-order nonlinear operators with quadratic nonlinearities.

In recent years, researchers have shown much interest in the study of fractional differential equations (FDE) [5–8]. Fractional calculus plays a significant role in the study of nonlinear phenomena. Fractional integrals and derivatives help in modelling complex systems involving long-term memory in time. Many anomalous diffusion processes are modelled using fractional derivatives. Numerical solutions to fractional PDEs like diffusion equation arising in transport phenomena, Klein–Gordon equations have been obtained recently [9,10]. Methods like Adomian decomposition method [11], homotopy analysis method [12], variational iteration method [13] etc. are used to find approximate solutions to nonlinear FDEs. There are very few methods in the literature to find exact solutions to nonlinear FDE, Lie symmetry analysis [14,15] being one of the widely used method. Recently, the invariant subspace method has been extended to find exact solutions to fractional PDEs [4,16–21]. Exact solution of time-fractional Burgers equation was found using this method [22,23]. Also invariance subspace method together with Lie group analysis was used to find particular solutions to FDEs [18]. A refined invariant subspace

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method was also introduced to solve evolution equations [24].

Although recent papers show that invariant subspace method can be used to find exact solutions of certain time-fractional PDEs [25], the problem of finding invariant subspaces for nonlinear operators of third and higher orders has not yet been studied extensively till now. With this motivation, in this paper we consider time-fractional PDEs involving nonlinear operators of third order with quadratic nonlinearities and find two-dimensional subspaces admitted by them. Using the invariant subspace method, we find exact solutions to the obtained FDEs wherever possible. In particular, we consider a general time-fractional evolution equation of the form

$$\frac{\partial^\alpha v}{\partial t^\alpha} = F[v], \quad t > 0, \tag{1}$$

where $v = v(x,t)$, $F[v] = F[x, v, v_x, v_{xx}, v_{xxx}]$, $(\partial^\alpha v/\partial t^\alpha)$, is the time-fractional Riemann–Liouville derivative of v of order α,

$$v_x = \frac{\partial v}{\partial x},\ v_{xx} = \frac{\partial^2 v}{\partial x^2},\ v_{xxx} = \frac{\partial^3 v}{\partial x^3} \tag{2}$$

and F is a third-order polynomial operator with quadratic nonlinearities. The aim of this paper is to describe operators F of the form given by (1) that admits two-dimensional (D) subspaces which are solutions of second-order linear ordinary differential equation (ODE) with constant coefficients. Subsequently, we solve the fractional PDE (1) for particular cases of the various operators F obtained, using the invariant subspace method. We consider that the results obtained are significant since apart from many well-known equations like time-fractional Rosenau–Hyman equation, Korteweg–de Vries equation, Benjamin equation, exact solutions to many other new fractional PDEs of third order of the form given by eq. (1) have also been obtained.

The plan of the article is as follows: In §2, we present some definitions, theorems and results required for the rest of the article. In §3, we describe in detail how the invariant subspace method is used to find exact solutions to time-fractional evolutionary PDEs. In §4, we explain in detail the main objective of this paper as mentioned above. In §5 we briefly summarise the results obtained. Since the operator considered here is of third order, it can admit subspaces of maximal dimension 7. In Appendices A, B, C, D, E we also give list of operators of the form (1) admitting 3D, 4D, 5D, 6D and 7D subspaces which form the solution space of nth-order (3rd, 4th, 5th, 6th, 7th, respectively) linear ODE with constant coefficients. We find that for 2D subspaces, the list of operators given in this paper is exhaustive.

2. Preliminaries

In this section, we present some basic definitions, theorems and results required for the rest of this article.

DEFINITION 1

The generalised Mittag–Leffler function [27] is defined as

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)} \times \frac{z^k}{k!},$$
$$z, \alpha, \beta, \rho \in \mathbb{C},\ \Re(\alpha) > 0, \tag{3}$$

where $(\rho)_k$ is the Pochammer symbol given by

$$(\rho)_0 = 1,\ (\rho)_k = \rho(\rho+1)(\rho+2)\cdots(\rho+k-1),\ k = 1, 2, \ldots$$

and $\Gamma(\cdot)$ is the Euler gamma function.

When $\rho = 1$, eq. (3) gives the two-parameter Mittag–Leffler function

$$E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ z, \alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0. \tag{4}$$

When $\rho = 1$, $\beta = 1$, eq. (3) gives the one-parameter Mittag–Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z, \alpha \in \mathbb{C},\ \Re(\alpha) > 0. \tag{5}$$

DEFINITION 2

Let $\alpha > 0$ and $a \in \mathbb{R}$. The Riemann–Liouville fractional derivative, $D^\alpha_{a+} f$, of order α is defined as [27]

$$D^\alpha_{a+} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \times \left[\int_a^t (t-x)^{m-\alpha-1} f(x)\mathrm{d}x\right],$$

where $m = \lceil \alpha \rceil$ ($\lceil \alpha \rceil$ is the smallest integer which is greater than or equal to α). If $\alpha = 0$, then we define,

$$D^0_{a+} f(t) = f(t).$$

Apart from the Riemann–Liouville definition, there are many other definitions of fractional derivative like Caputo derivative, Weyl's fractional derivative, Grünwald–Letnikov derivative etc. [26,27]. Throughout this paper we use the Riemann–Liouville definition of fractional derivative. Below we give some of the properties satisfied by Riemann–Liouville fractional derivative and other results used in solving FDEs in this article [27–29].

1. $D^{\alpha}_{a+}(f(t)+g(t)) = D^{\alpha}_{a+}(f(t)) + D^{\alpha}_{a+}(g(t))$ (6)

2. $D^{\alpha}_{a+}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}$,
$\alpha \geq 0,\ \lambda > -1,\ \lambda \neq \alpha - 1,\ t > 0.$ (7)

3. $D^{\alpha}_{a+}y(t) = 0,\ \ 0 < \alpha < 1,$ then,
$y(t) = c(t-a)^{\alpha-1}$, for any $c \in \mathbb{R}$. (8)

4. $$\frac{1}{\Gamma(\nu)}\int_0^t (t-u)^{\nu-1}E_{\alpha,\beta}(\lambda u^{\alpha})u^{\beta-1}\mathrm{d}u = t^{\beta+\nu-1}E_{\alpha,\beta+\nu}(\lambda t^{\alpha}),\quad \beta > 0,\ \nu > 0 \tag{9}$$
Hence, for $\beta = \alpha,\ \nu = 1,$ we have
$$\int_0^t u^{\alpha-1}E_{\alpha,\alpha}(\lambda u^{\alpha})\mathrm{d}u = t^{\alpha}E_{\alpha,\alpha+1}(\lambda t^{\alpha}). \tag{10}$$

5. The Laplace transform of the generalised Mittag–Leffler function $t^{\beta-1}E^{\rho}_{\alpha,\beta}(\lambda t^{\alpha})$ is given by
$$L[t^{\beta-1}E^{\rho}_{\alpha,\beta}(\lambda t^{\alpha})] = \frac{s^{\alpha\rho-\beta}}{(s^{\alpha}-\lambda)^{\rho}},\ \Re(s) > 0,$$
$$\Re(\beta) > 0,\ \rho \in \mathbb{C},\ \lambda \in \mathbb{C},\ |\lambda s^{-\alpha}| < 1.$$
Hence, for $\rho = 1,\ \beta = \alpha$, we have
$$L[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})] = \frac{1}{(s^{\alpha}-\lambda)},$$
$$(\Re(s) > 0,\ \lambda \in \mathbb{C},\ |\lambda s^{-\alpha}| < 1). \tag{11}$$
For $\rho = 2,\ \beta = 2\alpha$, we have
$$L[t^{2\alpha-1}E^{2}_{\alpha,2\alpha}(\lambda t^{\alpha})] = \frac{1}{(s^{\alpha}-\lambda)^2},$$
$$(\Re(s) > 0,\ \lambda \in \mathbb{C},\ |\lambda s^{-\alpha}| < 1). \tag{12}$$

6. The convolution of two functions f and g is given by
$$(f * g)(t) = \int_0^t f(t-x)g(x)\mathrm{d}x = \int_0^t g(t-x)f(x)\mathrm{d}x.$$
By the convolution theorem of Laplace transforms, we have
$$L^{-1}[\hat{f}(s)\hat{g}(s)] = (f * g)(t), \tag{13}$$
where
$$\hat{f}(s) = L[f(t)] \quad \text{and} \quad \hat{g}(s) = L[g(t)].$$
Let
$$f(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}).$$
Then,
$$(f * f)(t) = \int_0^t f(t-x)f(x)\mathrm{d}x.$$
Hence,
$$(f * f)(t) = \int_0^t (t-x)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-x)^{\alpha})x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^{\alpha})\mathrm{d}x = L^{-1}\left[\frac{1}{(s^{\alpha}-\lambda)^2}\right],$$
(using eqs (11) and (13)). Using eq. (12), we have
$$\int_0^t (t-x)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-x)^{\alpha})x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})\mathrm{d}x = t^{2\alpha-1}E^{2}_{\alpha,2\alpha}(\lambda t^{\alpha}). \tag{14}$$

7. The fractional differential equation of order $\alpha(\alpha > 0)$, given by
$$(D^{\alpha}_{a+}y)(t) = \lambda(t-a)^{\beta}[y(t)]^2,$$
$$(t > a,\ \lambda, \beta \in \mathbb{R},\ \lambda \neq 0), \tag{15}$$
for $\alpha + \beta < 1$, has an explicit solution of the form,
$$y(t) = \frac{\Gamma(1-\alpha-\beta)}{\lambda\Gamma(1-2\alpha-\beta)}(t-a)^{-(\alpha+\beta)}. \tag{16}$$

DEFINITION 3

Let $\alpha \in (0, 1],\ f : [a, b] \to \mathbb{R}$ and $t_0 \in [a, b]$. Then $f(t)$ is said to be α analytic at t_0 if there exists an interval $N(t_0)$ such that, for all $t \in N(t_0)$, $f(t)$ can be expressed as power series of $(t-t_0)^{\alpha}$ [27]. That is,
$$f(t) = \sum_{n=0}^{\infty} c_n(t-t_0)^{n\alpha},\ c_n \in \mathbb{R},$$
this series being absolutely convergent for $|t - t_0| < \rho$, $\rho > 0$ is the radius of convergence of the series.

DEFINITION 4

Let $\alpha \in (0, 1]$. A point $t_0 \in [a, b]$ is said to be an α-ordinary point of the equation
$$y^{(n\alpha)}(t) + \sum_{k=0}^{n-1} a_k(t)y^{(k\alpha)}(t) = 0, \tag{17}$$
if the functions $a_k(t)$ $(k = 0, 1, 2, \ldots, n-1)$ are α-analytic in t_0 [27]. A point $t_0 \in [a, b]$ which is not α-ordinary is said to be α-singular.

DEFINITION 5

Let $t_0 \in [a, b]$ be an α-singular point of eq. (17). Then, t_0 is said to be a regular α-singular point of this equation, if the functions $(t-t_0)^{(n-k)\alpha} a_k(t)$, $(k = 0, 1, \ldots, n-1)$ are α-analytic in t_0 [27].

Theorem 1. [27]. *Let $\alpha > 0$, $\lambda \in \mathbb{R}$ and $f(t)$ be a given function defined on $\mathbb{R}_+$ then*

$$(D^{\alpha}_{a+}y)(t) - \lambda y(t) = f(t), \ (D^{\alpha-1}_{a+}y)(a+) = C, \quad (C \in \mathbb{R}), \ t > 0$$

has a solution of the form

$$y(t) = C(t-a)^{\alpha-1}E_{\alpha,\alpha}[\lambda(t-a)^{\alpha}] + \int_0^t [(t-x)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-x)^{\alpha})] f(x)\mathrm{d}x,$$

provided the above integral is convergent.

Theorem 2. [27]. *Let $t_0 \geq a$ be a regular α-singular point of the equation*

$$(t-t_0)^{\alpha}(D^{\alpha}_{a+}y)(t) + q(t)y(t) = 0 \tag{18}$$

and let

$$q(t) = \sum_{n=0}^{\infty} q_n (t-t_0)^{n\alpha}, \quad \text{for } t \in [a, b].$$

Then there exists the solution

$$y(t, \alpha, s_1) = (t-t_0)^{s_1}\sum_{n=0}^{\infty} b_n (t-t_0)^{n\alpha},$$

of eq. (18) on a certain interval to the right of t_0. Here, b_0 is a non-zero arbitrary constant, $s_1 > -1$ is the real solution to the equation

$$\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} + q_0 = 0$$

and the coefficients b_n are given by the following recurrence formula:

$$b_n = -\frac{\Gamma(n\alpha+s-\alpha+1)}{\Gamma(n\alpha+s+1)}\sum_{l=0}^{n-1} b_l q_{n-l}, \quad n \geq 1.$$

3. Invariant subspace method for time-fractional partial differential equations

In this section, we explain in detail how the invariant subspace method used in solving PDEs can be extended to solve FDEs [4,16,18]. In this paper, we assume 'a' given in Definition 2 as 0. To make notations simpler, we denote the operator $D^{\alpha}_0 f(t)$ as $D^{\alpha} f(t)$.

Consider a general time-fractional evolutionary PDE of the form

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = F[v], \quad t > 0, \tag{19}$$

where $v = v(x, t)$ and F is a sufficiently smooth kth-order differential operator given by

$$F[v] = F[x, v, v_x, \ldots, v_x^{(k)}]. \tag{20}$$

Here,

$$v_x = \frac{\partial v}{\partial x}, \quad v_{xx} = \frac{\partial^2 v}{\partial x^2}, \ldots, \quad v_x^{(k)} = \frac{\partial^k v}{\partial x^k},$$

$\frac{\partial^{\alpha} v}{\partial t^{\alpha}}$ is the fractional derivative of v of order α with respect to t.

Let

$$W_n = \langle g_1(x), g_2(x), \ldots, g_n(x)\rangle, \tag{21}$$

be the linear span of n ($n \geq 1$) linearly independent functions $\{g_1(x), g_2(x), \ldots, g_n(x)\}$. That is, if $v \in W_n$, then,

$$v = \sum_{i=1}^{n} r_i g_i(x), \quad r_i \in \mathbb{C}, \quad i = 1, 2, \ldots, n. \tag{22}$$

The space W_n is said to be invariant under F if,

$$F[W_n] \subseteq W_n, \tag{23}$$

and F is said to admit W_n, that is,

$$F\left[\sum_{i=1}^{n} r_i g_i(x)\right] = \sum_{i=1}^{n} \xi_i(r_1, r_2, \ldots, r_n) g_i(x),$$

$$\text{for any } (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n. \tag{24}$$

Here $\{\xi_i\}$ are the expansion coefficients of $F[v] \in W_n$ on the basis of $\{g_i\}$. If the linear subspace W_n is invariant under F, then eq. (19) has a solution of the form

$$v(x, t) = \sum_{i=1}^{n} r_i(t) g_i(x), \tag{25}$$

where the coefficients $\{r_i(t)\}$ satisfy the system of time-fractional ordinary differential equations (ODEs) given by

$$D^{\alpha} r_i(t) = \xi_i(r_1(t), \ldots, r_n(t)), \quad i = 1, 2, 3, \ldots, n. \tag{26}$$

The proof of (26) follows from the linearity property of fractional derivatives and also because the functions $\{g_1, g_2, \ldots, g_n\}$ are linearly independent. Hence, to solve a given fractional differential equation of the form (19) by invariant subspace method, we find an invariant subspace admitted by the nonlinear operator F and solve the corresponding system of fractional ODEs for the expansion coefficients given by (26). It has been shown that if a linear subspace W_n is invariant under a differential operator of the form (20) of order k, then $n \leq 2k+1$ [4].

4. Invariant subspace method for nonlinear operators of third order

Consider a scalar time-fractional PDE of the form

$$\frac{\partial^\alpha v}{\partial t^\alpha} = F[v], \quad 0 < \alpha < 1,\ t > 0, \tag{27}$$

where $v = v(x,t)$ and F is a sufficiently smooth third-order nonlinear operator given by

$$F[v] = F[x, v, v_x, v_{xx}, v_{xxx}]. \tag{28}$$

Here, the Riemann-Liouville fractional derivative is considered. In particular, we assume F as a polynomial of the form

$$\begin{aligned} F[v] = {} & c_1 v^2 + c_2 v_1^2 + c_3 v_2^2 + c_4 v_3^2 + c_5 v v_1 + c_6 v v_2 \\ & + c_7 v v_3 + c_8 v_1 v_2 + c_9 v_1 v_3 + c_{10} v_2 v_3 + c_{11} v_1 \\ & + c_{12} v_2 + c_{13} v_3 + c_{14} v + c_{15}, \end{aligned} \tag{29}$$

where $c_i \in \mathbb{R}$ for $i = 1, 2, \ldots, 15$. Here,

$$\begin{aligned} v_1 &= v_x = \frac{\partial v}{\partial x}, \\ v_2 &= v_{xx} = \frac{\partial^2 v}{\partial x^2}, \\ v_3 &= v_{xxx} = \frac{\partial^3 v}{\partial x^3}. \end{aligned} \tag{30}$$

Now, we proceed to describe the operators of the form (29) admitting 2D invariant subspaces which are solutions of a linear second-order ordinary differential equation with constant coefficients.

2D subspaces

Now we describe third-order operators of the form (29) admitting 2D subspaces

$$W_2 = \langle g_1(x), g_2(x) \rangle, \tag{31}$$

where $\{g_1(x), g_2(x)\}$ forms a fundamental set of solution for the linear ODE

$$\mathfrak{L}[v] = v_2 - a_1 v_1 - a_2 v = 0, \tag{32}$$

where $\{a_1, a_2\}$ are arbitrary real constants and v_1, v_2 as in eq. (30). Now the invariance condition given by (23) for the subspace W_2 with respect to F is

$$\mathfrak{L}[F[v]] = 0,\ \text{given } \mathfrak{L}[v] = 0. \tag{33}$$

Equation (33) can be rewritten as

$$D^2 F - a_1 D F - a_2 F = 0, \tag{34}$$

where D is the total derivative operator given by

$$D = \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial v} + [a_1 v_1 + a_2 v] \frac{\partial}{\partial v_1}. \tag{35}$$

The invariance condition (34) leads to the following under-determined system of equations in c_i's and a_i's.

$$a_2 c_{15} = 0, \tag{36a}$$

$$\begin{aligned} & 3a_2c_5 + 4a_1^2a_2c_8 + 3a_2a_1c_6 + 3a_2^2c_8 + 14a_1^3a_2^2c_4 \\ & + 10a_1a_2^3c_4 + 3a_2^2c_7 + 10a_1^2a_2^2c_{10} + 4a_1a_2c_2 \\ & + 4a_1^3a_2c_3 + 4a_1^5a_2c_4 + 4a_1^3a_2c_9 + 3a_2a_1^2c_7 \\ & + 4a_1^4a_2c_{10} + 3a_2^3c_{10} + 7a_1a_2^2c_9 + 6a_1a_2^2c_3 = 0, \end{aligned} \tag{36b}$$

$$\begin{aligned} & a_2^2c_9 + 2a_1c_5 + a_2^3c_4 + 2a_1^3c_8 + 2c_3a_2^2 \\ & + 2c_6a_2 + 2a_1^4c_9 + 2a_1^5c_{10} + 2a_1^2c_2 \\ & + 10c_4a_1^2a_2^2 + a_2c_2 + 2a_1^2c_6 + 2c_1 \\ & + 4c_7a_1a_2 + 2a_1^6c_4 + 5a_2a_1^2c_3 \\ & + 7a_2a_1^3c_{10} + 2a_1^4c_3 \\ & + 5a_2a_1^2c_9 + 9a_2a_1^4c_4 + 5c_{10}a_2^2a_1 \\ & + 3a_2a_1c_8 + 2a_1^3c_7 = 0, \end{aligned} \tag{36c}$$

$$\begin{aligned} & 2a_1a_2^2c_8 + 2a_1^2a_2^2c_9 + 2a_1^3a_2^2c_{10} + a_2^3c_3 \\ & + 2a_1^4a_2^2c_4 + a_1a_2^2c_7 \\ & + a_2c_1 + a_2^2c_6 + 3a_1a_2^3c_{10} + 2a_2^4c_4 + 2a_2^3c_9 + 2a_2^2c_2 \\ & + 5a_1^2a_2^3c_4 + 2a_1^2a_2^2c_3 = 0. \end{aligned} \tag{36d}$$

Solving the eqs (36a)–(36d) for c_i's and a_i's, we find the following operators and the corresponding subspaces admitted by them. Maple software has been used to perform computations. The solutions obtained on solving the system of eqs (36a)–(36d) are given as separate cases and the corresponding FDE's obtained are solved wherever possible.

Case 1

$$\begin{aligned} & a_1 = 0,\ a_2 = 0, \\ & c_1 = -a_1^6c_4 - a_1^5c_{10} - a_1^4c_3 - a_1^4c_9 - a_1^3c_7 \\ & \quad - a_1^3c_8 - a_1^2c_2 - a_1^2c_6 - a_1c_5. \end{aligned}$$

Hence we have,

$$\begin{aligned} F = {} & c_{10}v_2v_3 + c_2v_1^2 + c_3v_2^2 + c_4v_3^2 \\ & + c_5vv_1 + c_6vv_2 + c_7vv_3 \\ & + c_8v_1v_2 + c_9v_1v_3 + c_{11}v_1 + c_{12}v_2 \\ & + c_{13}v_3 + c_{14}v + c_{15}, \end{aligned} \tag{37}$$

$$W_2 = \langle 1, x \rangle.$$

Now we proceed to solve the FDE given by eq. (27) with F given by eq. (37). Let,

$$v(x,t) = r_1(t) + r_2(t)x. \tag{38}$$

Substituting (38) in eq. (37) and collecting the expansion coefficients, we get

$$\xi_1(r_1(t), r_2(t)) = c_2 r_2^2(t) + c_5 r_1(t) r_2(t) + c_{11} r_2(t) + c_{14} r_1(t) + c_{15}, \tag{39}$$

$$\xi_2(r_1(t), r_2(t)) = c_5 r_2^2(t) + c_{14} r_2(t). \tag{40}$$

Using the invariance subspace method outlined in §3, we get the following system of fractional ODEs for the expansion coefficients:

$$D^\alpha r_1(t) = c_2 r_2^2(t) + c_5 r_1(t) r_2(t) + c_{11} r_2(t) + c_{14} r_1(t) + c_{15}, \tag{41}$$

$$D^\alpha r_2(t) = c_5 r_2^2(t) + c_{14} r_2(t). \tag{42}$$

In general, it is difficult to solve the above system. However, for the following subcases, eqs (41) and (42) can be solved explicitly.

(i) $c_2 = 0, c_5 = 0$
In this case, systems (41) and (42) reduce to

$$D^\alpha r_1(t) = c_{11} r_2(t) + c_{14} r_1(t) + c_{15}, \tag{43}$$

$$D^\alpha r_2(t) = c_{14} r_2(t). \tag{44}$$

Solving eq. (44) using Theorem (1), we have

$$r_2(t) = t^{\alpha-1} E_{\alpha,\alpha}(c_{14} t^\alpha). \tag{45}$$

Substituting $r_2(t)$ given by (45) in eq. (43) we have

$$D^\alpha r_1(t) - c_{14} r_1(t) = c_{11}[t^{\alpha-1} E_{\alpha,\alpha}(c_{14} t^\alpha)] + c_{15}. \tag{46}$$

Solving eq. (46) using Theorem 1 gives

$$\begin{aligned} r_1(t) &= \int_0^t [c_{11}(t-u)^{\alpha-1} E_{\alpha,\alpha}(c_{14}(t-u)^\alpha) + c_{15}] \\ &\quad \times u^{\alpha-1} E_{\alpha,\alpha}(c_{14} u^\alpha) \mathrm{d}u \\ &= c_{11} \int_0^t (t-u)^{\alpha-1} E_{\alpha,\alpha}(c_{14}(t-u)^\alpha) u^{\alpha-1} \\ &\quad \times E_{\alpha,\alpha}(c_{14} u^\alpha) \mathrm{d}u \\ &\quad + c_{15} \int_0^t u^{\alpha-1} E_{\alpha,\alpha}(c_{14} u^\alpha) \mathrm{d}u. \end{aligned} \tag{47}$$

Using eq. (14), the first integral on the right-hand side of eq. (47) is evaluated as

$$\int_0^t (t-u)^{\alpha-1} E_{\alpha,\alpha}(c_{14}(t-u)^\alpha) u^{\alpha-1} E_{\alpha,\alpha}(c_{14} u^\alpha) \mathrm{d}u = t^{2\alpha-1} E^2_{\alpha,2\alpha}(c_{14} t^\alpha). \tag{48}$$

Using eq. (10), the second integral on the right-hand side of eq. (47) is evaluated as

$$\int_0^t u^{\alpha-1} E_{\alpha,\alpha}(c_{14} u^\alpha) \mathrm{d}u = t^\alpha E_{\alpha,\alpha+1}(c_{14} t^\alpha). \tag{49}$$

$$r_1(t) = c_{11}[t^{2\alpha-1} E^2_{\alpha,2\alpha}(c_{14} t^\alpha)] + c_{15}\, t^\alpha\, E_{\alpha,\alpha+1}(c_{14} t^\alpha). \tag{50}$$

Hence, an explicit solution for the FDE (27) in this case is given by

$$\begin{aligned} v(x,t) &= c_{11}[t^{2\alpha-1} E^2_{\alpha,2\alpha}(c_{14} t^\alpha)] \\ &\quad + c_{15}\, t^\alpha E_{\alpha,\alpha+1}(c_{14} t^\alpha) \\ &\quad + x[t^{\alpha-1} E_{\alpha,\alpha}(c_{14} t^\alpha)]. \end{aligned} \tag{51}$$

To illustrate, we consider the fractional PDE

$$\frac{\partial^\alpha v}{\partial t^\alpha} = v v_3 - v_1 v_2 + v + 1. \tag{52}$$

Hence, by eq. (51) an exact solution for (52) is given by

$$v(x,t) = t^\alpha E_{\alpha,\alpha+1}(t^\alpha) + x t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha). \tag{53}$$

Also, for $\alpha = 1$, eq. (52) has an exact solution given by

$$v(x,t) = \mathrm{e}^t + x\mathrm{e}^t - 1. \tag{54}$$

We also observe that eq. (53) coincides with eq. (54) for $\alpha = 1$. Figure 1 shows the plot of solutions given by eq. (53) for $\alpha = 0.25$, $\alpha = 0.45$, $\alpha = 0.7$, $\alpha = 1$, respectively.

(ii) $c_5 \neq 0,\ c_2 = 0,\ c_{11} = 0,\ c_{14} = 0,\ c_{15} = 0$
In this case, eqs (41) and (42) reduce to

$$D^\alpha r_1(t) = c_5 r_1(t) r_2(t), \tag{55}$$

$$D^\alpha r_2(t) = c_5 r_2^2(t). \tag{56}$$

Using eq. (16) and solving eq. (56) we have

$$r_2(t) = \frac{\Gamma(1-\alpha)}{c_5 \Gamma(1-2\alpha)} t^{-\alpha}, \quad t > 0,\ c_5 \in \mathbb{R},\ c_5 \neq 0,\ \alpha \neq \frac{1}{2}. \tag{57}$$

Substituting $r_2(t)$ given by eq. (57) in (55), we have

$$t^\alpha D^\alpha r_1(t) - \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} r_1(t) = 0. \tag{58}$$

$t = 0$ is a regular α-singular point of eq. (58). Hence, to solve this we use Theorem 2. Here,

$$q(t) = \sum_{n=0}^\infty q_n t^{n\alpha} = -\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \tag{59}$$

$$\Longrightarrow q_0 = -\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \text{ and } q_n = 0,\ n = 1, 2, \ldots \tag{60}$$

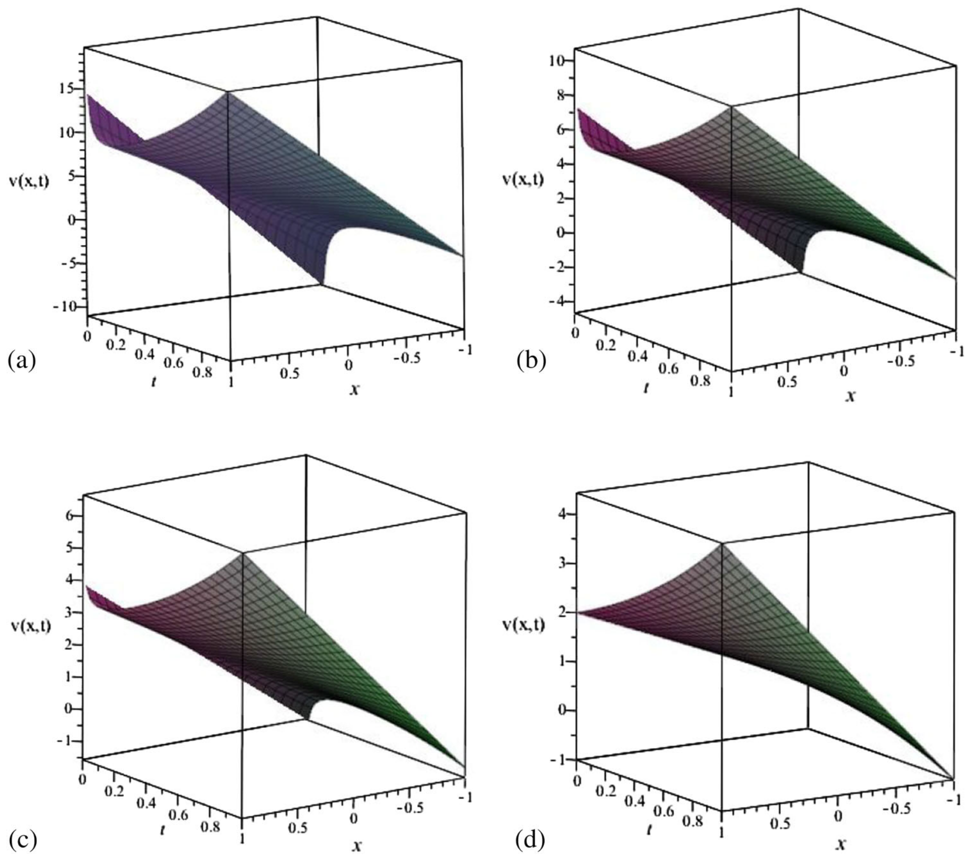

Figure 1. Graphical representation. (**a**) α = 0.25, (**b**) α = 0.45, (**c**) α = 0.7 and (**d**) α = 1.

Hence, a solution for FDE (58) is given by

$$r_1(t) = t^{s_1}\sum_{n=0}^{\infty} b_n t^{n\alpha}, \tag{61}$$

under the conditions, $s_1 > -1$, is a solution to equation,

$$\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} - \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} = 0$$

and the coefficients b_n are given by

$$b_n = -\frac{\Gamma(n\alpha+s-\alpha+1)}{\Gamma(n\alpha+s+1)}\sum_{l=0}^{n-1} b_l q_{n-l}, \quad n \geq 1.$$

Hence, we have

$$s_1 = -\alpha, \quad b_n = 0, \quad n = 1, 2, 3, \ldots.$$

$$\Longrightarrow r_1(t) = b_0 t^{-\alpha}, \tag{62}$$

where b_0 is an arbitrary constant.

Hence, an explicit solution for the FDE (27) in this case is given by

$$v(x,t) = b_0 t^{-\alpha} + x\left[\frac{\Gamma(1-\alpha)}{c_5\Gamma(1-2\alpha)}t^{-\alpha}\right], \quad t > 0. \tag{63}$$

Table 1 gives some of the known time-fractional PDEs that fall under Case 1 along with their explicit solutions. Benjamin equation models propagation of waves and Rosenau–Hyman equation describes the effect of nonlinear dispersion in the pattern formation in liquid drops. To the best of our knowledge, only approximate solutions to the time-fractional Rosenau–Hyman equation and Benjamin equation have been obtained [30,31]. Here we provide an exact solution for the same. In general, KdV equation, PKdV equation and KdV–Burgers equation are integrable equations admitting soliton solutions. Also, explicit solution to these equations in time-fractional form is available in the literature for other parametric values [16]. We feel that the exact solutions obtained in this paper are significant since they can be used to study the precision of other numerical techniques when a more general form of the same equation is considered.

Case 2

$a_1 \neq 0,\ a_2 = 0,$

Table 1. Invariant subspace $\langle 1, x\rangle$.

Name of the method	Parametric values*	Equation	Explicit solution
KdV	$c_5 = -b$, $c_{13} = -a$	$\frac{\partial^\alpha v}{\partial t^\alpha} = -av_3 - bvv_1$	$v(x,t) = b_0 t^{-\alpha} - x\left[\frac{\Gamma(1-\alpha)}{b\Gamma(1-2\alpha)}t^{-\alpha}\right]$
KdV–Burger	$c_5 = -1$, $c_{12} = b$, $c_{13} = -a$	$\frac{\partial^\alpha v}{\partial t^\alpha} = -av_3 + bv_2 - vv_1$	$v(x,t) = b_0 t^{-\alpha} - x\left[\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}t^{-\alpha}\right]$
Rosenau–Hyman	$c_5 = 1$, $c_7 = 1$, $c_8 = 3$	$\frac{\partial^\alpha v}{\partial t^\alpha} = vv_3 + 3v_1v_2 + vv_1$	$v(x,t) = b_0 t^{-\alpha} + x\left[\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}t^{-\alpha}\right]$
Benjamin	$c_5 = -1$, $c_{12} = H$, $c_{13} = \delta$, $(\delta > 0)$	$\frac{\partial^\alpha v}{\partial t^\alpha} = \delta v_3 + Hv_2 - vv_1$	$v(x,t) = b_0 t^{-\alpha} - x\left[\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}t^{-\alpha}\right]$

*$c_i = 0$, for $i = 1, \ldots, 15$, except otherwise specified.
Potential KdV (PKdV), $\frac{\partial^\alpha v}{\partial t^\alpha} = -v_3 - a(v_1)^2$ has a solution $v(x,t) = \left[-ab_0^2\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)} + b_0 x - \frac{1}{4a}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}x^2\right]t^{-\alpha}$, $\alpha \in (0,1) - \frac{1}{2}$. More details about the 3D invariant subspaces are given in Appendix A.

$$c_1 = -a_1^6c_4 - a_1^5c_{10} - a_1^4c_3 - a_1^4c_9 - a_1^3c_7 - a_1^3c_8 - a_1^2c_2 - a_1^2c_6 - a_1c_5.$$

Hence, we have

$$F = (-a_1^6c_4 - a_1^5c_{10} - a_1^4c_3 - a_1^4c_9 - a_1^3c_7 - a_1^3c_8 - a_1^2c_2 - a_1^2c_6 - a_1c_5)v^2 + c_{10}v_2v_3 + c_2v_1^2 + c_3v_2^2 + c_4v_3^2 + c_5vv_1 + c_6vv_2 + c_7vv_3 + c_8v_1v_2 + c_9v_1v_3 + c_{11}v_1 + c_{12}v_2 + c_{13}v_3 + c_{14}v + c_{15}, \tag{64}$$

$W_2 = \langle 1, e^{a_1x}\rangle$.
Let

$$v(x,t) = r_1(t) + e^{a_1x}r_2(t). \tag{65}$$

Substituting eq. (65) in eq. (64), we have the system of FDEs for the expansion coefficients as

$$D^\alpha r_1(t) = k_1r_1^2(t) + c_{14}r_1(t) + c_{15}, \tag{66}$$
$$D^\alpha r_2(t) = k_2r_1(t)r_2(t) + k_3r_2(t), \tag{67}$$

where

$$k_1 = -[a_1^6c_4 + a_1^5c_{10} + a_1^4c_3 + a_1^4c_9 + a_1^3c_7 + a_1^3c_8 + a_1^2(c_2 + c_6) + a_1c_5], \tag{68}$$
$$k_2 = -2[a_1^6c_4 + a_1^5c_{10} + a_1^4c_3 + a_1^4c_9 + a_1^3c_8 + a_1^2c_2] - [a_1^3c_7 + a_1^2c_6 + a_1c_5], \tag{69}$$
$$k_3 = a_1^3c_{13} + a_1^2c_{12} + a_1c_{11} + c_{14}. \tag{70}$$

Let

$$r_1(t) = p_1t^\lambda + p_2, \tag{71}$$

where p_1, p_2, λ, are constants to be determined. Substituting $r_1(t)$ given by eq. (71) in eq. (66) we have

$$p_1\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha} + p_2\frac{t^{-\alpha}}{\Gamma(1-\alpha)} = k_1[{p_1}^2t^{2\lambda} + {p_2}^2 + 2p_1p_2t^\lambda] + c_{14}[p_1t^\lambda + p_2] + c_{15}. \tag{72}$$

By solving eq. (72), we get

$$\lambda = -\alpha,$$
$$c_{15} = -(k_1{p_2}^2 + c_{14}p_2), \tag{73}$$
$$p_1 = \frac{1}{k_1}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}, \quad k_1 \neq 0 \tag{74}$$
$$p_2 = \frac{c_{14}}{k_1}\frac{(\Gamma(1-\alpha))^2}{(\Gamma(1-2\alpha) - 2(\Gamma(1-\alpha))^2)}, \quad \alpha \neq \frac{1}{2}. \tag{75}$$

Hence, eq. (67) reduces to

$$t^\alpha D^\alpha r_2(t) - [k_2p_1 + (k_2p_2 + k_3)t^\alpha]r_2(t) = 0. \tag{76}$$

$t = 0$ is a regular α-singular point of eq. (76). We use Theorem 2 to solve this. We have

$$q(t) = \sum_{n=0}^{\infty} q_nt^{n\alpha} = -k_2p_1 - (k_2p_2 + k_3)t^\alpha,$$
$$\Longrightarrow q_0 = -k_2p_1,$$
$$q_1 = -k_2p_2 - k_3, \quad q_n = 0, \quad n = 2, 3, \ldots$$

and

$$r_2(t) = t^{s_1}\sum_{n=0}^{\infty} b_nt^{n\alpha},$$

under the conditions, $s_1 > -1$, is a solution to equation,

$$\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} - k_2p_1 = 0 \tag{77}$$

and the coefficients b_n are given by

$$b_n = \frac{\Gamma(n\alpha + s - \alpha + 1)}{\Gamma(n\alpha + s + 1)} \times (k_2 p_2 + k_3) b_{n-1}.$$

Hence,

$$r_2(t) = t^{s_1} \sum_{n=0}^{\infty} \left[\frac{\Gamma(n\alpha + s_1 - \alpha + 1)}{\Gamma(n\alpha + s_1 + 1)} \times (k_2 p_2 + k_3) b_{n-1} \right] t^{n\alpha}. \quad (78)$$

An explicit solution to eq. (27) in this case is given by

$$v(x,t) = [p_1 t^{-\alpha} + p_2] + \mathrm{e}^{a_1 x} \times \left[t^{s_1} \sum_{n=0}^{\infty} \left[\frac{\Gamma(n\alpha + s_1 - \alpha + 1)}{\Gamma(n\alpha + s_1 + 1)} \times (k_2 p_2 + k_3) b_{n-1} \right] t^{n\alpha} \right], \quad (79)$$

where p_1, p_2, k_1, k_2, k_3 are given by eqs (74), (75), (68), (69) and (70), respectively.

Case 3

$$a_1 = 0,\ a_2 > 0,\ c_{15} = 0,$$
$$c_1 = -[c_4 a_1^2 a_2^2 + c_{10} a_2^2 a_1 + c_7 a_1 a_2 + c_3 a_2^2 + c_6 a_2],$$
$$c_2 = -a_1^4 c_4 - a_1^3 c_{10} - 2a_1^2 a_2 c_4 - c_3 a_1^2 - a_1^2 c_9 - c_{10} a_1 a_2 - a_2^2 c_4 - c_8 a_1 - a_2 c_9,$$
$$c_5 = -2a_1^3 a_2 c_4 - 2c_{10} a_1^2 a_2 - 2a_1 a_2^2 c_4 - a_1^2 c_7 - 2c_3 a_2 a_1 - c_9 a_1 a_2 - c_{10} a_2^2 - c_6 a_1 - a_2 c_7 - c_8 a_2.$$

Hence, we have

$$F = -(a_2^2 c_3 + a_2 c_6) v^2 + c_{10} v_2 v_3 - (a_2^2 c_4 + a_2 c_9) v_1^2 + c_3 v_2^2 + c_4 v_3^2 - (a_2^2 c_{10} + a_2 c_7 + a_2 c_8) v v_1 + c_6 v v_2 + c_7 v v_3 + c_8 v_1 v_2 + c_9 v_1 v_3 + c_{11} v_1 + c_{12} v_2 + c_{13} v_3 + c_{14} v, \quad (80)$$

$$W_2 = \langle \mathrm{e}^{\sqrt{a_2} x}, \mathrm{e}^{-\sqrt{a_2} x} \rangle.$$

Let

$$v(x,t) = \mathrm{e}^{\sqrt{a_2} x} r_1(t) + \mathrm{e}^{-\sqrt{a_2} x} r_2(t). \quad (81)$$

Substituting eq. (81) in eq. (80) and collecting the expansion coefficients, we have the following system of FDEs:

$$D^\alpha r_1(t) = k_1 r_1(t), \quad (82)$$
$$D^\alpha r_2(t) = k_2 r_2(t), \quad (83)$$

where

$$k_1 = c_{13} {a_2}^{3/2} + c_{11} {a_2}^{1/2} + c_{12} a_2 + c_{14},$$
$$k_2 = -(c_{13} {a_2}^{3/2} + c_{11} {a_2}^{1/2}) + c_{12} a_2 + c_{14}.$$

Using Theorem 1 and solving eqs (82) and (83) we have

$$r_1(t) = t^{\alpha-1} E_{\alpha,\alpha}(k_1 t^\alpha), \quad (84)$$
$$r_2(t) = t^{\alpha-1} E_{\alpha,\alpha}(k_2 t^\alpha). \quad (85)$$

Hence, an explicit solution to (27) in this case is given by

$$v(x,t) = (t^{\alpha-1} E_{\alpha,\alpha}(k_1 t^\alpha)) \mathrm{e}^{\sqrt{a_2} x} + (t^{\alpha-1} E_{\alpha,\alpha}(k_2 t^\alpha)) \mathrm{e}^{-\sqrt{a_2} x}. \quad (86)$$

Case 4

$$a_2 = -\frac{a_1^2}{4},$$
$$c_1 = -[c_4 a_1^2 a_2^2 + c_{10} a_2^2 a_1 + c_7 a_1 a_2 + c_3 a_2^2 + c_6 a_2],$$
$$c_{15} = 0,$$
$$c_2 = -[a_1^4 c_4 + a_1^3 c_{10} + 2a_1^2 a_2 c_4 + c_3 a_1^2 + a_1^2 c_9 + c_{10} a_1 a_2 + a_2^2 c_4 + c_8 a_1 + a_2 c_9],$$
$$c_5 = -[2a_1^3 a_2 c_4 + 2c_{10} a_1^2 a_2 + 2a_1 a_2^2 c_4 + a_1^2 c_7 + 2c_3 a_2 a_1 + c_9 a_1 a_2 + c_{10} a_2^2 + c_6 a_1 + a_2 c_7 + c_8 a_2].$$

Hence, we have

$$F = \left(-\frac{1}{16} a_1^6 c_4 - \frac{1}{16} a_1^5 c_{10} - \frac{1}{16} a_1^4 c_3 + \frac{1}{4} a_1^3 c_7 + \frac{1}{4} a_1^2 c_6 \right) v^2 - \left(\frac{9}{16} a_1^4 c_4 + \frac{3}{4} a_1^3 c_{10} + a_1^2 c_3 + \frac{3}{4} a_1^2 c_9 + a_1 c_8 \right) v_1^2 + \left(\frac{3}{8} a_1^5 c_4 + \frac{7}{16} a_1^4 c_{10} + \frac{1}{2} a_1^3 c_3 + \frac{1}{4} a_1^3 c_9 - \frac{3}{4} a_1^2 c_7 \right) v v_1 + \left(\frac{1}{4} a_1^2 c_8 - c_6 a_1 \right) v v_1 + c_{10} v_2 v_3 + c_9 v_1 v_3 + c_{13} v_3 + c_{14} v + c_4 v_3^2 + c_3 v_2^2 + c_{11} v_1 + c_{12} v_2 + c_6 v v_2 + c_7 v v_3 + c_8 v_1 v_2, \quad (87)$$

$$W_2 = \langle \mathrm{e}^{\frac{a_1 x}{2}}, x \mathrm{e}^{\frac{a_1 x}{2}} \rangle.$$

Let

$$v(x,t) = \mathrm{e}^{\frac{a_1 x}{2}} r_1(t) + x \mathrm{e}^{\frac{a_1 x}{2}} r_2(t). \quad (88)$$

Substituting $v(x,t)$ given by eq. (88) in eq. (87) and collecting the expansion coefficients, we get the following system of FDEs:

$$D^{\alpha}r_1(t)=k_1r_1(t)+k_2r_2(t), \tag{89}$$
$$D^{\alpha}r_2(t)=k_1r_2(t), \tag{90}$$

where

$$k_1=\frac{c_{11}a_1}{2}+\frac{c_{12}{a_1}^2}{4}+\frac{c_{13}{a_1}^3}{8}+c_{14},$$
$$k_2=c_{12}a_1+\frac{3}{4}c_{13}{a_1}^2+c_{11}.$$

Solving eq. (90) using Theorem 1 we have

$$r_2(t)=t^{\alpha-1}E_{\alpha,\alpha}(k_1t^{\alpha}). \tag{91}$$

Substituting $r_2(t)$ given by eq. (91) in eq. (89),

$$D^{\alpha}r_1(t)-k_1r_1(t)=k_2(t^{\alpha-1}E_{\alpha,\alpha}(k_1t^{\alpha})),$$
$$r_1(t)=k_2\int_0^t(t-u)^{\alpha-1}E_{\alpha,\alpha}(k_1(t-u)^{\alpha})\times(u^{\alpha-1}E_{\alpha,\alpha}(k_1u^{\alpha}))\mathrm{d}u. \tag{92}$$

Using eq. (14) we have

$$r_1(t)=k_2t^{2\alpha-1}E^2_{\alpha,2\alpha}(k_1t^{\alpha}). \tag{93}$$

Hence, an explicit solution to (27) in this case is

$$v(x,t)=k_2t^{2\alpha-1}E^2_{\alpha,2\alpha}(k_1t^{\alpha})\mathrm{e}^{\frac{a_1x}{2}}+(t^{\alpha-1}E_{\alpha,\alpha}(k_1t^{\alpha}))x\mathrm{e}^{\frac{a_1x}{2}}. \tag{94}$$

Case 5

$$a_1=0,\ a_2=-p^2,\ c_{15}=0,$$
$$c_1=-[c_4a_1^2a_2^2+c_{10}a_2^2a_1+c_7a_1a_2+c_3a_2^2+c_6a_2],$$
$$c_2=-[a_1^4c_4+a_1^3c_{10}+2a_1^2a_2c_4+c_3a_1^2+c_{10}a_1a_2+a_1^2c_9+a_2^2c_4+c_8a_1+a_2c_9],$$
$$c_5=-[2a_1^3a_2c_4+2c_{10}a_1^2a_2+2a_1a_2^2c_4+2c_3a_2a_1+a_1^2c_7+c_9a_1a_2+c_{10}a_2^2+c_6a_1+a_2c_7+c_8a_2].$$

Hence, in this case

$$\begin{aligned}F=&(-c_3p^4+c_6p^2)v^2+c_{10}v_2v_3\\&+(-c_4p^4+c_9p^2)v_1^2+c_3v_2^2\\&+(-c_{10}p^4+c_7p^2+c_8p^2)vv_1\\&+c_6vv_2+c_7vv_3+c_8v_1v_2\\&+c_4v_3^2+c_9v_1v_3+c_{11}v_1+c_{12}v_2\\&+c_{13}v_3+c_{14}v,\end{aligned} \tag{95}$$

$$W_2=\langle\cos(px),\sin(px)\rangle.$$

Let

$$v(x,t)=r_1(t)\cos(px)+r_2(t)\sin(px). \tag{96}$$

Substituting $v(x,t)$ given by eq. (96) in eq. (95) we get

$$D^{\alpha}r_1(t)=k_1r_1(t)+k_2r_2(t), \tag{97}$$
$$D^{\alpha}r_2(t)=-k_2r_1(t)+k_1r_2(t), \tag{98}$$

where

$$k_1=c_{14}-c_{12}p^2,\quad k_2=c_{11}p-c_{13}p^3.$$

Solving eqs (97) and (98) we get

$$r_1(t)=t^{\alpha-1}[E_{\alpha,\alpha}((k_1+ik_2)t^{\alpha})+E_{\alpha,\alpha}((k_1-ik_2)t^{\alpha})], \tag{99}$$
$$r_2(t)=it^{\alpha-1}[E_{\alpha,\alpha}((k_1+ik_2)t^{\alpha})-E_{\alpha,\alpha}((k_1-ik_2)t^{\alpha})]. \tag{100}$$

Hence, an explicit solution to (27) in this case is

$$\begin{aligned}v(x,t)=&t^{\alpha-1}[E_{\alpha,\alpha}((k_1+ik_2)t^{\alpha})\\&+E_{\alpha,\alpha}((k_1-ik_2)t^{\alpha})]\cos(px)\\&+it^{\alpha-1}[E_{\alpha,\alpha}((k_1+ik_2)t^{\alpha})\\&-E_{\alpha,\alpha}((k_1-ik_2)t^{\alpha})]\sin(px).\end{aligned} \tag{101}$$

Case 6

$$a_2=\frac{-2a_1^2}{9},\ a_1=3,$$
$$c_1=\frac{160}{729}a_1^6c_4+\frac{8}{27}a_1^5c_{10}+\frac{32}{81}a_1^4c_3+\frac{28}{81}a_1^4c_9+\frac{2}{9}a_1^3c_7+\frac{4}{9}a_1^3c_8+\frac{4}{9}a_1^2c_2+\frac{2}{9}a_1^2c_6,$$
$$c_5=-\left[\frac{112}{243}a_1^5c_4+\frac{52}{81}a_1^4c_{10}+\frac{8}{9}a_1^3c_3+\frac{22}{27}a_1^3c_9\right]-\left[\frac{7}{9}a_1^2c_7+\frac{10}{9}a_1^2c_8+\frac{4}{3}a_1c_2+c_6a_1\right],$$
$$c_{15}=0.$$

Hence, we have

$$\begin{aligned}F=&(160c_4+72c_{10}+32c_3+28c_9+6c_7\\&+12c_8+4c_2+2c_6)v^2\\&-(112c_4+52c_{10}+24c_3+22c_9\\&+7c_7+10c_8+4c_2+3c_6)vv_1\\&+c_{10}v_2v_3+c_2v_1^2+c_3v_2^2+c_4v_3^2\\&+c_6vv_2+c_7vv_3+c_8v_1v_2\\&+c_9v_1v_3+c_{11}v_1+c_{12}v_2+c_{13}v_3+c_{14}v,\end{aligned} \tag{102}$$

$$W_2=\langle\mathrm{e}^x,\mathrm{e}^{2x}\rangle.$$

Let

$$v(x,t)=r_1(t)\mathrm{e}^x+r_2(t)\mathrm{e}^{2x}. \tag{103}$$

Substituting (103) in eq. (102) we get

$$D^{\alpha}r_1(t)=k_1r_1(t), \tag{104}$$
$$D^{\alpha}r_2(t)=k_2r_1^2(t)+k_3r_2(t), \tag{105}$$

where

$$k_1=c_{11}+c_{12}+c_{13}+c_{14},$$

$k_2 = 21c_{10} + c_2 + 9c_3 + 49c_4 + 3c_8 + 7c_9,$
$k_3 = 2c_{11} + 4c_{12} + 8c_{13} + c_{14}.$

For two particular subcases, we are able to solve eqs (104) and (105) explicitly.

(i) $c_2 = -(21c_{10} + 9c_3 + 49c_4 + 3c_8 + 7c_9)$
In this case, systems (104) and (105) become

$$D^\alpha r_1(t) = k_1 r_1(t), \tag{106}$$
$$D^\alpha r_2(t) = k_3 r_2(t). \tag{107}$$

Using Theorem 1 we get

$$r_1(t) = t^{\alpha-1} E_{\alpha,\alpha}(k_1 t^\alpha), \tag{108}$$
$$r_2(t) = t^{\alpha-1} E_{\alpha,\alpha}(k_3 t^\alpha). \tag{109}$$

Hence, an explicit solution for the above system of FDE is given by

$$v(x,t) = [t^{\alpha-1} E_{\alpha,\alpha}(k_1 t^\alpha)]\mathrm{e}^x + [t^{\alpha-1} E_{\alpha,\alpha}(k_3 t^\alpha)]\mathrm{e}^{2x}. \tag{110}$$

(ii) $c_{14} = -(c_{11} + c_{12} + c_{13})$
In this case, the system of FDEs (104) and (105) becomes

$$D^\alpha r_1(t) = 0, \tag{111}$$
$$D^\alpha r_2(t) = k_2 r_1^2(t) + k_3 r_2(t). \tag{112}$$

Using eq. (8) we have

$$r_1(t) = t^{\alpha-1}. \tag{113}$$

Substituting (113) in (112) we have

$$D^\alpha r_2(t) - k_3 r_2(t) = k_2 t^{2\alpha-2}. \tag{114}$$

Using Theorem 1 we have

$$r_2(t) = k_2 \int_0^t (t-u)^{\alpha-1} E_{\alpha,\alpha}(k_3 (t-u)^\alpha) u^{2\alpha-2} \mathrm{d}u. \tag{115}$$

Evaluating the above integral using eq. (9) we have

$$r_2(t) = k_2 \Gamma(2\alpha-1)[t^{3\alpha-2} E_{\alpha,3\alpha-1}(k_3 t^\alpha)]. \tag{116}$$

An explicit solution for (27) in this case is given by

$$v(x,t) = t^{\alpha-1}\mathrm{e}^x + k_2 t^{3\alpha-2} E_{\alpha,3\alpha-1}(k_3 t^\alpha) \times \Gamma(2\alpha-1)\mathrm{e}^{2x}, \alpha \neq \frac{1}{2}. \tag{117}$$

Case 7

$$a_2 = \frac{-2a_1^2}{9}, \quad a_1 = -3,$$
$$c_1 = \frac{160}{729}a_1^6 c_4 + \frac{8}{27}a_1^5 c_{10} + \frac{32}{81}a_1^4 c_3 + \frac{28}{81}a_1^4 c_9 + \frac{2}{9}a_1^3 c_7 + \frac{4}{9}a_1^3 c_8 + \frac{4}{9}a_1^2 c_2 + \frac{2}{9}a_1^2 c_6,$$
$$c_5 = -\left[\frac{112}{243}a_1^5 c_4 + \frac{52}{81}a_1^4 c_{10} + \frac{8}{9}a_1^3 c_3 + \frac{22}{27}a_1^3 c_9\right] - \left[\frac{7}{9}a_1^2 c_7 + \frac{10}{9}a_1^2 c_8 + \frac{4}{3}a_1 c_2 + c_6 a_1\right],$$
$$c_{15} = 0.$$

Hence, we have

$$F = (160c_4 - 72c_{10} + 32c_3 + 28c_9 - 6c_7 - 12c_8 + 4c_2 + 2c_6)v^2 + (112c_4 - 52c_{10} + 24c_3 + 22c_9 - 7c_7 - 10c_8 + 4c_2 + 3c_6)vv_1 + c_{10}v_2v_3 + c_2v_1^2 + c_3v_2^2 + c_4v_3^2 + c_6vv_2 + c_7vv_3 + c_8v_1v_2 + c_9v_1v_3 + c_{11}v_1 + c_{12}v_2 + c_{13}v_3 + c_{14}v, \tag{118}$$

$$W_2 = \langle \mathrm{e}^{-x}, \mathrm{e}^{-2x} \rangle.$$

Let

$$v(x,t) = r_1(t)\mathrm{e}^{-x} + r_2(t)\mathrm{e}^{-2x}. \tag{119}$$

Substituting (119) in eq. (118) we get

$$D^\alpha r_1(t) = k_1 r_1(t), \tag{120}$$
$$D^\alpha r_2(t) = k_2 r_1^2(t) + k_3 r_2(t), \tag{121}$$

where

$k_1 = -c_{11} + c_{12} - c_{13} + c_{14},$
$k_2 = -21c_{10} + c_2 + 9c_3 + 49c_4 - 3c_8 + 7c_9,$
$k_3 = -2c_{11} + 4c_{12} - 8c_{13} + c_{14}.$

(i) $c_2 = -(-21c_{10} + 9c_3 + 49c_4 - 3c_8 + 7c_9)$
Using (110), an explicit solution in this case is

$$v(x,t) = t^{\alpha-1} E_{\alpha,\alpha}(k_1 t^\alpha)\mathrm{e}^{-x} + t^{\alpha-1} E_{\alpha,\alpha}(k_3 t^\alpha)\mathrm{e}^{-2x}. \tag{122}$$

(ii) $c_{14} = c_{11} - c_{12} + c_{13}$
Using (117), an explicit solution in this case is,

$$v(x,t) = t^{\alpha-1}\mathrm{e}^{-x} + k_2 t^{3\alpha-2} E_{\alpha,3\alpha-1}(k_3 t^\alpha) \times \Gamma(2\alpha-1)\mathrm{e}^{-2x}, \alpha \neq \frac{1}{2}. \tag{123}$$

5. Summary

In this paper, we consider a general FDE of the form (27), where F is a third-order polynomial operator with quadratic non-linearities. A third-order operator F of the form (27) can admit subspaces of maximal dimension 7. In this article, we have described all operators of the form (27) admitting 2D subspaces which forms solution space of a second-order ODE with constant coefficients. Subsequently, we have illustrated the effectiveness of the invariant subspace method by finding explicit solutions to FDE (1) for some particular cases of F obtained. We have also shown that an explicit solution of some time-fractional differential equations like KdV equation, potential KdV equation, KdV–Burgers equation, Rosenau–Hyman equation and Benjamin equation can be found using this method. Also in the appendices we have given a list of operators of the form (27) that admits 3D, 4D, 5D, 6D and 7D subspaces which are solutions of nth-order ODE (3rd, 4th, 5th, 6th, 7th) respectively with constant coefficients. For 2D subspaces, we find that the list of operators given in this article, as described by eq. (27) is exhaustive. For 3D, 4D, 5D, 6D and 7D cases we do not claim the list is exhaustive.

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Appendix A

We describe third-order operators of the form (29) admitting n-dimensional subspaces

$$W_n = \langle g_1(x), g_2(x), \ldots, g_n(x)\rangle, \quad n = 3, 4, 5, 6, 7, \tag{A.1}$$

where $\{g_1(x), g_2(x), \ldots, g_n(x)\}$ forms a fundamental set of solution for the linear ODE

$$\mathfrak{L}[v] = v_n - a_1 v_{n-1} - \cdots - a_{n-1}v_1 - a_n v = 0, \tag{A.2}$$

where $\{a_1, a_2, \ldots, a_n\}$ are arbitrary real constants and $v_j = \partial^j v/\partial x^j$. Now the invariance condition given by (23) for the subspace W_n with respect to F is

$$\mathfrak{L}[F[v]] = 0, \text{ given } \mathfrak{L}[v] = 0. \tag{A.3}$$

Equation (A.3) can be rewritten as

$$D^n F - a_1 D^{n-1} F - \cdots - a_{n-1} DF - a_n F = 0, \tag{A.4}$$

where D is the total derivative operator given by

$$D = \frac{\partial}{\partial x} + v_1\frac{\partial}{\partial v} + \cdots + v_{n-1}\frac{\partial}{\partial v_{n-2}} + [a_1 v_{n-1} + \cdots + a_{n-1}v_1 + a_n v]\frac{\partial}{\partial v_{n-1}}. \tag{A.5}$$

Now we give the operators of the form (27) and the corresponding invariant subspace obtained by solving the invariance condition (A.4) for $n = 3, 4, 5, 6, 7$.

3D Subspaces

Case 1

$a_1 = 0,\ a_3 = 0,$
$c_1 = -a_2^3 c_4 - a_2^2 c_3 - a_2^2 c_9 - a_2 c_2 - a_2 c_6,$
$c_5 = -a_2^2 c_{10} - c_7 a_2 - a_2 c_8,$
$W_3 = \langle 1, \mathrm{e}^{\sqrt{a_2}x}, \mathrm{e}^{-\sqrt{a_2}x}\rangle.$

Case 2

$a_1 = 0,\ a_2 = -p^2,\ a_3 = 0,$
$c_1 = c_4 p^6 - c_3 p^4 - c_9 p^4 + c_2 p^2 + c_6 p^2,$
$c_5 = -c_{10} p^4 + c_7 p^2 + c_8 p^2,$
$W_3 = \langle 1, \cos(px), \sin(px)\rangle.$

Case 3

$a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ c_1 = 0,\ c_5 = 0,$
$W_3 = \langle 1, x, x^2\rangle.$

Case 4

$a_1 = 3,\ a_2 = -\dfrac{2a_1^2}{9},\ a_3 = 0,$
$c_1 = 160c_4 + 72c_{10} + 32c_3 + 28c_9 + 6c_7 + 12c_8 + 4c_2 + 2c_6,$
$c_5 = -112c_4 - 52c_{10} - 24c_3 - 22c_9 - 7c_7 - 10c_8 - 4c_2 - 3c_6,$
$W_3 = \langle 1, \mathrm{e}^{x}, \mathrm{e}^{2x}\rangle.$

Case 5

$a_1 = -3,\ a_2 = -\dfrac{2{a_1}^2}{9},\ a_3 = 0,$
$c_1 = 160c_4 - 72c_{10} + 32c_3 + 28c_9 - 6c_7 - 12c_8 + 4c_2 + 2c_6,$
$c_5 = 112c_4 - 52c_{10} + 24c_3 + 22c_9 - 7c_7 - 10c_8 + 4c_2 + 3c_6,$
$W_3 = \langle 1, \mathrm{e}^{-x}, \mathrm{e}^{-2x}\rangle.$

Case 6

$a_1 = 0,\ a_2 = 7,\ a_3 = -6,$
$c_2 = \dfrac{7c_{10} + c_8}{2} - 49c_4 - 7c_9,$
$c_1 = -36c_4 + 126c_{10} + 6c_7 + 18c_8,\ c_3 = \dfrac{7c_{10} + c_8}{2},$
$c_5 = -42c_{10} - 6c_8 + 84c_4 + 6c_9 - 7c_7,$
$c_6 = -36c_{10} - 6c_8,\ c_{15} = 0,$

$W_3 = \langle e^{x}, e^{2x}, e^{-3x}\rangle$.

Case 7

$$a_2 = -\frac{a_1^2}{4},\ a_3{=}0,\ c_6{=}\frac{a_1^4c_4 + a_1^3c_{10} + a_1^2c_3}{4} - c_7a_1,$$

$$c_1 = 0,\ c_2{=}\frac{7a_1^6c_4{+}8a_1^5c_{10}{+}8a_1^4c_3{+}4a_1^4c_9}{16a_1^2},\ c_{15}{=}0,$$

$$c_5 = -\left(\frac{a_1^6c_4 + a_1^5c_{10} + a_1^4c_3 - 2a_1^3c_7}{8a_1}\right),$$

$$c_8 = -\left(\frac{4a_1^4c_4 + 5a_1^3c_{10} + 6a_1^2c_3 + 4a_1^2c_9}{4a_1}\right),$$

$W_3 = \langle 1, e^{\frac{a_1x}{2}}, xe^{\frac{a_1x}{2}}\rangle$.

Case 8

$a_1 = 0,\ a_2 = 7,\ a_3 = 6,$
$c_1 = -[36c_4 + 126c_{10} + 6c_7 + 18c_8],$

$$c_2 = -\left[\frac{7c_{10}}{2} + \frac{c_8}{2} + 49c_4 + 7c_9\right],$$

$$c_3 = -\left[\frac{7c_{10} + c_8}{2}\right],$$

$c_5 = -[42c_{10} + 6c_8 + 84c_4 + 6c_9 + 7c_7],$
$c_6 = 36c_{10} + 6c_8,\ c_{15} = 0,$
$W_3 = \langle e^{-x}, e^{-2x}, e^{3x}\rangle$.

Case 9

$a_1 = 4,\ a_2 = -5,\ a_3 = 2,$
$c_1 = 16c_{10} + 4c_3 + 60c_4 - 2c_7,$
$c_2 = 36c_{10} + 9c_3 + 119c_4 + 5c_9,$
$c_5 = -48c_{10} - 12c_3 - 172c_4 + 5c_7 - 2c_9$
$c_6 = 48c_4 + 14c_{10} + 4c_3 - 4c_7,$
$c_8 = -19c_{10} - 6c_3 - 56c_4 - 4c_9,$
$W_3 = \langle e^{x}, xe^{x}, e^{2x}\rangle$.

Case 10

$a_1 = 2,\ a_2 = 1,\ a_3 = -2,$
$c_1 = 84c_4 + 38c_{10} + 16c_3 + 12c_9 + 6c_8 + 2c_7,$
$c_2 = 7c_{10} + 3c_3 + 15c_4 + c_8 + c_9,$
$c_5 = 32c_4 + 13c_{10} + 6c_3 + 4c_9 - c_7 + c_8,$
$c_6 = -68c_4 - 31c_{10} - 14c_3 - 10c_9 - 2c_7 - 5c_8,$
$c_{15} = 0,$
$W_3 = \langle e^{x}, e^{-x}, e^{2x}\rangle$.

Case 11

$a_1 = -2,\ a_2 = 1,\ a_3 = 2,$
$c_1 = 84c_4 - 38c_{10} + 16c_3 + 12c_9 - 6c_8 - 2c_7,$
$c_2 = -7c_{10} + 3c_3 + 15c_4 - c_8 + c_9,$
$c_5 = -32c_4 + 13c_{10} - 6c_3 - 4c_9 - c_7 + c_8,$
$c_6 = -68c_4 + 31c_{10} - 14c_3 - 10c_9 + 2c_7 + 5c_8,$
$c_{15} = 0,$
$W_3 = \langle e^{x}, e^{-x}, e^{-2x}\rangle$.

Case 12

$a_1 = -7,\ a_2 = -14,\ a_3 = -8,$

$$c_2 = 133c_{10} - 30c_3 - 588c_4 + \frac{11}{2}c_8 - \frac{49}{2}c_9,$$

$c_1 = 8c_7 - 6336c_4 + 1232c_{10} - 224c_3$
$\quad - 168c_9 + 24c_8,$
$c_5 = -6496c_4 + 1288c_{10} - 240c_3$
$\quad - 188c_9 + 14c_7 + 28c_8,$
$c_6 = -896c_4 + 176c_{10} - 32c_3 - 28c_9 + 7c_7 + 4c_8,$
$c_{15} = 0,$
$W_3 = \langle e^{-x}, e^{-2x}, e^{-4x}\rangle$.

Case 13

$a_1 = 7,\ a_2 = -14,\ a_3 = 8,$
$c_1 = -8c_7 - 6336c_4 - 1232c_{10} - 224c_3$
$\quad - 168c_9 - 24c_8,$

$$c_2 = -133c_{10} - 30c_3 - 588c_4 - \frac{11}{2}c_8 - \frac{49}{2}c_9,$$

$c_5 = 6496c_4 + 1288c_{10} + 240c_3$
$\quad + 188c_9 + 14c_7 + 28c_8,$
$c_6 = -896c_4 - 176c_{10} - 32c_3 - 28c_9 - 7c_7 - 4c_8,$
$c_{15} = 0,$
$W_3 = \langle e^{x}, e^{2x}, e^{4x}\rangle$.

Case 14

$a_1 = 6,\ a_2 = -11,\ a_3 = 6,$
$c_1 = -6c_7 - 2844c_4 - 666c_{10} - 144c_3$
$\quad - 108c_9 - 18c_8,$
$c_2 = -95c_{10} - 25c_3 - 361c_4 - 5c_8 - 19c_9,$
$c_5 = 2760c_4 + 669c_{10}{+}150c_3{+}120c_9{+}11c_7{+}21c_8,$
$c_6 = -324c_4 - 81c_{10} - 18c_3 - 18c_9$
$\quad - 6c_7 - 3c_8,\ c_{15} = 0,$
$W_3 = \langle e^{x}, e^{2x}, e^{3x}\rangle$.

Case 15

$a_1 = -6,\ a_2 = -11,\ a_3 = -6,$
$c_1 = 6c_7 - 2844c_4 + 666c_{10}{-}144c_3{-}108c_9{+}18c_8,$
$c_2 = 95c_{10} - 25c_3 - 361c_4 + 5c_8 - 19c_9,$
$c_5 = -2760c_4 + 669c_{10} - 150c_3$
$\quad - 120c_9 + 11c_7 + 21c_8,$

$c_6 = -324c_4 + 81c_{10} - 18c_3 - 18c_9 + 6c_7 + 3c_8,$
$c_{15} = 0,$
$W_3 = \langle e^{-x}, e^{-2x}, e^{-3x} \rangle.$

Appendix B

4D Subspaces

Case 1

$$a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ c_1 = 0,\ c_2 = 0,\ c_5 = 0,$$
$$c_7 = -\frac{c_6}{a_1},\ c_8 = -a_1 c_9,\ c_{10} = -\left(\frac{a_1^2 c_4 + c_3}{a_1}\right),$$
$$W_4 = \langle 1, x, x^2, e^{a_1 x} \rangle.$$

Case 2

$$a_2 = p^2, a_3 = -p^2 a_1, a_4 = 0,$$
$$c_1 = -p^2 \frac{a_1 c_3 p^2 + a_1 c_6}{a_1},$$
$$c_2 = -p^2 (c_4 p^2 + c_9),$$
$$c_5 = \frac{a_1^3 c_4 p^4 + a_1^3 c_9 p^2 + a_1 c_3 p^4 + a_1 c_6 p^2}{a_1^2},$$
$$c_7 = \frac{a_1^3 c_4 p^2 - a_1 c_3 p^2 - a_1 c_6}{a_1^2},$$
$$c_{10} = -\frac{a_1^2 c_4 + c_3}{a_1},$$
$$c_8 = -\left(\frac{a_1^2 c_4 p^2 + a_1^2 c_9 - c_3 p^2}{a_1}\right),$$
$$W_4 = \langle 1, e^{a_1 x}, e^{px}, e^{-px} \rangle.$$

Case 3

$$a_2 = -p^2,\ a_3 = p^2 a_1,\ a_4 = 0,$$
$$c_1 = \frac{p^2(-a_1 c_3 p^2 + a_1 c_6)}{a_1},\ c_{10} = -\left(\frac{a_1^2 c_4 + c_3}{a_1}\right),$$
$$c_2 = p^2(-c_4 p^2 + c_9),$$
$$c_5 = \left(\frac{a_1^3 c_4 p^4 - a_1^3 c_9 p^2 + a_1 c_3 p^4 - a_1 c_6 p^2}{a_1^2}\right),$$
$$c_7 = -\left(\frac{a_1^3 c_4 p^2 - a_1 c_3 p^2 + a_1 c_6}{a_1^2}\right),$$
$$c_8 = \left(\frac{a_1^2 c_4 p^2 - a_1^2 c_9 - c_3 p^2}{a_1}\right),$$
$$W_4 = \langle 1, e^{a_1 x}, \cos(px), \sin(px) \rangle.$$

Case 4

$$a_2 = 1,\ a_3 = -a_1,\ a_4 = 0,$$
$$c_1 = -a_1^4 c_4 - a_1^3 c_{10} - a_1^2 c_3 + a_1^2 c_4 + a_1 c_{10} - c_6,$$
$$c_2 = a_1^4 c_4 + a_1^3 c_{10} + a_1^2 c_3 - a_1^2 c_4 - a_1 c_{10} - c_3 - c_4 - c_9,$$
$$c_5 = \frac{2a_1^4 c_4 + 2a_1^3 c_{10} + 2a_1^2 c_3 + a_1^2 c_9 - a_1 c_{10} + c_6}{a_1},$$
$$c_7 = \left(\frac{2a_1^2 c_4 + a_1 c_{10} - c_6}{a_1}\right),$$
$$c_8 = -\left(\frac{2a_1^4 c_4 + 2a_1^3 c_{10} + 2a_1^2 c_3 + 2a_1^2 c_4 + a_1^2 c_9 + a_1 c_{10}}{a_1}\right),$$
$$W_4 = \langle 1, e^{x}, e^{-x}, e^{a_1 x} \rangle.$$

Case 5

$$a_1 = 2,\ a_2 = 1,\ a_3 = -2,\ a_4 = 0,$$
$$c_1 = \frac{2c_7 + 8c_8 - 2c_6 + 20c_3 + 52c_{10} + 116c_4 + 16c_9}{3},$$
$$c_2 = \frac{91c_4 + 4c_7 + 41c_{10} + 19c_3 + 7c_8 + 2c_6 + 17c_9}{3},$$
$$c_5 = \frac{8c_{10} + 4c_3 + 28c_4 - 2c_8 - 5c_7 + 2c_9 - c_6}{3},$$
$$W_4 = \langle 1, e^{x}, e^{-x}, e^{2x} \rangle.$$

Case 6

$$a_1 = -2,\ a_2 = 1,\ a_3 = 2,\ a_4 = 0,$$
$$c_1 = \frac{20c_3 - 2c_7 - 8c_8 - 2c_6 - 52c_{10} + 116c_4 + 16c_9}{3},$$
$$c_2 = \frac{7c_8 - 91c_4 + 4c_7 + 41c_{10} - 19c_3 - 2c_6 - 17c_9}{3},$$
$$c_5 = \frac{8c_{10} - 4c_3 - 28c_4 - 2c_8 - 5c_7 - 2c_9 + c_6}{3},$$
$$W_4 = \langle 1, e^{x}, e^{-x}, e^{-2x} \rangle.$$

Case 7

$$a_1 = 2p,\ a_2 = p^2,\ a_3 = -2p^3,\ a_4 = 0,$$
$$c_1 = -4c_4 p^6 + 2c_7 p^3,\ c_{10} = -\left(\frac{4c_4 p^2 + c_3}{2p}\right),$$
$$c_2 = -\frac{1}{4} p^2 (4c_4 p^2 + 4c_9),$$
$$c_5 = 4c_4 p^5 + 2c_9 p^3 - c_7 p^2,$$
$$c_6 = 4c_4 p^4 - c_3 p^2 - 2c_7 p,$$
$$c_8 = -2p^3 c_4 + \frac{1}{2} p c_3 - 2c_9 p,$$
$$W_4 = \langle 1, e^{px}, e^{-px}, e^{2px} \rangle.$$

Case 8

$$a_1 = 14,\ a_2 = 49,\ a_3 = 34,\ a_4 = 0,$$
$$c_1 = -34c_7 + 225420c_4 + 16184c_{10} + 1156c_3,$$
$$c_2 = 41699c_4 + 3150c_{10} + 225c_3 - 49c_9,$$
$$c_5 = 14280c_{10} + 1020c_3 + 196588c_4 - 34c_9 - 49c_7,$$
$$c_6 = -14280c_4 - 986c_{10} - 68c_3 - 14c_7,$$
$$c_8 = -7252c_4 - 469c_{10} - 30c_3 - 14c_9,$$
$$W_4 = \langle 1, \mathrm{e}^{-x}, \mathrm{e}^{-2x}, \mathrm{e}^{17x}\rangle.$$

Case 9

$$a_1 = -14,\ a_2 = 49,\ a_3 = -34,\ a_4 = 0,$$
$$c_1 = 34c_7 + 225420c_4 - 16184c_{10} + 1156c_3,$$
$$c_2 = 41699c_4 - 3150c_{10} + 225c_3 - 49c_9,$$
$$c_5 = 14280c_{10} - 1020c_3 - 196588c_4 + 34c_9 - 49c_7,$$
$$c_6 = -14280c_4 + 986c_{10} - 68c_3 + 14c_7,$$
$$c_8 = 7252c_4 - 469c_{10} + 30c_3 + 14c_9,$$
$$W_4 = \langle 1, \mathrm{e}^{x}, \mathrm{e}^{2x}, \mathrm{e}^{-17x}\rangle.$$

Case 10

$$a_1 = 0,\ a_2 > 0,\ a_3 = 0,\ a_4 = 0,$$
$$c_1 = 0,\ c_2 = -a_2^2c_4 - a_2c_3 - a_2c_9,$$
$$c_5 = -c_7a_2,\ c_6 = 0,$$
$$c_{10} = -\frac{c_8}{a_2},$$
$$W_4 = \langle 1, x, \mathrm{e}^{\sqrt{(a_2)}x}, \mathrm{e}^{-\sqrt{(a_2)}x}\rangle.$$

Case 11

$$a_1 = 0,\ a_2 = -p^2,\ a_3 = 0,\ a_4 = 0,$$
$$c_1 = 0, c_2 = -c_4p^4 + c_3p^2 + c_9p^2,\ c_5 = c_7p^2,$$
$$c_6 = 0,\ c_{10} = \frac{c_8}{p^2},$$
$$W_4 = \langle 1, x, \cos(px), \sin(px)\rangle.$$

Case 12

$$a_2 = 4a_1^2,\ a_3 = -4a_1^3,\ a_4 = 0,$$
$$c_2 = -4a_1^4c_4 + 6a_1^3c_{10} - \frac{5}{2}a_1^2c_9 + \frac{3}{2}a_1c_8,$$
$$c_1 = -48a_1^6c_4 - 24a_1^5c_{10} - 16a_1^4c_3 - 6a_1^4c_9 - 6a_1^3c_8 - 4a_1^2c_6,$$
$$c_5 = 128a_1^5c_4 + 64a_1^4c_{10} + 32a_1^3c_3 + 16a_1^3c_9 + 12a_1^2c_8 + 4a_1c_6,$$
$$c_7 = -\left(\frac{32a_1^4c_4 + 20a_1^3c_{10} + 8a_1^2c_3 + 4a_1^2c_9 + 4a_1c_8 + c_6}{a_1}\right),$$
$$W_4 = \langle 1, \mathrm{e}^{a_1x}, \mathrm{e}^{2a_1x}, \mathrm{e}^{-2a_1x}\rangle.$$

Case 13

$$a_1 = 7,\ a_2 = -14,\ a_3 = 8,\ a_4 = 0,$$
$$c_1 = -\frac{7216}{7}c_{10} - \frac{1312}{7}c_3 - 136c_9 - \frac{136}{7}c_8 + \frac{8}{7}c_6 - 5312c_4,$$
$$c_2 = -588c_4 - 133c_{10} - 30c_3 - \frac{49}{2}c_9 - \frac{11}{2}c_8,$$
$$c_5 = 936c_{10} + 176c_3 + 4704c_4 + 20c_8 + 132c_9 - 2c_6,$$
$$c_7 = -\frac{176}{7}c_{10} - \frac{32}{7}c_3 - 128c_4 - \frac{1}{7}c_6 - \frac{4}{7}c_8 - 4c_9,$$
$$W_4 = \langle 1, \mathrm{e}^{x}, \mathrm{e}^{2x}, \mathrm{e}^{4x}\rangle.$$

Case 14

$$a_1 = -7,\ a_2 = -14,\ a_3 = -8,\ a_4 = 0,$$
$$c_2 = -588c_4 + 133c_{10} - 30c_3 - \frac{49}{2}c_9 + \frac{11}{2}c_8,$$
$$c_1 = \frac{7216}{7}c_{10} - \frac{1312}{7}c_3 - 136c_9 + \frac{136}{7}c_8 + \frac{8}{7}c_6 - 5312c_4$$
$$c_5 = 936c_{10} - 176c_3 - 4704c_4 + 20c_8 - 132c_9 + 2c_6,$$
$$c_7 = -\frac{176}{7}c_{10} + \frac{32}{7}c_3 + 128c_4 + \frac{1}{7}c_6 - \frac{4}{7}c_8 + 4c_9,$$
$$W_4 = \langle 1, \mathrm{e}^{-x}, \mathrm{e}^{-2x}, \mathrm{e}^{-4x}\rangle.$$

Case 15

$$a_1 = 6,\ a_2 = -11,\ a_3 = 6,\ a_4 = 0,$$
$$c_2 = -361c_4 - 95c_{10} - 25c_3 - 19c_9 - 5c_8,$$
$$c_1 = -2520c_4 - 585c_{10} - 126c_3 - 90c_9 - 15c_8 + c_6,$$
$$c_5 = \frac{1041}{2}c_{10} + 117c_3 + 2166c_4 + \frac{31}{2}c_8 + 87c_9 - \frac{11}{6}c_6,$$
$$c_7 = -\frac{27}{2}c_{10} - 3c_3 - 54c_4 - \frac{1}{6}c_6 - \frac{1}{2}c_8 - 3c_9,$$
$$W_4 = \langle 1, \mathrm{e}^{x}, \mathrm{e}^{2x}, \mathrm{e}^{3x}\rangle.$$

Case 16

$$a_1 = -6,\ a_2 = -11,\ a_3 = -6,\ a_4 = 0,$$
$$c_2 = -361c_4 + 95c_{10} - 25c_3 - 19c_9 + 5c_8,$$
$$c_1 = -2520c_4 + 585c_{10} - 126c_3 - 90c_9 + 15c_8 + c_6,$$
$$c_5 = \frac{1041}{2}c_{10} - 117c_3 - 2166c_4 + \frac{31}{2}c_8 - 87c_9 + \frac{11}{6}c_6,$$
$$c_7 = -\frac{27}{2}c_{10} + 3c_3 + 54c_4 + \frac{1}{6}c_6 - \frac{1}{2}c_8 + 3c_9,$$

$W_4 = \langle 1, e^{-x}, e^{-2x}, e^{-3x} \rangle.$

Appendix C

5D Subspaces

Case 1

$a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ c_1 = 0,$
$c_2 = -\frac{3}{4}c_6,\ c_5 = 0,\ c_8 = -\frac{1}{2}c_7,$
$W_5 = \langle 1, x, x^2, x^3, x^4 \rangle.$.

Case 2

$a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ c_1 = 0,$
$c_{10} = -\frac{2a_1^3 c_4 + a_1 c_9 + c_7}{a_1^2},$
$c_2 = \frac{2}{3}c_7 a_1,\ c_3 = \frac{3a_1^3 c_4 + 3a_1 c_9 + 5c_7}{3a_1},$
$c_5 = 0,\ c_6 = -c_7 a_1,\ c_8 = -a_1 c_9 - \frac{4}{3}c_7,$
$W_5 = \langle 1, x, x^2, x^3, e^{a_1 x} \rangle.$

Case 3

$a_1 = 0,\ a_2 > 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,$
$c_1 = 0,\ c_{10} = 0,\ c_2 = -\frac{1}{2}c_9 a_2,$
$c_3 = -c_4 a_2 - \frac{1}{2}c_9,\ c_5 = 0,\ c_6 = 0,\ c_7 = 0,\ c_8 = 0,$
$W_5 = \langle 1, x, x^2, e^{\sqrt{a_2}x}, e^{-\sqrt{a_2}x} \rangle.$

Case 4

$a_1 = 0,\ a_2 = -p^2,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,$
$c_1 = 0,\ c_{10} = 0,\ c_2 = \frac{1}{2}c_9 p^2,\ c_3 = c_4 p^2 - \frac{1}{2}c_9,$
$c_5 = 0,\ c_6 = 0,\ c_7 = 0,\ c_8 = 0,$
$W_5 = \langle 1, x, x^2, \cos(px), \sin(px) \rangle.$

Case 5

$a_1 = 2,\ a_2 = -1,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,$
$c_1 = 0,\ c_2 = -c_4 + \frac{1}{2}c_3,\ c_5 = 0,$
$c_6 = -2c_{10} - c_3 - 4c_4,$
$c_7 = 2c_{10} + c_3 + 4c_4,\ c_8 = 8c_4 + 3c_{10},$
$c_9 = -8c_4 - 4c_{10} - \frac{3}{2}c_3,$
$W_5 = \langle 1, x, x^2, e^x, xe^x \rangle.$

Case 6

$a_1 = -2,\ a_2 = -1,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,$
$c_1 = 0,\ c_2 = -c_4 + \frac{1}{2}c_3,$
$c_6 = 2c_{10} - c_3 - 4c_4,\ c_7 = 2c_{10} - c_3 - 4c_4,$
$c_8 = -8c_4 + 3c_{10},\ c_9 = -8c_4 + 4c_{10} - \frac{3}{2}c_3,$
$W_5 = \langle 1, x, x^2, e^{-x}, xe^{-x} \rangle.$

Case 7

$a_1^2 + 4a_2 = 0,$
$a_1 = 2,\ a_2 = -1,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,$
$c_1 = 0,\ c_2 = c_4,\ c_3 = 4c_4,\ c_5 = 0,$
$c_6 = 0,\ c_7 = 0,\ c_8 = -4c_4,\ c_9 = 2c_4,\ c_{10} = -4c_4,$
$W_5 = \langle 1, x, x^2, e^x, xe^x \rangle.$

Case 8

$a_1^2 + 4a_2 > 0,$
$a_1 = 1,\ a_2 = 2,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,$
$c_1 = 0,\ c_2 = 4c_4,\ c_3 = c_4,\ c_5 = 0,$
$c_6 = 0,\ c_7 = 0,\ c_8 = 4c_4,\ c_9 = -4c_4,\ c_{10} = -2c_4,$
$W_5 = \langle 1, x, x^2, e^{-x}, e^{2x} \rangle.$

Case 9

$a_1^2 + 4a_2 < 0,$
$a_1 = 0,\ a_2 = -4,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,$
$c_1 = 0,\ c_2 = 16c_4,\ c_3 = 0,\ c_5 = 0,$
$c_6 = 0,\ c_7 = 0,\ c_8 = 0,\ c_9 = 8c_4,\ c_{10} = 0,$
$W_5 = \langle 1, x, x^2, \cos(2x), \sin(2x) \rangle.$

Case 10

$a_1 = 3,\ a_2 = -2,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,$
$c_1 = 0,\ c_2 = -8c_4 - 2c_{10},$
$c_3 = -\frac{5}{2}c_{10} - 6c_4,\ c_5 = 0,\ c_6 = 0,\ c_7 = 0,$
$c_8 = 24c_4 + 6c_{10},\ c_9 = -8c_4 - 2c_{10}$
$W_5 = \langle 1, x, x^2, e^x, e^{2x} \rangle.$

Case 11

$a_1 = -3,\ a_2 = -2,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,$
$c_1 = 0,\ c_2 = -8c_4 + 2c_{10},$
$c_3 = \frac{5}{2}c_{10} - 6c_4,\ c_5 = 0,\ c_6 = 0,\ c_7 = 0,$
$c_8 = -24c_4 + 6c_{10},\ c_9 = -8c_4 + 2c_{10},$
$W_5 = \langle 1, x, x^2, e^{-x}, e^{-2x} \rangle.$

Appendix D

6D Subspaces

Case 1

$a_1=0,\ a_2=0,\ a_3=0,\ a_4=0,\ a_5=0,\ a_6=0,$
$c_1=0,\ c_2=0,\ c_3=-\dfrac{3c_9}{4},\ c_5=0,\ c_6=0, c_7=0,$
$c_8=0, c_{10}=0, c_{14}=0,$
$W_6=\langle 1,x,x^2,x^3,x^4,x^5\rangle.$

Case 2

$a_2=0,\ a_3=0,\ a_4=0,\ a_5=0,\ a_6=0,\ c_1=0,$
$c_{10}=-\left(\dfrac{4c_3+3c_9}{2a_1}\right),\ c_2=-\dfrac{3a_1^2c_9}{4},$
$c_4=\left(\dfrac{4c_3+3c_9}{4a_1^2}\right),\ c_5=0,\ c_6=a_1^2c_9,$
$c_7=-c_9a_1,\ c_8=\dfrac{c_9a_1}{2},$
$c_{11}=-\left(\dfrac{a_1^3c_{13}+a_1^2c_{12}+c_{14}}{a_1}\right),$
$W_6=\langle 1,x,x^2,x^3,x^4,\mathrm{e}^{a_1x}\rangle.$

Case 3

$a_1^2+4a_2=0,$
$a_3=0,\ a_4=0,\ a_5=0,\ a_6=0,\ c_1=0,\ c_{10}=0,$
$c_2=0,\ c_3=0,$
$c_4=0,\ c_5=0,\ c_6=0,\ c_7=0,\ c_8=0,\ c_9=0,$
$c_{11}=-a_1^2c_{13}-a_1c_{12}-a_2c_{13},$
$c_{14}=-a_1a_2c_{13}-a_2c_{12},$
$W_6=\langle 1,x,x^2,x^3,\mathrm{e}^{\frac{a_1x}{2}},x\mathrm{e}^{\frac{a_1x}{2}}\rangle.$

Case 4

$a_1^2+4a_2>0,$
$a_3=0,\ a_4=0,\ a_5=0,\ a_6=0,\ c_1=0,$
$c_{10}=0,\ c_2=0,$
$c_3=0,\ c_4=0,\ c_5=0,\ c_6=0,$
$c_7=0,\ c_8=0,\ c_9=0,$
$c_{11}=-a_1^2c_{13}-a_1c_{12}-a_2c_{13},$
$c_{14}=-a_1a_2c_{13}-a_2c_{12},$
$a_1^2+4a_2=p^2,$
$W_6=\langle 1,x,x^2,x^3,\mathrm{e}^{\frac{(a_1+p)x}{2}},\mathrm{e}^{\frac{(a_1-p)x}{2}}\rangle.$

Case 5

$a_1^2+4a_2<0,$
$a_3=0,\ a_4=0,\ a_5=0,\ a_6=0,\ c_1=0,$
$c_{10}=0,\ c_2=0,$
$c_3=0,\ c_4=0,\ c_5=0,\ c_6=0,\ c_7=0,$
$c_8=0,\ c_9=0,$
$c_{11}=-a_1^2c_{13}-a_1c_{12}-a_2c_{13},$
$c_{14}=-a_1a_2c_{13}-a_2c_{12},$
$a_1^2+4a_2=-p^2,$
$W_6=\langle 1,x,x^2,x^3,\mathrm{e}^{\frac{a_1x}{2}}\cos(px),\mathrm{e}^{\frac{a_1x}{2}}\sin(px)\rangle.$

Appendix E

7D Subspaces

Case 1

$a_1=0,\ a_2=0,\ a_3=0,\ a_4=0,\ a_5=0,\ a_6=0,$
$a_7=0,\ c_1=0,\ c_2=0,\ c_3=0, c_4=0,\ c_5=0,$
$c_6=0,\ c_7=0, c_8=0,\ c_9=0,\ c_{10}=0,$
$W_7=\langle 1,x,x^2,x^3,x^4,x^5,x^6\rangle.$

Case 2

$a_1=0,\ a_2=7,\ a_3=0,$
$a_4=-\dfrac{49}{4},\ a_5=0,\ a_6=\dfrac{9}{2},$
$a_7=0,\ c_4=0,\ c_{10}=0,\ c_1=-\dfrac{9}{4}c_3,\ c_2=-\dfrac{5}{8}c_3,$
$c_5=0,\ c_6=\dfrac{9}{4}c_3,\ c_7=0,$
$c_8=0,\ c_9=-\dfrac{5}{4}c_3,$
$W_7=\langle 1,\mathrm{e}^{\sqrt{2}x},\mathrm{e}^{-\sqrt{2}x},\mathrm{e}^{\frac{x}{\sqrt{2}}},\mathrm{e}^{-\frac{x}{\sqrt{2}}},\mathrm{e}^{\frac{3x}{\sqrt{2}}},\mathrm{e}^{-\frac{3x}{\sqrt{2}}}\rangle.$

Case 3

$a_1=0,\ a_2=-7,\ a_3=0,\ a_4=-\dfrac{49}{4},\ a_5=0,$
$a_6=-\dfrac{9}{2},\ a_7=0,\ c_4=0,\ c_{10}=0, c_1=-\dfrac{9}{4}c_3,$
$c_2=\dfrac{5}{8}c_3,\ c_5=0,\ c_6=-\dfrac{9}{4}c_3,\ c_7=0,$
$c_8=0, c_9=-\dfrac{5}{4}c_3,$
$$W_7=\left\langle 1,\cos\sqrt{2}x,\sin\sqrt{2}x,\cos\frac{x}{\sqrt{2}},\ \sin\frac{x}{\sqrt{2}},\cos\frac{3x}{\sqrt{2}},\sin\frac{3x}{\sqrt{2}}\right\rangle.$$

Case 4

$a_1=7,\ a_2=-7,\ a_3=-35,\ a_4=56,\ a_5=28,$

$a_6=-48,\ a_7=0,\ c_1=24c_3,\ c_2=\frac{33}{2}c_3,\ c_4=0,$
$c_{10}=0,\ c_5=-8c_3,\ c_6=-21c_3,$
$c_7=5c_3,\ c_8=-\frac{11}{4}c_3,\ c_9=-\frac{5}{4}c_3,$
$W_7=\langle 1,\mathrm{e}^{x},\mathrm{e}^{-x},\mathrm{e}^{2x},\mathrm{e}^{-2x},\mathrm{e}^{3x},\mathrm{e}^{4x}\rangle.$

Case 5

$a_1=-7,\ a_2=-7,\ a_3=35,\ a_4=56,\ a_5=-28,$
$a_6=-48,\ a_7=0,\ c_7=-5c_3,\ c_8=\frac{11}{4}c_3,\ c_9=-\frac{5}{4}c_3,$
$c_1=24c_3,\ c_2=\frac{33}{2}c_3,\ c_4=0,\ c_{10}=0,\ c_5=8c_3,$
$c_6=-21c_3,$
$W_7=\langle 1,\mathrm{e}^{x},\mathrm{e}^{-x},\mathrm{e}^{2x},\mathrm{e}^{-2x},\mathrm{e}^{-3x},\mathrm{e}^{-4x}\rangle.$

Case 6

$a_1=0,\ a_2=-\frac{21}{4},\ a_3=0,\ a_4=-\frac{21}{4},\ a_5=0,$
$a_6=-1,\ a_7=0,\ c_9=8c_4,\ c_1=8c_4,\ c_2=16c_4,$
$c_5=0,\ c_6=4c_4,\ c_7=0,$
$c_8=0,\ c_{10}=0,\ c_3=\frac{1}{2}c_4,$
$W_7=\left\langle 1,\cos x,\sin x,\cos 2x,\sin 2x,\cos\frac{x}{2},\sin\frac{x}{2}\right\rangle.$

Case 7

$a_1=0,\ a_2=\frac{21}{4},\ a_3=0,\ a_4=-\frac{21}{4},\ a_5=0,$
$c_1=-8c_4,\ c_2=16c_4,\ c_3=-\frac{1}{2}c_4,\ c_5=0,$
$c_8=0,\ c_{10}=0,\ c_6=4c_4,\ c_7=0,\ a_6=1,\ a_7=0,$
$W_7=\langle 1,\mathrm{e}^{x},\mathrm{e}^{-x},\mathrm{e}^{2x},\mathrm{e}^{-2x},\mathrm{e}^{\frac{x}{2}},\mathrm{e}^{\frac{-x}{2}}\rangle.$

Case 8

$a_1=0,\ a_2=0,\ a_3=0,\ a_4=0,$
$a_5=0, a_6=0,\ a_7=0,$
$c_1=0,\ c_2=0,\ c_3=0,\ c_4=0,$
$c_5=0,\ c_6=0,\ c_7=0,$
$c_8=0,\ c_9=0,\ c_{10}=0,\ c_{15}=0,$
$W_7=\langle 1,x,x^2,x^3,x^4,x^5,x^6\rangle.$

Case 9

$a_1=0,\ a_2=7,\ a_3=0,$
$a_4=-\frac{49}{4},\ a_5=0,\ a_6=\frac{9}{2},$
$c_1=-\frac{9}{4}c_3,\ c_2=-\frac{5}{8}c_3,\ c_4=0,\ c_5=0,\ c_6=\frac{9}{4}c_3,$
$c_8=0,\ c_9=-\frac{5}{4}c_3,\ c_{10}=0,$
$c_{15}=0,\ a_7=0,\ c_7=0,$
$W_7=\langle 1,\mathrm{e}^{\sqrt{2}x},\mathrm{e}^{-\sqrt{2}x},\mathrm{e}^{\frac{x}{\sqrt{2}}},\mathrm{e}^{-\frac{x}{\sqrt{2}}},\mathrm{e}^{\frac{3x}{\sqrt{2}}},\mathrm{e}^{-\frac{3x}{\sqrt{2}}}\rangle.$

Case 10

$a_1=7,\ a_2=-7,\ a_3=-35,\ a_4=56,$
$a_5=28,\ a_6=-48,$
$a_7=0,\ c_1=24c_3,\ c_{10}=0,\ c_{15}=0,$
$c_2=\frac{33}{2}c_3,\ c_4=0,\ c_5=-8c_3,$
$c_6=-21c_3,\ c_7=5c_3,$
$c_8=-\frac{11}{4}c_3,\ c_9=-\frac{5}{4}c_3,$
$W_7=\langle 1,\mathrm{e}^{x},\mathrm{e}^{-x},\mathrm{e}^{2x},\mathrm{e}^{-2x},\mathrm{e}^{3x},\mathrm{e}^{4x}\rangle.$

Case 11

$a_1=0,\ a_2=7,\ a_3=0,\ a_4=-\frac{49}{4},\ a_5=0,\ a_6=\frac{9}{2},$
$a_7=0,\ c_1=-\frac{243}{10}c_4+\frac{9}{5}c_9,\ c_2=\frac{99}{4}c_4+\frac{1}{2}c_9,$
$c_3=-\frac{46}{5}c_4-\frac{4}{5}c_9, c_5=0,\ c_6=\frac{9}{5}c_4-\frac{9}{5}c_9,$
$c_7=0,\ c_8=0,\ c_{10}=0,\ c_{15}=0,$
$W_7=\langle 1,\mathrm{e}^{\sqrt{2}x},\mathrm{e}^{-\sqrt{2}x},\mathrm{e}^{\frac{x}{\sqrt{2}}},\mathrm{e}^{-\frac{x}{\sqrt{2}}},\mathrm{e}^{\frac{3x}{\sqrt{2}}},\mathrm{e}^{-\frac{3x}{\sqrt{2}}}\rangle.$

Case 12

$a_1=0,\ a_2=-7,\ a_3=0,\ a_4=-\frac{49}{4},\ a_5=0,$
$a_6=-\frac{9}{2},\ a_7=0, c_1=\frac{243}{10}c_4+\frac{9}{5}c_9,$
$c_2=\frac{99}{4}c_4-\frac{1}{2}c_9,$
$c_3=\frac{46}{5}c_4-\frac{4}{5}c_9,\ c_5=0,\ c_6=\frac{9}{5}c_4+\frac{9}{5}c_9,$
$c_7=0,\ c_8=0,\ c_{10}=0,\ c_{15}=0,$
$W_7=\left\langle 1,\cos\sqrt{2}x,\sin\sqrt{2}x,\cos\frac{x}{\sqrt{2}},\sin\frac{x}{\sqrt{2}},\cos\frac{3x}{\sqrt{2}},\sin\frac{3x}{\sqrt{2}}\right\rangle.$

Case 13

$a_1=0,\ a_2=-\frac{21}{4},\ a_3=0,\ a_4=-\frac{21}{4},\ a_5=0,$
$a_6=-1,\ a_7=0,\ c_9=8c_4\ c_1=8c_4,\ c_2=16c_4,$
$c_3=\frac{1}{2}c_4,\ c_5=0,$
$c_6=4c_4,\ c_7=0,\ c_8=0,\ c_{10}=0,\ c_{15}=0,$
$W_7=\left\langle 1,\cos x,\sin x,\cos 2x,\sin 2x,\cos\frac{x}{2},\sin\frac{x}{2}\right\rangle.$

Case 14

$a_1 = 0,\ a_2 = \dfrac{21}{4},\ a_3 = 0,\ a_4 = -\dfrac{21}{4},$

$a_5 = 0,\ a_6 = 1, a_7 = 0,\ c_9 = -8c_4,$

$c_1 = -8c_4,\ c_2 = 16c_4,\ c_{15} = 0,$

$c_3 = -\dfrac{1}{2}c_4,\ c_5 = 0,\ c_6 = 4c_4,$

$c_7 = 0,\ c_8 = 0,\ c_{10} = 0,$

$W_7 = \langle 1, \mathrm{e}^x, \mathrm{e}^{-x}, \mathrm{e}^{2x}, \mathrm{e}^{-2x}, \mathrm{e}^{\frac{x}{2}}, \mathrm{e}^{\frac{-x}{2}} \rangle.$

References

[1] M Lakshmanan and S Rajasekar, *Nonlinear dyanamics: Integrability, chaos and patterns* (Springer-Verlag, Berlin, Heidelberg, 2003)

[2] G W Bluman and S C Anco, *Symmetry and integration methods for differential equations* (Springer-Verlag, New York, 2002)

[3] S R Svirshchevskii, *Phys. Lett. A* **199**, 344 (1995)

[4] V A Galaktionov and S R Svirshchevskii. *Exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics* (Chapman and Hall/CRC, London, 2007)

[5] R Hilfer, *Applications of fractional calculus in physics* (World Scientific, Singapore, 2000)

[6] K Diethelm, *The analysis of fractional differential equations* (Springer Verlag, 2010)

[7] R Hermann, *Fractional calculus (An introduction to physicists)* (World Scientific Publishing, 2011)

[8] S Samco, A A Kilbas and O Marichev, *Fractional integrals and derivatives: Theory and applications* (Gordon and Breach Science, Switzerland, 1993)

[9] N H Tuan, Y E Aghdam, H Jafari and H Mesgarani, *Numer. Meth. Partial Diff. Eqns* **37(2)**, 1397 (2021)

[10] R M Ganji, H Jafari, M Kgarose and A Mohammadi, *Alex. Eng. J.* **60(5)**, 4563 (2021)

[11] Varsha Daftardar-Gejji and Hossein Jafari, *J. Math. Anal. Appl.* **301**, 508 (2005)

[12] H Jafari and S Seifi, *Commun. Nonlinear Sci. Numer. Simulat.* **14**, 1962 (2009)

[13] Guo-Cheng Wu and Dumitru Baleanu, *Adv. Diff. Eqns* **2013**, 18 (2013)

[14] R Sahadevan and P Prakash, *Choas Solitons Fractals* **104**, 107 (2017)

[15] Maryam Yourdkhany and Mehdi Nadjafikhah, *J. Geom. Phys.* **153**, 103661 (2020)

[16] R Sahadevan and T Bakkyaraj, *Fract. Calc. Appl. Anal.* **18(1)** (2015)

[17] Sangita Choudhary and Varsha Daftardar-Gejji, *Fract. Calc. Appl. Anal.* **20(2)** (2017)

[18] R K Gazizov and A A Kasatkin, *Comput. Math. Appl.* **66**, 576 (2013)

[19] R Sahadevan and P Prakash, *Nonlinear Dynam.* **85(1)**, 659 (2016)

[20] M S Hashemi, *Chaos Solitons Fractals* **107**, 161 (2018)

[21] Sangita Choudhary and Varsha Daftardar-Gejji, *Int. J. Model. Simul. Sci. Comput.* **10(1)**, 1941010 (2019)

[22] P Artale Harris and R Garra, *Nonlinear Stud.* **20**, 471 (2013)

[23] R Sahadevan and T Bakkyaraj, *J. Math. Anal. Appl.* **393**, 341 (2012)

[24] W X Ma, *Sci. China Math.* **55**, 1769 (2012)

[25] P Prakash, *Pramana - J. Phys.* **94**, 103 (2020)

[26] K S Miller and B Ross, *An introduction to the fractional calculus and fractional differential equations* (Wiley, New York, 1993)

[27] A A Kilbas, H M Srivastava and J J Trujillo, *Theory and applications of fractional differential equations* (Elsevier, 2006)

[28] T R Prabhakar, *Yokohama Math. J.* **19**, 7 (1971)

[29] I Podlubny, *Fractional differential equations* (Academic Press, San Diego, CA, 1999)

[30] P Veeresha, D G Prakasha, N Magesh, A John Christopher and Deepak Umrao Sarwe, *J. Ocean Eng. Sci.* **6**, 265 (2021)

[31] O S Iyiola, G O Ojo and O Mmaduabuchi, *Alex. Eng. J.* **55**, 1655 (2016)