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Electromagnetism according to geometric algebra: An appropriate and comprehensive formulation

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Abstract. While the Maxwell's equations describe the electric and magnetic fields developed by electrically charged particles and/or currents, the Lorentz force law completes the picture of classical electromagnetic theory by defining the force acting on a localised charge (or charge distribution) moving in the field. However, the standard formulation using vector algebra suffers from several inadequacies and unwarranted features. Clifford's geometric algebra or more specifically space–time algebra, i.e geometric algebra in 4D Minkowski space–time, provides an elegant, compactified and comprehensive description by removing the discrepancies of the earlier formulation. It provides an invariant description, in the appropriate space–time setting, in terms of the combined electromagnetic field without reference to any inertial system. Moreover, using elementary geometric calculus, it facilitates direct analytical introduction of the putative concept of magnetic monopole and renders the equations for both the constituent fields, symmetric and inhomogeneous. In terms of the single space–time force equation, space–time algebra also encapsulates both the Lorentz force equation and the electromagnetic power equation.

Keywords. Electromagnetism; Maxwell's equations; Lorentz force law; Clifford's geometric algebra; multivectors; relativity; space–time algebra; dual symmetry; energy–momentum conservation laws.

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1. Introduction

Fragmented descriptions of electricity and magnetism in terms of several important empirical laws were elegantly unified by Maxwell in ten eponymic field equations [1]. The field description also provided a much better approach than those furnished in terms of electric and magnetic forces. For example, the action-at-a-distance paradigm for the electric force (Coulomb's law) implies an instantaneous propagation, which is not consistent with the laws of relativity. The field paradigm, on the other hand, mediates the influence with a finite velocity of interaction. Furthermore, the field is endowed with energy, linear momentum and angular momentum – and hence appears as a real physical entity.

With the modification of Ampére's law for the time-dependent field [2], Maxwell's unification and Heaviside's subsequent compactification of the theory of *Electromagnetism*, introducing Gibbs–Helmholtz's 3D vector algebra (VA), are presented by the following four (scalar, vector, pseudovector and pseudoscalar) equations:

$$\nabla \cdot \mathbf{e} = \frac{\rho_e}{\epsilon_0} \text{ (Gauss' law)}, \tag{1}$$

$$\nabla \times \mathbf{b} - c^{-2}\partial_t \mathbf{e} = \mu_0 \mathbf{j}_e \text{ (Ampère–Maxwell equation)}, \tag{2}$$

$$\nabla \times \mathbf{e} + \partial_t \mathbf{b} = 0 \text{ (Faraday's law)}, \tag{3}$$

$$\nabla \cdot \mathbf{b} = 0 \text{ (Gauss' law of magnetism)}, \tag{4}$$

where the electrostatic charge density ρ_e and the electric current density $\mathbf{j}_e$ represent the source of the electromagnetic (EM) field; ϵ_0 and μ_0 are respectively the absolute permittivity and permeability of free space and $c = (\mu_0\epsilon_0)^{-1/2}$ is the speed of light in vacuum. Vector fields $\mathbf{e}$ and $\mathbf{b}$ are the conventional electric and magnetic fields and we use lower case symbols for scalars and bold lower cases for vectors throughout the article. The vector differential operator ∇ of the vector calculus is given by

$$\nabla = \hat{\alpha}^i \frac{\partial}{\partial x_i} \equiv \hat{\alpha}_i \frac{\partial}{\partial x_i}$$

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and

$$\partial_t = \frac{\partial}{\partial t}.$$

$\hat{\alpha}_i$'s are the three orthonormal basis vectors (Euclidean), with $\hat{\alpha}_i \cdot \hat{\alpha}_j = \delta_{ij}$ (the Kronecker delta). It may be noted that, Maxwell actually amended Ampère's static equation for the time-dependent fields using the continuity equation (A continuity equation is the mathematical expression for the local conservation of a physical quantity which can neither be created nor be destroyed and can only move from one place to another by a continuous flow.) for charge and current:

$$\partial_t \rho_e + \nabla \cdot \mathbf{j}_e = 0. \tag{5}$$

Equations (1)–(4) represent Maxwell's microscopic equations involving the primary fields **e** and **b** created by the electrons and nuclei in the medium. Ever since Lorentz proposed his theory of electrons, these equations are also called Maxwell–Lorentz equations. The macroscopic Maxwell's equations, on the other hand, use macroscopic average of the primary fields. Moreover, the intensity of electric field is replaced with the electric induction (or the electric displacement vector) field **d** in eq. (1) and in eq. (2) the induction of magnetic field with the magnetic field intensity **h**, derived from the respective average primary field incorporating the material properties of the medium [2]. The standard classical electromagnetism is usually described by the four microscopic equations along with the Lorentz force law expressing the electromagnetic force on a point charge (or a continuous charge distribution) moving with uniform velocity through an external EM field. The divergence-less, solenoidal magnetic field implies the absence of any magnetic charge (monopole).

According to the Helmholtz theorem of vector calculus, a vector field is completely determined by its divergence and curl. However, as Griffiths has further explained [3], a unique definition of EM field(s) typically requires that, the divergence and the curl of the vector field together with the field itself go to zero at infinity (far away from all charges). Moreover, the two divergence equations (1) and (4) actually act as equations of constraints and so all the eight coupled equations and six field variables are not at all independent. This becomes evident with the application of relevant identities of vector calculus. The two source-free Maxwell equations, (3) and (4), represent two identities (the so-called Bianchi identities of the vector calculus) involving the electric and magnetic fields. Expressing the solenoidal field **b** (in eq. (4)) as

$$\mathbf{b} = \nabla \times \mathbf{a}_e \tag{6}$$

and substituting this into the other source-free equation (3), we get $\nabla \times (\mathbf{e} + \partial_t \mathbf{a}_e) = 0$. This equation can be solved, again identically, by writing

$$\mathbf{e} = -\nabla\phi_e - \partial_t \mathbf{a}_e. \tag{7}$$

The application of these identities finally provides an alternative representation of Maxwell's equations, involving two new field (potential) functions ϕ_e and $\mathbf{a}_e$. The scalar potential ϕ_e generated by static electric charge(s), describes only the conservative electric field, whereas the vector potential, generated by electric current(s), describes the magnetic field completely. A time-varying vector potential also induces a non-conservative electric field.

Putting eq. (7) into eq. (1) and both eqs (6) and (7) into eq. (2) one gets

$$\nabla^2\phi_e + \partial_t \nabla \cdot \mathbf{a}_e = -\frac{\rho_e}{\epsilon_0} \tag{8}$$

and

$$\begin{aligned} &\nabla \times (\nabla \times \mathbf{a}_e) = \mu_0 \mathbf{j}_e - c^{-2}\{\nabla(\partial_t\phi_e) + \partial_t^2 \mathbf{a}_e\} \\ &\Rightarrow (\nabla^2 \mathbf{a}_e - c^{-2}\partial_t^2 \mathbf{a}_e) - \nabla(\nabla \cdot \mathbf{a}_e + c^{-2}\partial_t \phi_e) \\ &= -\mu_0 \mathbf{j}_e, \\ &\text{(using the identity } \nabla \times (\nabla \times \mathbf{a}_e) \\ &= \nabla(\nabla \cdot \mathbf{a}_e) - \nabla^2 \mathbf{a}_e), \end{aligned} \tag{9}$$

– the alternative potential formulation of Maxwell's equations. Although the above two equations look clumsy, this new representation regulates redundant degrees of freedom and squarely reduces the number of equations to solve for the four components of ϕ_e and $\mathbf{a}_e$. Moreover, eqs (6) and (7) do not uniquely define the potentials and we are at liberty to modify and add extra terms to the potentials, as long as they do not affect the fields **e** and **b**. For example, if we modify ϕ_e and $\mathbf{a}_e$ to $\phi'_e = \phi_e + \partial_t\lambda$ and $\mathbf{a}'_e = \mathbf{a}_e - \nabla\lambda$, where λ is an arbitrary differentiable scalar field function, then ϕ'_e and $\mathbf{a}'_e$ lead to the same constituent EM fields **e** and **b**. This is precisely the 'gauge freedom' or 'gauge invariance' of electromagnetism and profiting from this non-uniqueness of the potentials, simplification in specific problems can be achieved. Two widely used gauge transformations are

$$\begin{aligned} &\text{Coulomb gauge:} \quad \phi_e(|\mathbf{r}| = \infty) = 0; \quad \nabla \cdot \mathbf{a}_e = 0, \\ &\text{Lorentz gauge:} \quad \nabla \cdot \mathbf{a}_e + c^{-2}\,\partial_t\phi_e = 0. \end{aligned} \tag{10}$$

Under these (gauge) transformations, eqs (8) and (9) respectively become

Coulomb gauge:

$$\begin{aligned} &\nabla^2\phi_e = -\frac{\rho_e}{\epsilon_0}; \\ &\nabla^2 \mathbf{a}_e - c^{-2}\{\partial_t^2 \mathbf{a}_e + \nabla\partial_t\phi_e\} = -\mu_0 \mathbf{j}_e, \end{aligned} \tag{11}$$

Lorentz gauge:

$$\nabla^2\phi_e - c^{-2}\partial_t^2\phi_e = -\frac{\rho_e}{\epsilon_0};$$
$$\nabla^2\mathbf{a}_e - c^{-2}\partial_t^2\mathbf{a}_e = -\mu_0\mathbf{j}_e. \quad (12)$$

In Coulomb gauge, the scalar potential satisfies the simple Poisson equation, which is just the Coulomb potential due to the instantaneous charge density $\rho(r,t)$ – hence the name, whereas the Lorentz gauge decouples ϕ_e and $\mathbf{a}_e$ and both are described symmetrically on equal footings by two inhomogeneous differential equations with d'Alembertian operator $\Box^2 = \nabla^2 - c^{-2}\partial_t^2$. Thus, the potential representation reduces the number of equations and the gauge freedom simplifies the equations in specific problems. In the study of fundamental interactions, gauge invariance has special importance. For the EM field theory also, it provides many testable consequences. The theoretical justification for charge conservation is further strengthened by being linked to this symmetry and it requires that the photons (excitations or quanta of the EM field) are massless. The strong experimental support that the photon has zero mass is also a good evidence that charge is conserved. Gauge invariance also forms the basis of electro-weak and strong interactions and aspects of electro-weak interaction will appear in a subsequent paper, where electromagnetism, fluid mechanics and quantum mechanics would be discussed side by side using space–time algebra (to be reviewed presently). Also, it should be emphasised here that, the vector potential causes phase change in quantum mechanical wave function (detectable in interference experiments) and the incredible Aharonov–Bohm effect suggests that the potentials are essential elements of physical descriptions!

Charge and current densities not only generate the EM field but also respond to the field. The Lorentz force law describes the effect (force) on charges in an EM field. The generation of the EM field by charges and currents according to the Maxwell's equation is one aspect; the response of the charge(s) to an EM field (according to the Lorentz law) is another. The 'Lorentz force' exerted by the EM field on the charged particle is the rate at which linear momentum is transferred from the EM field to the particle. Associated with it is the power which is the rate at which energy is transferred from the field to the particle. In 1881, while analysing the behaviour of the charged particles in cathode rays, Thomson derived the force on moving charged particles due to an external magnetic field which involved an incorrect factor of half. Subsequently, Heaviside using his vector formulation of Maxwell's field equations, had fixed the error and arrived at the correct form [4]. Applying Euler Lagrange equations for the Lagrangian

$$\mathcal{L} = \frac{m}{2}\mathbf{v}^2 + q_e(\mathbf{a}_e\cdot\mathbf{v} - \phi_e)$$

of a point charge q_e moving with velocity $\mathbf{v}$ in an EM field defined by the vector and scalar fields $\mathbf{a}_e$ and ϕ_e respectively and using Heaviside's version of the Maxwell equations, in 1895 Lorentz finally arrived at the correct form of the force law [5] that bears his name:

$$m\,\dot{\mathbf{v}} = q_e(\mathbf{e} + \mathbf{v}\times\mathbf{b}). \quad (13)$$

Poynting's earlier enunciation [6] of the conservation of energy for the EM field, in the form of a continuity equation, relates the work done on a charge distribution by the EM field to the decrease in the field energy, less the flowed out energy flux. Mathematically, this may be summarised in the form of a partial differential equation as

$$\partial_t u = \mathbf{j}_e\cdot\mathbf{e} \Rightarrow \partial_t u = -\partial_t\frac{\epsilon_0\mathbf{e}^2 + \mu_0^{-1}\mathbf{b}^2}{2} - \nabla\cdot\frac{\mathbf{e}\times\mathbf{b}}{\mu_0}$$
$$\Rightarrow \partial_t(u + u_{em}) + \nabla\cdot\mathbf{s} = 0. \quad (14)$$

The equation states that $\partial_t u$, if the rate of energy supplied to the system (of charges) by the EM field, is equal to the (rate of) decrease in the field energy, less energy flux flows out. The power developed by the EM field, i.e. the rate at which the fields do work on the charge/current density, is obtained from the Lorentz force density (In field theory, the concepts of conservation of momentum and energy apply over the spread-out field and the calculations involve volume integral over the whole field which imply conservation of total energy, momentum etc. of both field and matter and the final results are expressed in terms of densities of the relevant physical quantities.) $\mathbf{f}_\mathrm{L}$ as

$$\partial_t u = \mathbf{f}_\mathrm{L}\cdot\mathbf{v} = \rho_e(\mathbf{e} + \mathbf{v}\times\mathbf{b})\cdot\mathbf{v} = \rho_e\mathbf{v}\cdot\mathbf{e} = \mathbf{j}_e\cdot\mathbf{e}.$$

Here, $u_{em} = 2^{-1}(\epsilon_0\mathbf{e}^2 + \mu_0^{-1}\mathbf{b}^2)$ is the energy density of the EM field in vacuum and the term $\nabla\cdot\mathbf{s}$ represents the flux of energy density flowing out, $\mathbf{s} = \mu_0^{-1}(\mathbf{e}\times\mathbf{b})$, being the 'Poynting vector'. While conservation of energy and the Lorentz force law can give the general form of the theorem, Maxwell's equations are additionally required to derive the expression for the Poynting vector to complete the derivation.

Using the Maxwell's equations further, the Lorentz force density [2,3,7] is also expressed as

$$\begin{aligned}\mathbf{f}_L = \partial_t\mathbf{p} &= \rho_e\mathbf{e} + \mathbf{j}_e\times\mathbf{b}\\ &= \epsilon_0[(\nabla\cdot\mathbf{e})\mathbf{e} + (\mathbf{e}\cdot\nabla)\mathbf{e}]\\ &\quad + \mu_0^{-1}[(\nabla\cdot\mathbf{b})\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{b}]\\ &\quad - \nabla \mathrm{u}_{em} - c^{-2}\,\partial_t\mathbf{s}. \quad (15)\end{aligned}$$

Poynting [6] and subsequently Heaviside [8] have independently identified $\mathbf{s}$ as the flux of electromagnetic energy, along the direction of propagation of the EM wave. On the other hand, Thomson has pointed out [9] that an energy flux gives rise to electromagnetic momentum density proportional to the Poynting vector: $\mathbf{p}_{em} = c^{-2}\,\mathbf{s}$. Equations (14) and (15) express these two assertions respectively. Equation (14) is analogous with the continuity equation (5) expressing charge conservation where the Poynting vector represents the flow of energy in the same way as $\mathbf{j}_e$ representing the flow of charge in eq. (5), whereas eq. (15) implies that $c^{-2}\,\mathbf{s}$ represents the momentum density carried by the field. Thus, the Poynting vector $\mathbf{s}$ has appeared in two different roles as energy per unit area per unit time transported by the EM field, while $c^{-2}\,\mathbf{s}$ is the momentum density of the field. The EM field possesses both energy and momentum which can be exchanged with charged particles.

All the terms of the Lorentz force density, except the last one on the right-hand side (eq. (15)), together can be expressed as divergence of a second rank symmetric tensor, the so-called 'Maxwell's stress tensor':

$$\mathbf{f}_{\mathrm{L}} = \nabla \cdot \Gamma - c^{-2}\,\partial_t \mathbf{s},$$
$$\Gamma_{ij} = \epsilon_0 \left(e_i\, e_j - \frac{1}{2}\delta_{ij}\, \mathbf{e}^2 \right) + \mu_0^{-1} \left(b_i\, b_j - \frac{1}{2}\delta_{ij}\, \mathbf{b}^2 \right)$$

and like charge and energy conservation equations, the conservation of momentum (flux) in the differential form may be written as: $\partial_t(\mathbf{p}_{em} + \mathbf{p}) = \nabla \cdot \Gamma$. In the standard theory, eqs (14) and (15) are combined in a single energy–momentum conservation equation by equating the space–time divergence of the symmetric gauge-invariant stress tensor ($\Theta^{\alpha\beta}$), the space–space components of which are just the negative of Maxwell's stress tensor (Γ) components, to the contracted anti-symmetric EM field tensor ($F^{\beta\lambda}$) with the four-current density [2] as

$$\partial_\alpha \Theta^{\alpha\beta} = -c^{-1} F^{\beta\lambda}\, j_\lambda. \tag{16}$$

The Lorentz force is defined as the force acting on a moving charge (or equivalently on the current) due to an 'external' EM field and experience also vindicates this assertion. The use of Maxwell's equations in the detailed derivation of the 'Poynting's theorem' (eq. (14)), however, poses difficulties in having a consistent interpretation of the Poynting vector and the Lorentz force law [10,11]. Analysing two types of derivation of this theorem based on (i) the work done on a charge by an external field and (ii) the work done by the logically consistent total fields, introducing the self-field, Campos and Jiménez [11] have concluded that the theorem should be used with care in the presence of charges. The physical meaning of Poynting's theorem would be clear-cut only in regions free of charges and giving a satisfactory picture only in the case of a plane EM wave. They have finally argued that "any prescribed current, microscopic or macroscopic, requires the intervention of non-electromagnetic forces. Then we may conclude that a system of electromagnetic fields and charges is usually not a closed system." In other words, electromagnetism cannot be considered a complete theory. We also note another similar controversial issue related to the derivation of eq. (15) and defining the celebrated 'Maxwell's stress tensor'. Proper resolution of these problems may compel us to look beyond present-day electromagnetism!

Notwithstanding, away from the atomic world, the Lorentz force law, combined with classical laws of motion, fully explains the behaviour of charges in the presence of EM field and completes the picture of electromagnetism together with the Maxwell's equations. Using conventional Maxwell's theory with Kirchhoff's boundary-value formulation, Sciama [12] has concluded that the asymmetry in the time of the expanding Universe leads to the required retarded potentials, i.e. the observed irreversibility of radiation processes. A more recently published work has reported that Maxwell's equations can describe not only the conduction electrons and other inertial charges but also moving virtual charges forming surface currents (Tesla currents) in conductors [13]. Several experiments have already confirmed the existence of virtual (non-inertial) charges and Tesla currents which can be created and destructed under certain influences and violate the usual law of conservation of charge. The problem of studying the physics of Tesla currents is new and a description of their structure and the nature of the interaction with inertial charges and EM fields require further experimental and theoretical studies.

1.1 *Inadequacies of the standard formulation*

While all these offer a considerable theoretical enhancement, nevertheless, Heaviside's formulation suffers from several unwarranted shortcomings. Following Biot–Savart law, the magnetic field at a given point is conventionally defined according to VA by the cross product between two polar vectors – the vector current-element and the position vector $\mathbf{r}$ of that point. Hence, it is a pseudo (or an axial) vector, whose direction is convention-dependent and not absolute. Unlike a 'true' vector, it gains an additional sign flip under an improper rotation such as reflection. VA thus introduces, through its definition of cross product, chirality (a handedness), even when there is no chirality in the entity being modelled. The handedness thus introduced, gives rise to the so-called 'Pierre's puzzle' (In 1894, Pierre Curie analysed the relation between the symmetries of the electric and magnetic fields [21]. He pointed out that, the spurious handedness introduced by the definition of the cross product of VA produces an apparent asymmetry in the theory. However, every observable prediction of the electromagnetic theory involves an even number of applications of the right-hand rule, so that the second application always cancels the effect of the first [10,22]!). For the pseudovector representation of the magnetic field, the additional abstraction of the right-hand rule convention is also needed to interpret the part of the Lorentz force ($\mathbf{j}_e \times \mathbf{b}$) it produces. The magnetic field due to a current element, as presented in Biot–Savart law, is described appropriately by a bivector field given by Grassman's exterior (or wedge) product of the vectorial current element with $\mathbf{r}$. Electromagnetism has no chirality and Hestenes and others have revealed that Clifford's geometric algebra (GA), or more appropriately, the space–time algebra (STA, GA in 4D Minkowski space) provides correct and comprehensive description of electromagnetism and a host of other physical theories [14,15].

Both Lorentz force equation and Maxwell's equations imply that the electric and magnetic fields are relative, not only on the choice of electromagnetic sources, but also on the choice of an observer – a choice of a reference system. Ivezic [16] has shown that Heaviside's formulation of electromagnetism according to VA is not covariant under Lorentz transformation. Following Einstein's observation [17] on the relative strength of electric and magnetic fields respectively due to a stationary and a moving magnet relative to a conductor, it is reasonable to have a single EM field variable that transforms in a consistent way depending on the relative motion, rather than writing the electric and magnetic fields as 'independent' vector quantities. GA quite readily offers such a representation for the EM field. The real strength of the single Maxwell's equation of GA is thus more than its compactness and simplicity – it provides appropriate relativistic generalisation of the traditional Maxwell's equations of VA.

VA deals only with scalars and vectors and hence does not contain geometric description beyond points and lines. It should be noted that, the cross product, triple product etc. of vectors and hence VA itself can be defined only in 3D. Generalisation to any other dimension is not possible – in two dimensions, no third dimension exists to accommodate the (cross) product vector, and in higher dimensions there are too many orthogonal directions [18]! GA, on the other hand, is now increasingly recognised as the natural algebra for describing physics of n-space. In fact, GA provides an immensely powerful mathematical framework for a profound description of the most advanced concepts in theoretical physics such as classical mechanics, electromagnetism, fluid mechanics, theory of relativity, quantum mechanics, computer science etc. It is also claimed that its superior geometric intuition is straightforward and simple enough to be taught in high schools replacing Heaviside–Gibbs VA [19,20]! The present study is intended to facilitate a comprehensive introduction to the theory of electromagnetism in the appropriate framework of GA and to provide a broad based exposure to the advanced undergraduate students.

1.2 *Clifford's geometric algebra*

In 1876, Clifford unified the works of Hamilton and Grassmann to form the foundation of a new algebra of mathematical physics with several distinctive features, called geometric algebra. Using the associative exterior or wedge ($\wedge$) product of Grassmann algebra, two, three or any number of linearly independent vectors in a given dimension can be 'wedged' together to produce higher grade multivectors – bivector, trivector etc. Instead of four unit bases of usual VA of 3D space, GA is spanned by the $2^3 = 8$ multivector unit bases:

$$1,\ \{\hat{\alpha}_i\},\ \{\hat{\sigma}_{ij}\ (= \hat{\alpha}_i \wedge \hat{\alpha}_j)\}\quad \text{and}\quad I_3\ (= \hat{\alpha}_1\,\hat{\alpha}_2\,\hat{\alpha}_3),$$

with $i, j = 1, 2, 3$ and $i \neq j$. 1 and I_3 ($I_3^2 = -1$) are unit scalar and pseudoscalar respectively and form a pair of unit dual bases. Similarly, $\hat{\alpha}_i$ and $\hat{\sigma}_{jk}$ ($= I_3\,\hat{\alpha}_i$) form another pair of unit dual bases. Clifford combined the inner and exterior products of vectors and multivectors to define a new associative product – the 'geometric product' (The geometric product of any two vectors $\mathbf{u}$ and $\mathbf{v}$ is designed to contain all the information about the relative directions of the two vectors and is defined as: $\mathbf{u}\,\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}$, where $\mathbf{u}\wedge\mathbf{v} = u_i\, v_j\, \hat{\sigma}_{ij}$.) and allows any linear combination of scalar, vector(s), bivector(s) and higher grade multivectors to be a member of his geometric algebra. The algebraic formulation provides a seamless extension to any dimension. Hestenes elaborated Clifford's work to show how it unites "vectors, spinors and complex numbers into a single mathematical system with a comprehensive geometric significance" [23] and revealed how the Pauli and Dirac algebras are embedded in geometric algebra. Eventually, it turned out that as a unified mathematical language it provides a much simplified representation in many cases [14,24]. In addition to the usual dimensional analysis in checking physical formulae, as we are always taught, GA provides an additional structural check for the calculated quantity to have the correct geometric order from a scalar, vector, bivector, trivector etc. [25].

In GA, an arbitrary element of the algebra is called a multivector and multivectors in which all elements have the same grade r are usually written as $\mathbf{M}_r$. Multivectors of definite grade (bivectors, trivectors etc.) are represented here with bold capitals and calligraphics represent general multivectors (or 'clifs'). A general multivector in 3D may be expressed as: $\mathcal{M} = \Sigma_{r=0}^{3}\mathbf{M}_r = s_1 + \mathbf{v}_1 + \mathbf{B} + s_2 I_3 \equiv s_1 + \mathbf{v}_1 + \mathbf{v}_2 I_3 + s_2 I_3$, where s, $\mathbf{v}$, $\mathbf{B}$ represent scalars, vectors, bivectors respectively, $\mathbf{B} = \mathbf{M}_2$ and trivector $\mathbf{T}$ $(= \mathbf{M}_3) = s_2 I_3$, being the highest grade element is a pseudoscalar. The magnetic field, angular momentum, angular velocity, torque of a force field etc. – all are appropriately represented by bivectors in GA [18]. Also, $\langle\mathcal{M}\rangle_r$ represents the grade selection operator and picks up the grade-r component from $\mathcal{M}$ and $\langle\mathcal{M}\rangle_r \equiv \mathbf{M}_r$.

2. Electromagnetism according to geometric algebra

Physical theories are more conveniently and economically described with GA. The strength of GA may also be argued using 'Occam's Razor' as it offers a simpler and economic mathematical model for the description of physical theories, naturally extending from one to two, to higher dimensions. The effectiveness of this algebra is amply demonstrated as it offers a consummated unified formulation by encapsulating the usual four Maxwell's equations of VA in a single, compact equation for the combined electromagnetic field.

Replacing the magnetic field pseudovector $\mathbf{b}$ with the appropriate magnetic field bivector $\mathbf{B}$ $(= \mathbf{b}I_3 = I_3\mathbf{b}$, where its dual $\mathbf{b}$ in 3D GA [14] is now a 'true' vector but otherwise equivalent to the 'pseudovector $\mathbf{b}$' of the previous formulation of VA and the operator curl $(\nabla\times)$ with the exterior differential operator $(\nabla\wedge \equiv I_3\nabla\times)$, I_3 $(= \hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3)$ being the unit pseudoscalar in 3D, elementary geometric calculus modifies the set of equations (1)–(4) to

$$\nabla\cdot\mathbf{e} = \frac{\rho_e}{\epsilon_0}, \tag{17}$$

$$\nabla\cdot\mathbf{B} + c^{-2}\partial_t\mathbf{e} = -\mu_0\mathbf{j}_e, \tag{18}$$

$$\nabla\wedge\mathbf{e} + \partial_t\mathbf{B} = 0; \tag{19}$$

$$\nabla\wedge\mathbf{B} = 0. \tag{20}$$

This four geometrically distinct equations – scalar, vector, bivector and pseudoscalar respectively – are combined in GA and represent a genuine unification in the form:

$$(c^{-1}\partial_t + \nabla)\mathcal{F} = \frac{\rho_e}{\epsilon_0} - c\mu_0\mathbf{j}_e = c\mu_0\mathcal{J}_e, \tag{21}$$

where both the combined EM field and the source are described by general mixed grade multivectors (clifs). The field $\mathcal{F} = \mathbf{e} + c\mathbf{B}$, as the sum of the electric field vector and the magnetic field bivector multiplied by c – a 'parabivector', while the source $\mathcal{J}_e = c\rho_e - \mathbf{j}_e$ is a 'paravector' – a scalar (ρ_e multiplied by c) minus a vector ($\mathbf{j}_e$). The operator on the left-hand side of eq. (21) may be termed as a paravector differential operator and $\nabla\mathcal{F} = \nabla\cdot\mathcal{F} + \nabla\wedge\mathcal{F}$. Also, since $(c^{-1}\partial_t + \nabla)(-c^{-1}\partial_t + \nabla) = \Box^2$ (d'Alembertian), it has an inverse $(-c^{-1}\partial_t + \nabla)\Box^{-2}$ – involving the integral operator $\Box^{-2}$. Hence, the inverse of eq. (21), $\mathcal{F} = c\mu_0(c^{-1}\partial_t + \nabla)^{-1}\mathcal{J}_e$, provides the solution for the combined EM field $\mathcal{F}$. The gauge invariance of the parabivector EM field is also apparent and in Lorentz gauge, the pair of equations (12) for ϕ_e and $\mathbf{a}_e$ can be combined in a single equation for the paravector potential $\mathcal{A} = \phi_e - c\mathbf{a}_e$ as: $\Box^2\mathcal{A} = -c\mu_0\mathcal{J}_e$. In this formulation, the Lorentz force can be obtained from the vector (grade 1) term of the geometric product $\mathcal{J}_e\mathcal{F}$ as: $c^{-1}\langle\mathcal{J}_e\mathcal{F}\rangle_1 = \langle c^{-1}(c\rho_e - \mathbf{j}_e)(\mathbf{e} + c\mathbf{B})\rangle_1 = \rho_e\mathbf{e} - \mathbf{j}_e\cdot\mathbf{B}$ $(\equiv \rho_e\mathbf{e} + \mathbf{j}_e\times\mathbf{b}$, $\mathbf{b}$ being the dual vector of the magnetic field bivector $\mathbf{B}$). Historically, similar complex vector field was considered by Riemann and Silberstein in some of the earlier investigations of electromagnetism. Bialynicki-Birula and Bialynicki-Birula have reviewed [26] the role of the Riemann–Silberstein vector field in classical and quantum theories of electromagnetism.

It should be noted here that the unification of the separate equations for divergence and curl in electromagnetism in a single equation (21) is non-trivial – the unified equation can be inverted directly to determine the EM field. Also, both the constituent electric and magnetic fields being completely left–right symmetric, the formulation dispenses with the so-called left-hand and right-hand rules and offers resolution to the Pierre's puzzle in this context. The apparent chirality turns out to be actually an artifact of the standard formulation of EM theory according to VA.

Unlike the equations in source-free space ($\rho_e = 0$, $\mathbf{j}_e = 0$), the inhomogeneous Maxwell's equations are not symmetric in the constituent (electric and magnetic) fields due to the absence of any 'magnetic charge'. Maxwell's equations, in a sense, 'beg for magnetic monopole (Magnetic monopoles (isolated magnetic north and south poles, analogues to two types of electric charges) were first predicted by Paul Dirac in 1931 in his work on quantum electrodynamics. Its existence is motivated mainly by two considerations: (i) quantisation of electric charge – the presence of even a single magnetic monopole would require electric charge to be quantised! and (ii) to render Maxwell's equations symmetric.) to exist' [3]. Another interesting feature of the formulation with GA is that, with the inclusion of pseudoscalar magnetic charge density and the corresponding bivector current density, it consistently symmetrises Maxwell's equations (17)–(20). It may be noted that in this formulation, the dual of the electric vector field is a bivector ($I_3\,\mathbf{e}$), whereas the dual of the magnetic bivector is a vector ($-I_3\,\mathbf{B}$). Also the duals of the scalar electric charge and the vector electric current densities are pseudoscalar and bivector, representing respectively the magnetic charge and magnetic current densities. The representation in GA thus aptly depicts electromagnetic duality and provides a proper framework for the investigation of the traits of the posited monopole. Further elucidation of this aspect will be gained in the following description of the theory with space–time algebra STA).

2.1 *GA on Minkowski space – Space–time algebra*

GA finally allows an appropriate description of electrodynamics and special theory of relativity by extending the algebra of 3D space to the algebra of 4D Minkowski space–time [27], with the replacement of the Euclidean metric by the Minkowski metric. This is achieved using four basis vectors $\hat{\alpha}_\mu$, satisfying $\hat{\alpha}_\mu \cdot \hat{\alpha}_\nu = \frac{1}{2}(\hat{\alpha}_\mu\hat{\alpha}_\nu + \hat{\alpha}_\nu\hat{\alpha}_\mu) = \eta_{\mu\nu} = \mathrm{diag}(-+++)$, generates the sequence $(-+++)$ of algebraic signs on the main diagonal – the signature of the flat space–time metric in which $\hat{\alpha}_0^2 = -1 = -\hat{\alpha}_k^2$ (the opposite signature of $(+---)$ is also used in the literature). All through, Greek indices run from 0 to 3 and Latin indices run from 1 to 3. For the entire set of basis vectors $\{\hat{\alpha}_\mu\}$, the geometric product is defined as $\hat{\alpha}_\mu\,\hat{\alpha}_\nu = \hat{\alpha}_\mu \wedge \hat{\alpha}_\nu,\ \mu \neq \nu; = \hat{\alpha}_\mu \cdot \hat{\alpha}_\nu$, $\mu = \nu$. Here; $\hat{\alpha}_k$'s are evidently similar to the orthogonal basis vectors of the usual VA of 3D space and $\hat{\alpha}_0$ is the time-like basis vector of the 4D space–time. The corresponding space–time algebra is spanned by the $2^4 = 16$ multivector unit bases:

$$1,\ \{\hat{\alpha}_\mu\},\ \{\hat{\alpha}_j\,\hat{\alpha}_0,\ \hat{\alpha}_j\,\hat{\alpha}_k,\},\ \{I_4\,\hat{\alpha}_\mu\}$$

and

$$I_4\ (= \hat{\alpha}_0\,\hat{\alpha}_1\,\hat{\alpha}_2\,\hat{\alpha}_3),$$

with $\mu = 0, 1, 2, 3;\ \ j, k = 1, 2, 3$ and $j \neq k$. 1 and I_4 ($I_4^2 = -1$) are unit scalar and pseudoscalar respectively and forms a pair of unit dual bases. The dual sets of $\hat{\alpha}_j\,\hat{\alpha}_0$ (time-like) and $\hat{\alpha}_j\,\hat{\alpha}_k$ (space-like) bases together represent six orthogonal space–time bivector bases. Also, $I_4\,\hat{\alpha}_\mu$, the dual of $\hat{\alpha}_\mu$, represent four trivector bases of which $I_4\,\hat{\alpha}_0 = \hat{\alpha}_1\,\hat{\alpha}_2\,\hat{\alpha}_3$, may be identified as the unit pseudoscalar I_3 of the associated 3D space. One can also use the reciprocal space–time bases $\hat{\alpha}^\mu$ which is defined as: $\hat{\alpha}^\mu \cdot \hat{\alpha}_\nu = \delta^\mu_\nu \Rightarrow \hat{\alpha}^0 = -\hat{\alpha}_0,\ \hat{\alpha}^k = \hat{\alpha}_k$. So, there is no need for the covariant and contravariant distinction for the spatial components and we use subscript notations for these components.

Any arbitrary space–time vector is represented as $\bar{\mathbf{u}} = \hat{\alpha}_\nu\,u_\mu$. For example, the space–time position vector $\bar{\mathbf{r}}$ ($= \hat{\alpha}_0\,c\,t + \mathbf{r}$) and the 'ordinary' space–time velocity

$$\bar{\mathbf{v}} = \frac{\mathrm{d}\bar{\mathbf{r}}}{\mathrm{d}t} = \hat{\alpha}_0 c + \mathbf{v};$$

$\mathbf{r}$ and $\mathbf{v}$ ($= \mathrm{d}\mathbf{r}/\mathrm{d}t$) are the corresponding 3D (spatial) vectors. (The 'proper' or the frame-independent space–time velocity

$$\bar{\mathbf{w}} = \frac{\mathrm{d}\bar{\mathbf{r}}}{\mathrm{d}\tau} = \bar{\mathbf{v}}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\,\bar{\mathbf{v}} \Rightarrow \bar{\mathbf{w}}^2 = -c^2,$$

the minus (or plus) sign depends on the choice of the metric signature of space–time, τ being the 'proper' time elapsed on the clock of the travelling object and γ ($= 1/\sqrt{1-\beta^2}$) is the relativistic (or Lorentz) stretch factor with $\beta = c^{-1}|\mathbf{v}|$.) For an object at rest ($\mathbf{v} = 0$), its four-velocity is parallel to the direction of the time axis $\hat{\alpha}_0$, the temporal component being c. With $\bar{\mathbf{u}}$, the product $\bar{\mathbf{u}}\,I_4 = -I_4\,\bar{\mathbf{u}}$ is a space–time trivector – the dual of $\bar{\mathbf{u}}$. The wedge product between the space–time position vector $\bar{\mathbf{r}}$ and the space–time momentum vector (or the energy-momentum four vector) $\bar{\mathbf{p}} = m\,\bar{\mathbf{v}}$, define the space–time angular momentum bivector as

$$\mathbf{L} = \bar{\mathbf{r}} \wedge \bar{\mathbf{p}} = \frac{\bar{\mathbf{r}}\bar{\mathbf{p}} - \bar{\mathbf{p}}\bar{\mathbf{r}}}{2}$$
$$= (c^{-1}\varepsilon\mathbf{r} \wedge \hat{\alpha}_0 + c\,t\hat{\alpha}_0 \wedge \mathbf{p}) + \mathbf{r} \wedge \mathbf{p},$$

where $\varepsilon = m\,c^2$ is the rest energy and $\mathbf{p} = m\,\mathbf{v}$.

The geometric product equips the vector space of GA with an algebraic structure that provide a very powerful tool to unify and describe in a single formalism the vector, complex and the spin algebras [23]. Rotors (to be discussed presently), arbitrary elements in even subalgebra of 3D GA, produce rotations in the plane of the generating bivector [15]. They allow smooth interpolations between arbitrary orientations and share identical algebra with the spinors. Moreover, rotations in space–time can be of either type. The term $\mathbf{r} \wedge \mathbf{p}$ generates the usual spatial rotations of 3D, whereas the first term within the first bracket is a bivector of opposite signature that generates Lorentz boost. The invariance of angular momentum in space–time includes both spatial rotation and boost components. The dual of a space–time bivector $\mathbf{L}$ is $\mathbf{L}\,I_4 = I_4\,\mathbf{L}$ – a space–time bivector. An arbitrary space–time multivector is expressed as $\mathcal{M} = \Sigma_{r=0}^4 \mathbf{M}_r = s_1 + \mathbf{v}_1 + \mathbf{B} + \mathbf{T} + s_2 I_4 \equiv s_1 + \mathbf{v}_1 + \mathbf{B} + \mathbf{v}_2 I_4 + s_2 I_4$.

3. Space–time descriptions of electrodynamics with STA

In STA, replacing the paravector differential operator of eq. (21) by 4D space–time vector differential operator $\Box = \hat{\alpha}^{\mu}\partial_{x_{\mu}} = -\hat{\alpha}_0 c^{-1}\partial_t + \nabla$ (just as the three sides of the 'nabla' operator may be considered to represent the three ordinary spatial dimensions, the four sides of the 'box' operator can also be taken to represent the four dimensions of space–time) and the EM field accordingly by a general space–time bivector of the form: $\mathbf{F} = \mathbf{e} \wedge \hat{\alpha}_0 - c\,\mathbf{B}$ (also called 'Faraday space–time bivector' EM field), the set of Maxwell's equations (17)–(20) in STA takes a similar form of eq. (21) as

$$\Box\mathbf{F} \equiv (-\hat{\alpha}_0 c^{-1}\partial_t + \nabla)\mathbf{F} = c\mu_0\bar{\mathbf{j}}_e, \tag{22}$$

where the source field is accordingly represented by the space–time current density vector $\bar{\mathbf{j}}_e = \hat{\alpha}_0 c\,\rho_e + \mathbf{j}_e$ (for a charge distribution ρ_e moving with spatial velocity $\mathbf{v}$, the space–time current density $\bar{\mathbf{j}}_e$ is obtained by multiplying ρ_e with the space–time velocity $\bar{\mathbf{v}}$ $(= \hat{\alpha}_0\, c + \mathbf{v})$ as $\bar{\mathbf{j}}_e = \hat{\alpha}_0\, c\rho_e + \rho_e\,\mathbf{v}$).

The operator $\Box$ may be recognised as the Dirac operator in vector form, defined originally as the square root of the d'Alembertian to find a first-order relativistically invariant wave equation for the electron – the four (4×4) Dirac matrices γ^{μ} are simply replaced by four space–time basis vectors $\hat{\alpha}_{\mu}$. It represents the space–time gradient and for a check, we note: $\Box^2 = (-\hat{\alpha}_0 c^{-1}\partial_t + \nabla)\cdot(-\hat{\alpha}_0 c^{-1}\partial_t + \nabla) = -c^{-2}\partial_t^2 + \nabla^2$ is the d'Alembertian. (In representing the d'Alembertian, wide variation in notation within the literature may be found. Here, we represent the d'Alembertian by $\Box^2$ and its appropriate square root $\Box$ as the space–time gradient operator). The operator $\Box$, contrary to the impression given by conventional accounts of relativistic quantum theory, is not specially applicable to spin-$1/2$ wave equations only. It is equally apt in describing EM field equations and fluid mechanics (to be discussed in another paper). We note that the left-hand side of eq. (22):

$$\begin{aligned}\Box\mathbf{F} &= \Box\cdot\mathbf{F} + \Box\wedge\mathbf{F} \\ &= (-\hat{\alpha}_0 c^{-1}\partial_t + \nabla)\cdot(\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B}) \\ &\quad + (-\hat{\alpha}_0 c^{-1}\partial_t + \nabla)\wedge(\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B}) \\ &= -c^{-1}\partial_t\mathbf{e} + \hat{\alpha}_0\,\nabla\cdot\mathbf{e} - c\nabla\cdot\mathbf{B} \\ &\quad + \hat{\alpha}_0\wedge\partial_t\mathbf{B} + \hat{\alpha}_0\wedge\nabla\wedge\mathbf{e} - c\nabla\wedge\mathbf{B},\end{aligned} \tag{23}$$

contains both the vector $(\langle\Box\mathbf{F}\rangle_1 = \Box\cdot\mathbf{F} = -c^{-1}\partial_t\mathbf{e} + \hat{\alpha}_0\,\nabla\cdot\mathbf{e} - c\,\nabla\cdot\mathbf{B})$ and trivector $(\langle\Box\mathbf{F}\rangle_3 = \Box\wedge\mathbf{F} = \hat{\alpha}_0\wedge\partial_t\mathbf{B} + \hat{\alpha}_0\wedge\nabla\wedge\mathbf{e} - c\nabla\wedge\mathbf{B})$ parts, whereas the right-hand side of eq. (22) contains only a space–time vector. Hence, equating vectors with time-like and space-like bases of the two sides of the equation separately, and the trivector term on its left-hand side to zero, one gets back the set of equations (17)–(20).

The gauge invariance condition is also evidently satisfied for the EM field bivector $\mathbf{F}$. Also, since $\Box\wedge\mathbf{F} = 0$, it can be defined in terms of the space–time vector potential $\bar{\mathbf{a}}_e = -\hat{\alpha}_0\phi_e - c\,\mathbf{a}_e$, as $\mathbf{F} = \Box\wedge\mathbf{a}_e$ and in Lorentz gauge $(\Box\cdot\bar{\mathbf{a}}_e = 0)$ we can as well take $\mathbf{F} \equiv \Box\bar{\mathbf{a}}_e$. In potential representation, eq. (22) then becomes $\Box^2\bar{\mathbf{a}}_e = c\,\mu_0\bar{\mathbf{j}}_e$. As we shall see, the potential formulation facilitates a Lagrangian formulation of the electromagnetic field theory and also offers a profound dual symmetric description of electromagnetism. Next, we write the six-component Faraday bivector EM field according to STA as

$$\begin{aligned}&\mathbf{F} = \mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{b}I_3 = \mathbf{E} - c\,\mathbf{B};\;\; \mathbf{E} = \mathbf{e}\wedge\hat{\alpha}_0, \\ &\mathbf{B} = \mathbf{b}I_3 \Rightarrow \mathbf{b} = -I_3\mathbf{B} \\ &\quad = F^{10}\,\hat{\alpha}_1\hat{\alpha}_0 + F^{20}\,\hat{\alpha}_2\hat{\alpha}_0 + F^{30}\,\hat{\alpha}_3\hat{\alpha}_0 + F^{12}\,\hat{\alpha}_1\hat{\alpha}_2 \\ &\qquad + F^{23}\,\hat{\alpha}_2\hat{\alpha}_3 + F^{31}\,\hat{\alpha}_3\hat{\alpha}_1,\end{aligned} \tag{24}$$

where the electric field components having time-like bases $\hat{\alpha}_j\,\hat{\alpha}_0$ are F^{j0} $(= E^{j0} = e_j)$ and the part containing the magnetic field components F^{jk} $(= -c\,B^{jk} = -c\,b_l)$ are with space-like bases $\hat{\alpha}_j\,\hat{\alpha}_k$ $(= -I_4\,\hat{\alpha}_l\,\hat{\alpha}_0)$; j, k and l in cyclic order. The dual of the space–time bivector $I_4\,\mathbf{F} = \mathbf{F}\,I_4$ is also a space–time bivector – dual of the time-like electric bivector $(\mathbf{E} = \mathbf{e}\wedge\hat{\alpha}_0)$ is a space-like bivector $I_4\,\mathbf{E}$, whereas the dual of the space-like magnetic bivector is time-like $I_4\,\mathbf{B}$. An appropriate unification and inherent duality of the two aspects of the EM field notwithstanding, the rich geometrical content of STA also depicts their deeper distinct characters. In the combined EM field representation, the actual electric vector field is the intersection of a time-like plane with a spatial surface in space–time, whereas the magnetic field is a space-like simple bivector plane.

Another important information about the field $\mathbf{F}$ can be obtained from its square:

$$\begin{aligned}\mathbf{F}^2 &= \mathbf{F}\mathbf{F} = \mathbf{F}:\mathbf{F} + \mathbf{F}\cdot\mathbf{F} + \mathbf{F}\wedge\mathbf{F} \\ &= \langle\mathbf{F}^2\rangle_0 + \langle\mathbf{F}^2\rangle_2 + \langle\mathbf{F}^2\rangle_4 \\ &= (\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B}) : (\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B}) \\ &\quad + (\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B})\cdot(\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B}) \\ &\quad + (\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B})\wedge(\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B}) \\ &= \mathbf{e}^2 + c^2\mathbf{B}^2 + 2c\,\hat{\alpha}_0\,\mathbf{e}\wedge\mathbf{B};\end{aligned}$$

since

$$(\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B}).(\mathbf{e}\wedge\hat{\alpha}_0 - c\,\mathbf{B}) = 0,$$

which defines a scalar and pseudoscalar pair of Lorentz invariant terms and $\langle\mathbf{F}^2\rangle_2$ is identically equal to zero (see Appendix I). Both the scalar Lagrangian density $(\mathbf{e}^2 + c^2\mathbf{B}^2 \equiv \mathbf{e}^2 - c^2\mathbf{b}^2)$ and the pseudoscalar $(2c\,\hat{\alpha}_0\,\mathbf{e}\wedge\mathbf{B} \equiv 2c\,\mathbf{e}\cdot\mathbf{b}\,I_4)$, which is related to the time evolution of magnetic helicity density $\mathbf{a}_e \wedge \mathbf{B}$, are independent of the reference frame in which they are measured. However, this is not true for the EM field energy density u_{em} $(= \epsilon_0(\mathbf{e}^2 - c^2\,\mathbf{B}^2)/2 \equiv \epsilon_0(\mathbf{e}^2 + c^2\,\mathbf{b}^2)/2\,)$ – which is an observer-dependent quantity.

3.1 *Lagrangian density of the EM field*

As an alternative, one can derive the Maxwell's equation (eq. (22)) from the Euler–Lagrange equation using appropriate Lagrangian density for the EM field. The electromagnetic interaction is known to conserve parity and the Lorentz invariant scalar $\mathbf{F} : \mathbf{F}$ $(= \langle \mathbf{F}^2 \rangle_0$ together with the interaction (due to the presence of the source) term $\bar{\mathbf{j}} \cdot \bar{\mathbf{a}}_e$ $(= \langle \bar{\mathbf{j}}\,\bar{\mathbf{a}}_e \rangle_0)$ appears as an automatic choice. We take the EM field Lagrangian density in appropriate unit as

$$\mathcal{L}_{em} = \epsilon_0\left(\left\langle \frac{\mathbf{F}^2}{2} + c\mu_0 \bar{\mathbf{j}}\,\bar{\mathbf{a}}_e \right\rangle_0\right) \equiv \epsilon_0\left\langle \frac{(\Box \wedge \bar{\mathbf{a}}_e)^2}{2} + c\mu_0\,\bar{\mathbf{j}}\,\bar{\mathbf{a}}_e \right\rangle_0. \tag{25}$$

In functional form we may write: $\mathcal{L}_{em} \equiv \mathcal{L}_{em}(\bar{\mathbf{a}}_e, \Box \wedge \bar{\mathbf{a}}_e)$ and the space–time Euler–Lagrange equation may be accordingly expressed as

$$\frac{\partial\,\mathcal{L}}{\partial\,\bar{\mathbf{a}}_e} - \Box \frac{\partial\,\mathcal{L}}{\partial(\Box \wedge \bar{\mathbf{a}}_e)} = 0.$$

Substitution of $\mathcal{L}_{em}$ from eq. (25) in the above equation finally reproduces the Maxwell's equation $\Box^2\,\bar{\mathbf{a}}_e = \Box \cdot \mathbf{F} = c\mu_0\,\bar{\mathbf{j}}$. The remaining equation also follows as: $\Box \wedge \mathbf{F} = \Box \wedge \Box \wedge \bar{\mathbf{a}}_e = 0$. The potential formulation thus facilitates formulation of a Lagrangian field theory of electromagnetism starting with a scalar-valued Lagrangian density. The Lagrangian approach also leads to the conservation laws of energy and momentum of the EM field. According to Noether's theorem [28], symmetries of the Lagrangian of a physical system (with respect to dynamical variables) identify conjugate conserved quantities. The conservation laws can be derived using the variational technique from the associated action integral $\mathcal{S}$ over any region for the space–time Lagrangian density, by enforcing the invariance condition $\delta\mathcal{S} = 0$ under various symmetries. For example, the invariance of action under gauge transformations (global) leads to the continuity equation implying charge conservation [29]. Similarly, the Lagrangian approach also leads to the conservation laws of energy and momentum of the EM field [15,30].

3.2 *Generalisation for magnetic monopoles*

From a second differentiation of eq. (22) with $\Box$,

$$\Box^2\mathbf{F} = c\mu_0\Box\bar{\mathbf{j}}_e = c\mu_0(\Box \cdot \bar{\mathbf{j}}_e + \Box \wedge \bar{\mathbf{j}}_e). \tag{26}$$

Separately equating scalar and bivector parts of eq. (26), we get back the equation of continuity (eq. (5)) expressing charge conservation:

$$\Box \cdot \bar{\mathbf{j}}_e = 0 \Rightarrow \partial_t \rho_e + \nabla \cdot \mathbf{j}_e = 0.$$

So, in reduced form, eq. (26) becomes

$$\Box^2\mathbf{F} = c\mu_0\Box \wedge \bar{\mathbf{j}}_e\,\cdot \tag{27}$$

An inclusive field equation may be written by using the fact that, under general conditions, a bivector field $\mathbf{F}$ can be derived from two independent vector and trivector fields, with the specific form:

$$\mathbf{F} = \Box \wedge \bar{\mathbf{a}}_e + \Box \cdot \mathbf{T}, \tag{28}$$

where $\mathbf{T}$ is the space–time trivector potential field in addition to the usual vector potential $\bar{\mathbf{a}}_e$. This may be regarded as a generalisation of the well-known Helmholtz theorem of vector analysis. Here also, the source field $\mathbf{T}$ is not uniquely defined by eq. (28) and retains a gauge freedom. In the Lorentz condition, $\Box \wedge \mathbf{T} = 0$ is taken with $\Box \cdot \bar{\mathbf{a}}_e = 0$. Expressing the trivector potential field in terms of a second independent space–time vector potential as: $\mathbf{T} = I_4\,\bar{\mathbf{a}}_m$, eq. (28) becomes

$$\mathbf{F} = \Box \wedge \bar{\mathbf{a}}_e + \Box \cdot (I_4\,\bar{\mathbf{a}}_m) = \Box \wedge \bar{\mathbf{a}}_e + I_4\,\Box \wedge \bar{\mathbf{a}}_m. \tag{29}$$

With appropriate Lorentz gauge, $\Box \cdot \bar{\mathbf{a}}_e = 0$ and $\Box \wedge \mathbf{T} = 0 = \Box \wedge (I_4\,\bar{\mathbf{a}}_m) = I_4\,\Box \cdot \bar{\mathbf{a}}_m \Rightarrow \Box \cdot \bar{\mathbf{a}}_m = 0$, eqs (28) and (29) read as $\mathbf{F} = \Box(\bar{\mathbf{a}}_e + \mathbf{T}) = \Box\bar{\mathbf{a}}_e + I_4\,\Box\bar{\mathbf{a}}_m$. Inserting eq. (29) into Maxwell's equation (22) and separating vector and trivector parts, we obtain the usual wave equation for the vector potential $\bar{\mathbf{a}}_e$, $\Box^2\bar{\mathbf{a}}_e = c\,\mu_0\bar{\mathbf{j}}_e$ and $I_4\,\Box^2\bar{\mathbf{a}}_m = 0 \Rightarrow \Box^2\bar{\mathbf{a}}_m = 0$. Since $\mathbf{T}$ is independent of the vector field $\bar{\mathbf{a}}_e$ (and hence of $\bar{\mathbf{j}}_e$), it can be left out of eq. (28) and consequently, $\bar{\mathbf{a}}_m$ from eq. (29). But more importantly, introducing a non-zero trivector source in this formulation, one can consistently accommodate a moving space–time pseudoscalar magnetic (monopole) charge density $I_4\,\rho_m$, producing a trivector (space–time) current density $I_4\,\rho_m\,\bar{\mathbf{v}}$ $(\equiv I_4\,\bar{\mathbf{j}}_m)$. The equation for the non-zero trivector potential field then leads to $I_4\,\Box^2\bar{\mathbf{a}}_m = c\,\mu_0\,I_4\,\bar{\mathbf{j}}_m \Rightarrow \Box^2\bar{\mathbf{a}}_m = c\,\mu_0\bar{\mathbf{j}}_m$, $\bar{\mathbf{a}}_m$ now may be identified as the space–time magnetic vector potential. Correspondingly, the Maxwell's equation takes the form:

$$\Box\mathbf{F} = \Box^2(\bar{\mathbf{a}}_e + I_4\,\bar{\mathbf{a}}_m) = c\,\mu_0(\bar{\mathbf{j}}_e + I_4\,\bar{\mathbf{j}}_m) = c\,\mu_0\,\mathcal{J}, \tag{30}$$

where $\mathcal{J} = \bar{\mathbf{j}}_e + I_4\,\bar{\mathbf{j}}_m$. We also note that, the electric charge density is a scalar quantity, whereas the pseudoscalar monopole (magnetic) charge density changes sign both under spatial inversion and time reversal. Equation (26) now modifies to

$$\Box^2\mathbf{F} = c\mu_0\Box(\bar{\mathbf{j}}_e + I_4\,\bar{\mathbf{j}}_m) \\ = c\mu_0[\Box\cdot(\bar{\mathbf{j}}_e + I_4\,\bar{\mathbf{j}}_m) + \Box\wedge(\bar{\mathbf{j}}_e + I_4\,\bar{\mathbf{j}}_m)]$$

and separately equating the scalar, pseudoscalar terms of right-hand side to zero, we get the pair of continuity equations:

$$\Box\cdot\bar{\mathbf{j}}_e = 0 \Rightarrow \partial_t\rho_e + \nabla\cdot\mathbf{j}_e = 0$$

and

$$\Box\cdot\bar{\mathbf{j}}_m = 0 \Rightarrow \partial_t\rho_m + \nabla\cdot\mathbf{j}_m = 0;\;\; (\mathbf{j}_m = \rho_m\mathbf{v}) \tag{31}$$

and retaining only the non-vanishing (bivector) terms on the right-hand side, the reduced field equation reads as

$$\Box^2\mathbf{F} = c\mu_0(\Box\wedge\bar{\mathbf{j}}_e + I_4\,\Box\wedge\bar{\mathbf{j}}_m).$$

Now, equating only the trivector terms of eq. (23) (last three terms on the right-hand side) with those of eq. (30), we get

$$\hat{\alpha}_0\wedge\partial_t\mathbf{B} + \hat{\alpha}_0\wedge\nabla\wedge\mathbf{e} - c\nabla\wedge\mathbf{B} = c\mu_0\,I_4\,\bar{\mathbf{j}}_m \\ = c\mu_0(c\,\rho_m\,I_3 + I_4\,\mathbf{j}_m).$$

Finally equating the space-like and time-like parts:

$$\partial_t\mathbf{B} + \nabla\wedge\mathbf{e} = c\,\mu_0\,I_3\,\mathbf{j}_m \tag{32}$$

$$\Rightarrow \partial_t\mathbf{b} + \nabla\times\mathbf{e} = c\,\mu_0\mathbf{j}_m,$$

and

$$-c\,\nabla\wedge\mathbf{B} = c^2\,\mu_0\rho_m I_3 \Rightarrow c\,\nabla\cdot\mathbf{b} = -\frac{\rho_m}{\epsilon_0}. \tag{33}$$

Equations (19) and (20) are accordingly modified and STA offers a consistent symmetrisation of Maxwell's equations to accommodate any possible presence of 'magnetic monopole' through the inclusion of higher multivector terms in the field source. Invoking invariance of Maxwell's equations, under duality transformation, similar modification can be guessed by grafting pseudoscalar magnetic charge and pseudovector current in the standard formulation. Heaviside has pointed that the Maxwell equations retain their form when suitably scaled magnetic and electric fields are interchanged [8]. It is generally accepted that particles of ordinary matters possess only electric charge or, equivalently, may be considered as having the same ratio of magnetic to electric charge [2]. In the latter case, using appropriate duality transformation, one can make $\rho_m = 0;\; \mathbf{j}_m = 0$ and write the usual Maxwell's equations (17)–(20). But for other, unstable particles the question of magnetic charge remains wide open and no definite evidence exists yet.

The formulation according to STA, on the other hand, directly shows a natural passage for the introduction of the magnetic monopole. Here we note that, the scalar electric charge and the pseudoscalar magnetic monopole form a dual. The vector electric current and the potential also clearly manifest duality with the trivector magnetic current and potential, respectively. STA thus offers a profound dual symmetric description of electromagnetism.

Despite the fact that no free monopole has yet been observed, it is predicted by grand unified theories – the predicted masses are generally far beyond the energies reachable in modern particle accelerators. However, some recent theories suggest masses that could be reached in near future and search for free magnetic monopoles at high energies are still ongoing. Recent investigations also indicate an effective flow of magnetic monopoles in a spin ice system, though they cannot exist outside the material in a free form [31]. Moreover, magnetic monopoles appear as a new candidate for dark matter. All these led some theorists to regard the existence of magnetic monopoles as one of the safest bets that one can make about the new physics to come!

The potential formulation is thus integrated into the presentation that allows an alternate solution path for the generalisation of magnetic monopoles. We have also seen that EM theory can be developed alternatively using the Lagrangian approach, aided by the potential formulation. The Lagrangian approach also leads to the conservation laws and thus provides an alternative formulation in GA for both the field equation and conservation laws of electromagnetism.

Since the EM fields propagate at the finite speed of light c, there is a delay between the cause and effect at two distinct space points and the concept of retarded potentials comes into play. The inhomogeneous wave equations, therefore cannot connect the propagating fields and their sources at the same time. In 1888, retarded potentials came into general use after Hertz's experiments on EM waves. In 1895, a further boost to the theory of retarded potentials came after Thomson's interpretation of data for electrons. However, the EM fields of a single source are not exclusively retarded but are time-symmetric and Maxwell's equations admit both retarded and advanced solutions. (Any electromagnetic potential arising as a solution of the classical Maxwell's field equations lying on the future light cone of space–time is termed as advanced potentials. The potential does not appear to have physical interpretation. However, some recent studies of advanced potentials claim new insight into the mathematical interpretation of special and general relativity.) The appearance of the causal, pure retardation is usually explained as the result of interference by time-symmetric exchanges with the cosmological gravitational field [12,32]. The causal solutions to the Maxwell's equation is more easily obtained in terms of retarded potentials both for a moving charge distribution or a point charge (Lienard–Wiechert potential) and differentiating them, rather than going directly to the fields [3]. This is another example of advantages of the potential formulation and using Green's functions [33] one gets a solution for eq. (22) with the advantage of the single Maxwell's equation of STA formulation.

3.3 *Space–time force equation for the EM field and the conservation laws*

STA also comprehensively describes the Lorentz force law along with the power equation in terms of a single space–time force equation. The space–time force density on the space–time current density element, or on equivalent moving charge distribution in the field, can be easily identified as the grade-1 (vector) term of the geometric product between the space–time current density multivector and the EM field bivector. Written explicitly:

$$\bar{\mathbf{f}} = \partial_t \bar{\mathbf{p}} = \hat{\alpha}_0\, c^{-1}\, \partial_t \mathrm{u} + \partial_t \mathbf{p} = c^{-1} \langle \mathcal{J}\mathbf{F}\rangle_1.$$

With the electrical source(s) only, i.e. $\mathcal{J} = \bar{\mathbf{j}}_e + I_4\, \bar{\mathbf{j}}_m \equiv \bar{\mathbf{j}}_e$ as in eq. (22), we get

$$\begin{aligned}\bar{\mathbf{f}} &= \partial_t \bar{\mathbf{p}} = \hat{\alpha}_0\, c^{-1}\, \partial_t \mathrm{u} + \mathbf{f}_\mathrm{L}; \quad (\mathbf{f}_\mathrm{L} \equiv \partial_t \mathbf{p})\\ &= \rho_e\, \mathbf{e} + c^{-1}\, \hat{\alpha}_0\, \mathbf{j}_e \cdot \mathbf{e} - \mathbf{j}_e \cdot \mathbf{B}, \end{aligned} \tag{34}$$

the expression for appropriate space–time force density in terms of both the source and the EM field variables. The temporal and spatial parts of the equation yield expressions for the power and the Lorentz force (density) given by eqs (14) and (15). Dispensing with tensors of the standard formulation (eq. (16)), the space–time force thus encapsulates the conservation laws for electromagnetic energy and momentum. Substituting $\bar{\mathbf{j}}_e$ with $\Box\, \mathbf{F}$ using eq. (22), we can also express $\bar{\mathbf{f}}$ in terms of the field variables only, as

$$\begin{aligned}\hat{\alpha}_0\, c^{-1}\, \partial_t \mathrm{u} + \partial_t \mathbf{p} &= \epsilon_0 \langle (\Box\, \mathbf{F})\mathbf{F}\rangle_1 = \epsilon_0 \{(\Box \cdot \mathbf{F}) \cdot \mathbf{F}\\ &\quad + (\Box \wedge \mathbf{F}) : \mathbf{F}\}\\ &= -\hat{\alpha}_0\, c^{-1} (\partial_t u_{em} + \nabla \cdot \mathbf{s})\\ &\quad + \epsilon_0 [(\nabla \cdot \mathbf{e})\mathbf{e} + (\mathbf{e} \cdot \nabla)\mathbf{e}]\\ &\quad - \mu_0^{-1} [(\nabla \wedge \mathbf{B}) : \mathbf{B}\\ &\quad + (\mathbf{B} \wedge \nabla) : \mathbf{B}] - \nabla \mathrm{u}_{em} - c^{-2}\, \partial_t \mathbf{s}, \end{aligned} \tag{35}$$

since $\Box \wedge \mathbf{F} = 0$ in this case (see Appendix II). Thus, the space–time force equation of STA (eq. (34)) replaces the tensorial energy–momentum conservation equation (16). Next, with the putative pseudoscalar magnetic monopole source only, $\mathcal{J} \equiv I_4\, \bar{\mathbf{j}}_m$ and in the absence of any electrical source, say, the space–time force, is given by

$$\begin{aligned}\bar{\mathbf{f}} &= \hat{\alpha}_0\, c^{-1}\, \partial_t \mathrm{u} + \mathbf{f}_\mathrm{L} = c^{-1} \langle I_4\, \bar{\mathbf{j}}_m (\mathbf{e} \wedge \hat{\alpha}_0 - c\mathbf{B})\rangle_1\\ &= -c^{-1}\, \bar{\mathbf{j}}_m\, I_4 : (\mathbf{e} \wedge \hat{\alpha}_0 - c\mathbf{B})\\ &= (\hat{\alpha}_0 \rho_m + c^{-1}\, \mathbf{j}_m) \cdot (\mathbf{e}\, I_3 + c\, I_4\, \mathbf{B})\\ &= -\rho_m\, c\, I_3\, \mathbf{B} + c^{-1} (\mathbf{j}_m \wedge \mathbf{e})\, I_3 + \mathbf{j}_m \wedge \mathbf{B}\, I_4\\ \Rightarrow \hat{\alpha}_0\, c^{-1}\, \partial_t \mathrm{u} &= \mathbf{j}_m \wedge \mathbf{B}\, I_4\end{aligned}$$

and

$$\mathbf{f}_\mathrm{L} = -\rho_m\, c\, I_3\, \mathbf{B} + c^{-1} (\mathbf{j}_m \wedge \mathbf{e})\, I_3. \tag{36}$$

Equations (34) and (36) state that magnetic fields do not work on electric charges but can do work on magnetic charges. Similarly, electric fields can do work only on electric charges.

Finally, with both the source terms:

$$\mathcal{J} = \bar{\mathbf{j}}_e + I_4\, \bar{\mathbf{j}}_m : \hat{\alpha}_0\, c^{-1}\, \partial_t \mathrm{u} = \hat{\alpha}_0\, c^{-1}\, \mathbf{j}_e \cdot \mathbf{e} + \mathbf{j}_m \wedge \mathbf{B}\, I_4$$

and

$$\mathbf{f}_L = \rho_e\, \mathbf{e} + \mathbf{j}_e \cdot \mathbf{B} - \rho_m\, c\, I_3\, \mathbf{B} + c^{-1} (\mathbf{j}_m \wedge \mathbf{e})\, I_3. \tag{37}$$

The part of the Lorentz force ($\mathbf{j}_e \cdot \mathbf{B}$) due to the magnetic field is now properly described in terms of the bivector representation of the magnetic field [33], removing the clumsy representation of VA. Interestingly, even from eqs (36) and (37) (with magnetic monopole source only and both the source terms, respectively) one gets the same eqs (14), (15) using the corresponding Maxwell's equations, $\Box \mathbf{F} = c\mu_0\, I_4\, \bar{\mathbf{j}}_m$ and $\Box \mathbf{F} = c\mu_0 (\bar{\mathbf{j}}_e + I_4\, \bar{\mathbf{j}}_m)$ respectively. This is actually an expression of the complete intrinsic dual symmetry of electromagnetism and choosing electric sources only appears as an arbitrary convention that essentially breaks the dual symmetry of the electric and magnetic fields.

STA also enables us to write appropriately the space–time energy–momentum density of the field $\bar{\mathbf{p}}_{em} = \hat{\alpha}_0\, c^{-1}\, \mathrm{u}_{em} + \mathbf{p}_{em} = \epsilon_0 [\hat{\alpha}_0\, (2\, c)^{-1} (\mathbf{e}^2 - c^2 \mathbf{B}^2) + \mathbf{B} \cdot \mathbf{e}]$, which can be represented in terms of the EM field bivector $\mathbf{F}$ as

$$\begin{aligned}\bar{\mathbf{p}}_{em} &= c^{-1} \left(\epsilon_0 \frac{\mathbf{e}^2 - c^2 \mathbf{B}^2}{2} \hat{\alpha}_0 + \frac{1}{\mu_0\, c} \mathbf{B} \cdot \mathbf{e} \right)\\ &= \frac{\epsilon_0}{2c} (\mathbf{F}\, \hat{\alpha}_0\, \mathbf{F}\, \hat{\alpha}_0) \hat{\alpha}_0 = -\frac{\epsilon_0}{2c} \mathbf{F}\, \hat{\alpha}_0\, \mathbf{F}. \end{aligned} \tag{38}$$

Now, $\Box \cdot \bar{\mathbf{p}}_{em} = (-\hat{\alpha}_0 c^{-1} \partial_t + \nabla) \cdot (\hat{\alpha}_0\, c^{-1}\, \mathrm{u}_{em} + \mathbf{p}_{em}) = c^{-2}\, \partial_t \mathrm{u}_{em} + \nabla \cdot \mathbf{p}_{em} = -c^{-2} \partial_t \mathrm{u}$ (from eq. (14)). The same result may be obtained also from eqs (38) and (35). Hence, with $\Box \cdot \bar{\mathbf{p}} = c^{-2}\, \partial_t \mathrm{u} + \nabla \cdot \mathbf{p}$, we can express the EM energy–momentum conservation law also in the form

$$\Box \cdot (\bar{\mathbf{p}} + \bar{\mathbf{p}}_{em}) = \nabla \cdot \mathbf{p}. \tag{39}$$

3.4 *Field angular momentum and the generator of rotations*

The EM field also possesses angular momentum and can also exchange it with a system of charged particles. Consider the geometric product, $\bar{\mathbf{r}}\bar{\mathbf{p}}_{em} = \bar{\mathbf{r}} \cdot \bar{\mathbf{p}}_{em} + \bar{\mathbf{r}} \wedge \bar{\mathbf{p}}_{em}$. The symmetric scalar part,

$$\phi = \bar{\mathbf{r}} \cdot \bar{\mathbf{p}}_{em} = \frac{\bar{\mathbf{r}}\bar{\mathbf{p}}_{em} + \bar{\mathbf{p}}_{em}\bar{\mathbf{r}}}{2} = \mathbf{r} \cdot \mathbf{p}_{em} - \mathrm{u}_{em} t$$

describes the phase of the electromagnetic wave field, whereas the antisymmetric bivector part gives the space–time angular momentum (density):

$$\begin{aligned}\mathbf{L}_{em} = \bar{\mathbf{r}} \wedge \bar{\mathbf{p}}_{em} &= \frac{\bar{\mathbf{r}}\bar{\mathbf{p}}_{em} - \bar{\mathbf{p}}_{em}\bar{\mathbf{r}}}{2} \\ &= (c^{-1}\mathrm{u}_{em}\, \mathbf{r} \wedge \hat{\alpha}_0 + c\, t\, \hat{\alpha}_0 \wedge \mathbf{p}_{em}) \\ &\quad + \mathbf{r} \wedge \mathbf{p}_{em} \\ &= c^{-1}\left\{ \varepsilon_0 \frac{(\mathbf{e}^2 - c^2\mathbf{B}^2)}{2} \mathbf{r} \wedge \hat{\alpha}_0 \right. \\ &\quad \left. + \frac{t\, \hat{\alpha}_0 \wedge (\mathbf{B} \cdot \mathbf{e})}{\mu_0} + \frac{\mathbf{r} \wedge (\mathbf{B} \cdot \mathbf{e})}{\mu_0 c} \right\}.\end{aligned}$$

In quantum mechanics, the angular momentum determines the behaviour of the wave function under rotation and the corresponding operator acts as the generator of rotations. The total angular momentum consists of two parts – orbital and spin angular momentum and the representation involve Hermitian operators satisfying appropriate commutation relations. The spin operator does not involve coordinates or momenta, hence, it commutes with linear momentum. This provides an elementary but perfectly general definition of spin as that part of angular momentum which commutes with linear momentum [34].

In classical theory, polarisation refers to the rotation of the electric field vector as a function of time at a fixed position in space and helicity corresponds to the degree of polarisation. In general, polarisation is elliptical and in special cases becomes circular or linear. In quantum field theory, on the other hand, the rotational content of the field – polarisation, helicity etc. are related to the intrinsic angular momentum (spin) of the field. Gauge invariance determines the polarisation and the helicity states of the electromagnetic field and the dual symmetry leads to helicity conservation in vacuum fields. Using GA, Baylis *et al* described the polarisation of a collimated beam of monochromatic electromagnetic radiation [35]. Aspects of electromagnetic field propagation and its rotational content will be discussed in our subsequent article.

A rotation through an angle θ is usually described about an axis $\hat{n}$ in 3D and executed through a 3×3 orthogonal matrix $\mathcal{O}(\hat{n}, \theta)$ having the determinant $+1$. The coordinates get transformed as $\mathbf{r} \rightarrow \mathbf{r}' = \mathcal{O}(\hat{n}, \theta)\, \mathbf{r}$. In standard formulations of QM, for every such rotation $\mathcal{O}(\hat{n}, \theta)$, the wave function $\psi(\mathbf{r}, t)$ of a system undergoes a unitary transformation: $\psi(\mathbf{r}, t) \rightarrow \psi'(\mathbf{r}, t) = \mathcal{U}(\hat{n}, \theta)\, \psi(\mathbf{r}, t)$. The unitary operator $\mathcal{U}(\hat{n}, \theta)$ can be written in terms of the vector Hermitian operator for the angular momentum. The usual angular momentum commutation relations can be derived directly from the above definition of angular momentum as the generator of rotations [36].

4. Generalised rotation with rotors in STA

It is already noted that, arbitrary elements in even subalgebra of GA, rotors, produce rotations and share identical algebra with the spinors. In addition to the usual rotations in three orthogonal spatial planes, they also produce Lorentz boost in 4D space–time continuum. GA thus reduces rotations and Lorentz transformations to algebraic multiplication and allows computations without matrices or tensors. As well, GA or more appropriately STA introduces spin without reference to quantum or relativistic physics and removes much of the mathematical divide among classical, quantum and relativistic physics.

Using the generalisation of the concept of exponential function for multivectors as introduced by Hestenes [27], the rotors in ordinary space are represented by elliptic functions as

$$\mathcal{R} = \exp\left(\frac{\hat{\mathbf{B}}|\theta|}{2}\right) = \cos\left(\frac{|\theta|}{2}\right) + \hat{\mathbf{B}}\, \sin\left(\frac{|\theta|}{2}\right), \quad (40)$$

with

$$\cos\left(\frac{|\theta|}{2}\right) = \hat{\mathbf{w}} \cdot \hat{\mathbf{u}},\ \ \sin\left(\frac{|\theta|}{2}\right) = |\hat{\mathbf{w}} \wedge \hat{\mathbf{u}}|$$

and

$$\hat{\mathbf{B}} = \frac{\hat{\mathbf{w}} \wedge \hat{\mathbf{u}}}{|\hat{\mathbf{w}} \wedge \hat{\mathbf{u}}|}.$$

The rotor expressed as the exponential of the bivector $\mathbf{B}$, generates a rotation through the bilinear expression $\mathbf{v}' = \mathcal{R}\, \mathbf{v}\, \mathcal{R}^{-1}$. The bivector encodes both the the direction (defined by the oriented plane of $\hat{\mathbf{B}}$) and the magnitude $\theta/2$ (the angle of rotation θ appears due to the bilinear form of the operation) and provides unambiguous specification of rotation in any dimension. The bivector representation of the angular momentum according to GA is thus inherently consistent with its definition as the generator of rotation. It is already noted that rotors can handle much more complex rotations and in the non-Euclidean space, the bivectors may possess a positive square and rotors are no longer elliptic but hyperbolic. The corresponding rotors for the bivectors containing a time-like component in three space–time surfaces of a 4D space–time continuum, produce Lorentz boost in addition to the usual rotations in three remaining spatial planes. Although it is less evident in lower dimensions, the bilinear transformation is much easier to handle than the one-sided rotation matrix.

4.1 *Lorentz boost*

In STA, all equations are studied in the appropriate space–time setting and obtaining electromagnetic fields that appear to different observers, gets simplified. Hestenes has pointed out that, STA describes "physical processes by equations which are invariant in the sense that they are not referred to any inertial system" [37]. The field equation is invariant under coordinate transformations, rather than covariant like the tensor form. It is also very convenient for handling accelerated charges and radiation [15]. However, measuremental observations, usually expressed in terms of variables corresponding to a particular inertial frame, requires reformulation of these invariant equations in terms of those variables. Lorentz transformations in special relativity relate observations made in different inertial frames.

Rotors produce Lorentz boost in 4D space–time continuum and this is exactly what is used in STA [27] with advantage, replacing the old space–time tensorial approach to Lorentz transformations. A Lorentz boost along 'any arbitrary direction of the observer velocity' $\mathbf{v} = \hat{\alpha}_j v_j$ is provided by the appropriate space–time rotor $\mathcal{R} = \exp(\hat{\mathbf{R}}\,\omega/2) \equiv (\cosh\omega/2 + \hat{\mathbf{R}}\sinh\omega/2)$, containing the unit space–time bivector $\hat{\mathbf{R}} = \hat{\alpha}_0\hat{\mathbf{v}}$ with positive square. Here, $\hat{\mathbf{v}} = |\mathbf{v}|^{-1}\mathbf{v}$ and belongs to the 3D subspace orthogonal to $\hat{\alpha}_0$. The relative speed $\beta = \tanh\omega$ is parametrised by the hyperbolic angle or rapidity ω and the relativistic stretch factor $\gamma = \cosh\omega$. The transformation between the basis vectors, defined in terms of $\mathcal{R}$ and its reversion $\mathcal{R}^\dagger$ is $\hat{\alpha}'_\mu = \mathcal{R}\,\hat{\alpha}_\mu\,\mathcal{R}^\dagger$, where $\hat{\alpha}'_\mu$ are the new basis vectors. Now,

$$\begin{aligned}\hat{\alpha}'_0 &= \mathcal{R}\hat{\alpha}_0\mathcal{R}^\dagger = \exp(\hat{\mathbf{R}}\,\omega/2)\hat{\alpha}_0\exp(-\hat{\mathbf{R}}\,\omega/2)\\ &= \hat{\alpha}_0\cosh\omega + \hat{\mathbf{v}}\sinh\omega\\ &= \gamma(\hat{\alpha}_0 + \beta\hat{\mathbf{v}}) \Rightarrow \hat{\alpha}'_0 = c^{-1}\gamma\bar{\mathbf{v}}\end{aligned} \tag{41}$$

and obviously, $\hat{\alpha}'_0\cdot\hat{\alpha}'_0 = -1$. In STA, the inertial rest frame of the observer is therefore, defined by the future-pointing unit time-like basis vector ($\hat{\alpha}'_0$) along its space–time velocity $\bar{\mathbf{v}}$, i.e. along the tangent to the world line of the observer. The transformed spatial bases are similarly given by

$$\begin{aligned}\hat{\alpha}'_j &= \mathcal{R}\hat{\alpha}_j\mathcal{R}^\dagger = \exp(\hat{\mathbf{R}}\,\omega/2)\hat{\alpha}_j\exp(-\hat{\mathbf{R}}\,\omega/2)\\ &= \hat{\alpha}_j + \gamma\,\beta|\mathbf{v}|^{-1}v_j\hat{\alpha}_0 + (\gamma-1)|\mathbf{v}|^{-1}v_j\,\hat{\mathbf{v}}.\end{aligned} \tag{42}$$

Note that, putting $\hat{\mathbf{v}} = \hat{\alpha}_k;\ |\mathbf{v}|^{-1}v_k = 1$ and $v_j = 0,\ j\neq k$, we get directly, $\hat{\alpha}'_0 = \gamma(\hat{\alpha}_0 + \beta\hat{\alpha}_k);\ \hat{\alpha}'_k = \gamma(\hat{\alpha}_k + \beta\hat{\alpha}_0)$ and for all $j\neq k,\ \hat{\alpha}'_j = \hat{\alpha}_j$.

The combined EM field represented by the space–time bivector $\mathbf{F}$ is clearly reference-frame independent. However, the decomposition into relative fields $\mathbf{E} \equiv -(\mathbf{F}\cdot\hat{\alpha}_0)\hat{\alpha}_0$ and $c\,\mathbf{B} \equiv (\mathbf{F}\wedge\hat{\alpha}_0)\hat{\alpha}_0$ still implicitly depends upon the chosen time axis $\hat{\alpha}_0$. From eqs (41) and (42) follow

$$\begin{aligned}\hat{\alpha}'_0\cdot\hat{\alpha}'_j &= \gamma(\hat{\alpha}_0+\beta\hat{\mathbf{v}})\cdot[|\mathbf{v}|^{-1}v_j\{\gamma(\hat{\mathbf{v}}\\ &\quad+\beta\hat{\alpha}_0)-\hat{\mathbf{v}}\}+\hat{\alpha}_j]\\ &= |\mathbf{v}|^{-1}v_j\{-\gamma^2\beta+\gamma^2\beta-\gamma\beta+\gamma\beta\} = 0\end{aligned}$$

and

$$\begin{aligned}\hat{\alpha}'_j\cdot\hat{\alpha}'_k &= [|\mathbf{v}|^{-1}v_j\{\gamma(\hat{\mathbf{v}}+\beta\hat{\alpha}_0)-\hat{\mathbf{v}}\}\\ &\quad+\hat{\alpha}_j]\cdot[|\mathbf{v}|^{-1}v_k\{\gamma(\hat{\mathbf{v}}+\beta\hat{\alpha}_0)-\hat{\mathbf{v}}\}+\hat{\alpha}_k]\\ &= |\mathbf{v}|^{-2}v_jv_k\{\gamma^2(1-\beta^2)+1-2\gamma\}\\ &\quad+2|\mathbf{v}|^{-2}v_jv_k(\gamma-1)+\delta_{jk}\\ &= |\mathbf{v}|^{-2}v_jv_k\{1+1-2\gamma+2(\gamma-1)\}+\delta_{jk}\\ &= \delta_{jk}.\end{aligned}$$

Thus, $\hat{\alpha}'_0$ and $\hat{\alpha}'_j$ are the new orthogonal basis vectors under the Lorentz boost along any arbitrary direction of $\hat{\mathbf{v}}$. The field bivector is the same for all observers, no matter how the components transform under a change of reference system. However, it can be easily related to a description of electric and magnetic (relative) fields in a given inertial system. Now, for the same electromagnetic field bivector $\mathbf{F}$ expressed in two different coordinate systems, we can write

$$\begin{aligned}\mathbf{F} &= F^{\mu\nu}\,\hat{\alpha}_\mu\,\hat{\alpha}_\nu = F'^{\mu\nu}\,\hat{\alpha}'_\mu\,\hat{\alpha}'_\nu\\ &= F'^{\mu\nu}\,\mathcal{R}\hat{\alpha}_\mu\mathcal{R}^\dagger\mathcal{R}\hat{\alpha}_\nu\mathcal{R}^\dagger = F'^{\mu\nu}\,\mathcal{R}\hat{\alpha}_\mu\,\hat{\alpha}_\nu\mathcal{R}^\dagger\end{aligned} \tag{43}$$

and using the reverse transformation on $\mathbf{F}$, we get

$$\mathcal{R}^\dagger\mathbf{F}\mathcal{R} = F^{\mu\nu}\,\mathcal{R}^\dagger\hat{\alpha}_\mu\,\hat{\alpha}_\nu\mathcal{R} = F'^{\mu\nu}\,\hat{\alpha}_\mu\,\hat{\alpha}_\nu = \mathbf{F}'(\text{say}). \tag{44}$$

From the reverse transformations of the basis vectors $\mathcal{R}^\dagger\hat{\alpha}_0\mathcal{R} = \gamma(\hat{\alpha}_0 - \beta\hat{\mathbf{v}})$ and $\mathcal{R}^\dagger\hat{\alpha}_j\mathcal{R} = \hat{\alpha}_j - \gamma\,\beta|\mathbf{v}|^{-1}v_j\hat{\alpha}_0 + (\gamma-1)|\mathbf{v}|^{-1}v_j\,\hat{\mathbf{v}}$, we get explicit expressions for the reverse transformation of six bivector bases (see Appendix III). Using these expressions in eq. (44), one finally gets

$$F'^{10} = \gamma\, F^{10} + (1-\gamma)|\mathbf{v}|^{-2}\{(v_1)^2 F^{10} + v_1 v_2 F^{20} + v_1 v_3\, F^{30}\} + \gamma\beta|\mathbf{v}|^{-1}\{v_3 F^{31} - v_2\, F^{12}\}, \quad (45)$$

$$F'^{20} = \gamma\, F^{20} + (1-\gamma)|\mathbf{v}|^{-2}\{v_1 v_2 F^{10} + (v_2)^2 F^{20} + v_2 v_3 F^{30}\} + \gamma\beta|\mathbf{v}|^{-1}\{v_1 F^{12} - v_3 F^{23}\} \quad (46)$$

and

$$F'^{30} = \gamma\, F^{30} + (1-\gamma)|\mathbf{v}|^{-2}\{v_1 v_3 F^{10} + v_2 v_3 F^{20} + (v_3)^2 F^{30}\} + \gamma\beta|\mathbf{v}|^{-1}\{v_2 F^{23} - v_1 F^{31}\} \quad (47)$$

for the time-like components of the Faraday bivector and the purely spatial components in the primed system are given by

$$F'^{23} = \gamma\, F^{23} + (1-\gamma)|\mathbf{v}|^{-2}\{v_1^2 F^{23} + v_1 v_2 F^{31} + v_1 v_3 F^{12}\} + \gamma\beta|\mathbf{v}|^{-1}(v_2 F^{30} - v_3 F^{20}), \quad (48)$$

$$F'^{31} = \gamma\, F^{31} + (1-\gamma)|\mathbf{v}|^{-2}\{v_1 v_2 F^{23} + v_2^2 F^{31} + v_2 v_3 F^{12}\} + \gamma\beta|\mathbf{v}|^{-1}(v_3 F^{10} - v_1 F^{30}) \quad (49)$$

and

$$F'^{12} = \gamma\, F^{12} + (1-\gamma)|\mathbf{v}|^{-2}\{v_1 v_3 F^{23} + v_2 v_3 F^{31} + v_3^2 F^{12}\} + \gamma\beta|\mathbf{v}|^{-1}(v_1 F^{20} - v_2 F^{10}). \quad (50)$$

Alternatively, resolving $\mathbf{F}$ with respect to the space–time unit bivector $\hat{\mathbf{R}} = \hat{\alpha}_0\,\hat{\mathbf{v}}$ as

$$\mathbf{F} = \mathbf{F}(\hat{\mathbf{R}}\,\hat{\mathbf{R}}) = (\mathbf{F} : \hat{\mathbf{R}})\hat{\mathbf{R}} + (\mathbf{F}\cdot\hat{\mathbf{R}}).\hat{\mathbf{R}} + (\mathbf{F}\wedge\hat{\mathbf{R}}) : \hat{\mathbf{R}}, \text{ since } \hat{\mathbf{R}}\,\hat{\mathbf{R}} = 1 = \mathbf{F}_{\|\hat{\mathbf{R}}} + \mathbf{F}_{\perp\hat{\mathbf{R}}},$$

we also get the above expressions as follows. The first term $(\mathbf{F} : \hat{\mathbf{R}})\hat{\mathbf{R}}$ on the right-hand side lies in the plane of $\hat{\mathbf{R}}$ (denoted by $\mathbf{F}_{\|\hat{\mathbf{R}}}$) and the second term consists of parts of $\mathbf{F}_{\perp\hat{\mathbf{R}}}$ orthogonal to the plane of $\hat{\mathbf{R}}$ and also having a common basis vector with $\hat{\mathbf{R}}$. The final term $(\mathbf{F}\wedge\hat{\mathbf{R}}) : \hat{\mathbf{R}}$ (also a part of $\mathbf{F}_{\perp\hat{\mathbf{R}}}$), exists for dimensions higher than 3 and lies in the plane touching the orthogonal plane of $\hat{\mathbf{R}}$ only at a single point. The boost, generated by $\hat{\mathbf{R}}$, operates only on the second term $(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}}$, leaving $(\mathbf{F} : \hat{\mathbf{R}})\hat{\mathbf{R}}$ and $(\mathbf{F}\wedge\hat{\mathbf{R}}) : \hat{\mathbf{R}}$ unchanged.

$$\Rightarrow \mathbf{F}' = \mathcal{R}^\dagger\mathbf{F}\mathcal{R} = \mathbf{F} - (\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}} + \mathcal{R}^\dagger\{(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}}\}\mathcal{R} \quad (51)$$

and we get all the expressions, eqs (45)–(50) from eqs (43) and (51) (see Appendix III).

Now, if we consider the velocity boost simply along the *x*-axis (say), i.e. $\hat{\mathbf{v}} = \hat{\alpha}_1$, then obviously $|\mathbf{v}|^{-1}v_1 = 1$, $v_2 = 0$ and $v_3 = 0$, and from the above, we get the simplified expressions for the transformed bivector components: $F'^{10} = F^{10}$, $F'^{23} = F^{23}$, $F'^{20} = \gamma(F^{20} + \beta F^{12})$, $F'^{30} = \gamma(F^{30} - \beta F^{31})$, $F'^{31} = \gamma(F^{31} - \beta F^{30})$ and $F'^{12} = \gamma(F^{12} + \beta F^{20})$, as obtained in [38,39].

4.2 *Space–time split*

Hestenes has also introduced a simple alternative algebraic device called the space–time split to relate 'proper' (invariant) descriptions of physical properties to relative descriptions of different inertial frames. The eight even grade unit bases in STA, which define the even subalgebra, are mapped to eight unit bases of the 'relative' 3D GA space. The even subalgebra contains the scalar, pseudoscalar and six bivector bases of which the three time-like bases $\hat{\sigma}_{j0} = \hat{\alpha}_j\,\hat{\alpha}_0$ are converted into three 'relative' vector bases $\hat{\sigma}_j$ ($\equiv \hat{\sigma}_{j0}$) and the remaining three bivector bases $\hat{\sigma}_{jk} = \hat{\alpha}_j\,\hat{\alpha}_k$, $j \neq k$ are taken as the three 'relative' bivector bases. This defines eight unit bases of the 'relative' 3D GA space with respect to the observer's rest frame defined by $\hat{\alpha}_0$ and is called the space–time split. Different observer velocity vectors define different space–time splits, i.e. split the space–time with respect to different observers.

The space–time split actually separates the invariant electromagnetic field represented by Faraday bivector $\mathbf{F}$ (eq. (43)) into parts which commute and anticommute with $\hat{\alpha}_0$ respectively as $\mathbf{F} = \frac{1}{2}\{(\mathbf{F} - \hat{\alpha}_0\mathbf{F}\hat{\alpha}_0) + (\mathbf{F} + \hat{\alpha}_0\mathbf{F}\hat{\alpha}_0)\}$, and decomposes it into separate electric vector (relative) and magnetic bivector (relative) fields in the 'relative three-dimensional vector space' with respect to an observer with velocity along $\hat{\alpha}_0$. The electromagnetic field, as in 3D GA, is represented by a 'parabivector in the relative space'. Here, the commuting part $\frac{1}{2}\{(\mathbf{F} - \hat{\alpha}_0\mathbf{F}\hat{\alpha}_0)\} = -c\mathbf{B}$ represents purely spatial magnetic bivector and the anticommuting part $\frac{1}{2}\{(\mathbf{F}+\hat{\alpha}_0\mathbf{F}\hat{\alpha}_0)\} = \mathbf{e}\hat{\alpha}_0 = e_j\hat{\sigma}_j$ ($\equiv F^{j0}\hat{\sigma}_{j0}$). This shows clearly that the split into separate electric and magnetic fields depends on the observer velocity ($\hat{\alpha}_0$). Observers in relative motions, for example, a second observer in the rest frame basis vectors $\hat{\alpha}'_\mu = \mathcal{R}\hat{\alpha}_\mu\mathcal{R}^\dagger$, see different fields appropriate with velocity $\hat{\alpha}'_0$ (according to eq. (41)). From eq. (43) or (44), the components of the electric and the magnetic fields, measured by this observer may be easily obtained, respectively as $e'_j = \hat{\alpha}'_j\hat{\alpha}'_0 : \mathbf{F} = \hat{\alpha}_j\hat{\alpha}_0 : \mathbf{F}'$ and $c\,b'_j = \hat{\alpha}'_k\hat{\alpha}'_l : \mathbf{F} = \hat{\alpha}_k\hat{\alpha}_l : \mathbf{F}'$.

5. Concluding remarks

The advantages of using the geometric algebra, or more specifically the space–time algebraic formulation of electromagnetic theory are manifold. Apart from removing several inadequacies of the standard formulation according to VA, the provision for the inclusion of higher grade source term(s) in STA offers a natural passage for the introduction of the magnetic monopole. The formulation clearly illustrates the complete intrinsic dual symmetry of electromagnetism. Also, the space–time force (density), defined in STA, combines the electromagnetic energy and momentum conservation equation without tensors (which are actually grafted in the standard theory of VA). Moreover, GA uses the spinorial form replacing the old space–time tensorial approach to Lorentz transformations. Rotations in time-like planes represent boosts. Lorentz boost along any arbitrary direction of the observer velocity is deduced in the present work. All these theories are described in appropriate forms and terms and the paper is intended to provide a broad based exposure to the advanced undergraduate students.

In classical electromagnetism, the field(s) and the interactions are fully described by the Maxwell's equation and the Lorentz force law. However, in real materials, charged particles may couple with other physical forces and electromagnetic forces and equations are not the entire picture. Complex transport equations, for example the Boltzmann equation, the Fokker–Planck equation or the Navier–Stokes equations, must be solved to determine the time and spatial responses of charges in developing electrohydrodynamics, fluid dynamics, magnetohydrodynamics, superconductivity, stellar evolution etc.

Applications of space–time algebra to a wide range of physical theories are increasingly appearing as comprehensive, yet convenient and more economic descriptions removing the mathematical divide among classical, quantum and relativistic physics to a great extent. The Pauli and Dirac algebras of quantum mechanics are matrix representations of the geometric algebra of space and space–time, respectively. Hestenes has elucidated the geometric structure of the successive quantum theories of Schrödinger, Pauli and Dirac [40]. Spinors and operators can be manipulated without introducing any matrix representation or coordinate system in STA. It also naturally promises an elegant formulation of relativistic quantum electrodynamics (QED).

The term, eddy current, in electricity comes from analogous currents seen in fluid dynamics, causing localised areas of turbulence known as eddies giving rise to persistent vortices. Somewhat similarly, eddy currents can take time to build up and can persist for very short times in conductors due to their inductance. On the other hand, the Biot–Savart law, which is usually known to describe the magnetic field generated by a stationary electric current, is also used in aerodynamics to calculate the velocity induced by the vortex field. However, in comparison to the electromagnetic case, the roles of vortex and magnetic fields are reversed.

Formal similarities between physical concepts, notions and in mathematical structures, specially between the theories of electromagnetism and fluid mechanics are already noted in the prevalent literatures. Maxwell himself has pointed out parallels between the electromagnetic vector potential and the fluid velocity field $\mathbf{v}$. Similarity with the mathematical structures of electromagnetism and fluid mechanics may be readily perceived from the definitions of the vorticity field ($\mathbf{W} = \nabla\wedge\mathbf{v}$) and the Lamb's vector $\mathbf{l} = \mathbf{W}\cdot\mathbf{v} = -\nabla\Phi - \partial_t\mathbf{v}$ (in the inviscid case). The corresponding equations for the vorticity bivector field $\nabla\wedge\mathbf{W} = 0$ and for the Lamb's vector field ($\nabla\cdot\mathbf{l} = -\nabla^2\Phi = \rho_f$), are respectively, exactly similar to that for the magnetic and the electric fields described by the vector and the scalar potentials $\mathbf{a}_e$ and ϕ.

In subsequent papers, advantages gained from using GA in electromagnetism will be extended to the study of fluid mechanics and quantum mechanics. As pointed out by Feynman *et al* [10], electromagnetism is much easier than fluid mechanics and we have discussed electromagnetism first to understand the complications of fluid mechanics better. On the other hand, a new approach to electrodynamics, based on a fluidic viewpoint, developed a Navier–Stokes-like equation by using appropriate electromotive force. Recent works using new concepts, suggest among others, possible applications in producing electric fields of the required configuration in plasma medium [41].

Appendix I: Derivation of $\mathbf{F}^2$ ($\langle\mathbf{F}^2\rangle_0$, $\langle\mathbf{F}^2\rangle_2$ and $\langle\mathbf{F}^2\rangle_4$)

Since $\hat{\alpha}_j\,\hat{\alpha}_0 : \hat{\alpha}_k\,\hat{\alpha}_0 = \delta_{jk}$; $\hat{\alpha}_j\,\hat{\alpha}_0 : \hat{\alpha}_k\,\hat{\alpha}_l = 0$ and $\hat{\alpha}_j\,\hat{\alpha}_k : \hat{\alpha}_l\,\hat{\alpha}_m = \delta_{jm}\,\delta_{kl} - \delta_{jl}\,\delta_{km}$, from eq. (24) we get

$$\langle\mathbf{F}^2\rangle_0 = \mathbf{F} : \mathbf{F} = (F^{10})^2 + (F^{20})^2 + (F^{30})^2 - \{(F^{23})^2 + (F^{31})^2 + (F^{12})^2\} = \mathbf{e}^2 - c^2\mathbf{b}^2 \equiv \mathbf{e}^2 + c^2\mathbf{B}^2.$$

Again, since $\hat{\alpha}_j\,\hat{\alpha}_0\cdot\hat{\alpha}_k\,\hat{\alpha}_0 = \hat{\alpha}_j\,\hat{\alpha}_k = -\hat{\alpha}_k\,\hat{\alpha}_0\cdot\hat{\alpha}_j\,\hat{\alpha}_0$; $\hat{\alpha}_j\,\hat{\alpha}_0\cdot\hat{\alpha}_k\,\hat{\alpha}_l = \hat{\alpha}_l\,\hat{\alpha}_0\,\delta_{jk} - \hat{\alpha}_k\,\hat{\alpha}_0\,\delta_{jl} = -\hat{\alpha}_k\,\hat{\alpha}_l.\hat{\alpha}_j\,\hat{\alpha}_0$ and $\hat{\alpha}_j\,\hat{\alpha}_k.\hat{\alpha}_l\,\hat{\alpha}_m = \hat{\alpha}_j\,\hat{\alpha}_m\delta_{kl} - \hat{\alpha}_k\,\hat{\alpha}_m\delta_{jl} = 0$, it follows

$$\langle\mathbf{F}^2\rangle_2 = \mathbf{F}\cdot\mathbf{F}$$

$$= F^{10} F^{20} \cdot 0 + F^{10} F^{30} \cdot 0 + F^{20} F^{30} \cdot 0 + F^{10} F^{23} \cdot 0 + F^{10} F^{31} \cdot 0 + F^{10} F^{12} \cdot 0 + F^{20} F^{23} \cdot 0 + F^{20} F^{31} \cdot 0 + F^{20} F^{12} \cdot 0 + F^{30} F^{23} \cdot 0 + F^{30} F^{31} \cdot 0 + F^{30} F^{12} \cdot 0 + 0 + \cdots + 0 = 0.$$

Finally,

$$\begin{aligned}\langle \mathbf{F}^2\rangle_4 &= \mathbf{F}\wedge\mathbf{F}\\ &= F^{10} F^{23}(\hat\alpha_1\hat\alpha_0\hat\alpha_2\hat\alpha_3 + \hat\alpha_2\hat\alpha_3\hat\alpha_1\hat\alpha_0)\\ &\quad + F^{20} F^{31}(\hat\alpha_2\hat\alpha_0\hat\alpha_3\hat\alpha_1 + \hat\alpha_3\hat\alpha_1\hat\alpha_2\hat\alpha_0)\\ &\quad + F^{30} F^{12}(\hat\alpha_3\hat\alpha_0\hat\alpha_1\hat\alpha_2 + \hat\alpha_1\hat\alpha_2\hat\alpha_3\hat\alpha_0)\\ &= -2(F^{10} F^{23} + F^{20} F^{31} + F^{30} F^{12})\hat\alpha_0\hat\alpha_1\hat\alpha_2\hat\alpha_3\\ &= 2c\,\mathbf{e}\cdot\mathbf{b}\, I_4 \equiv 2c\,\hat\alpha_0\,\mathbf{e}\wedge\mathbf{B}.\end{aligned}$$

Appendix II: Derivation of $(\Box \cdot \mathbf{F})\cdot\mathbf{F}$

$$\begin{aligned}(\Box\cdot\mathbf{F})\cdot\mathbf{F} &= \{(-\hat\alpha_0 c^{-1}\partial_t + \nabla)\cdot(\mathbf{e}\wedge\hat\alpha_0 - c\,\mathbf{B})\}\\ &\quad \cdot(\mathbf{e}\wedge\hat\alpha_0 - c\,\mathbf{B}) = (-c^{-1}\partial_t\mathbf{e} + \hat\alpha_0\nabla\cdot\mathbf{e}\\ &\quad - c\nabla\cdot\mathbf{B})\cdot(\mathbf{e}\wedge\hat\alpha_0 - c\,\mathbf{B})\\ &= -\hat\alpha_0 c^{-1}(\partial_t\mathbf{e}).\mathbf{e} + (\partial_t\mathbf{e}).\mathbf{B}\\ &\quad + (\nabla\cdot\mathbf{e})\mathbf{e} + 0 - \hat\alpha_0\, c\,(\nabla\cdot\mathbf{B}).\mathbf{e}\\ &\quad + c^2(\nabla\cdot\mathbf{B}).\mathbf{B} = -\hat\alpha_0 c^{-1}\{(\partial_t\mathbf{e}).\mathbf{e}\\ &\quad + c^2(\nabla\cdot\mathbf{B}).\mathbf{e}\} + (\partial_t\mathbf{e}).\mathbf{B}\\ &\quad + (\nabla\cdot\mathbf{e})\mathbf{e} + c^2(\nabla\cdot\mathbf{B}).\mathbf{B}\\ &= -\hat\alpha_0 c^{-1}\{2^{-1}\partial_t(\mathbf{e}^2 - c^2\mathbf{B}^2)\\ &\quad + c^2\nabla\cdot(\mathbf{B}\cdot\mathbf{e})\} - \partial_t(\mathbf{B}\cdot\mathbf{e}) - 2^{-1}\nabla\mathbf{e}^2\\ &\quad + (\mathbf{e}.\nabla)\mathbf{e} + (\nabla\cdot\mathbf{e})\mathbf{e} + 2^{-1}c^2\nabla\mathbf{B}^2 - c^2(\mathbf{B}\wedge\nabla):\mathbf{B}\\ \Rightarrow \bar{\mathbf{f}} &= \epsilon_0\langle(\Box\mathbf{F})\mathbf{F}\rangle_1 \equiv \epsilon_0\,(\Box\cdot\mathbf{F})\cdot\mathbf{F}\\ &= -c^{-1}\,\hat\alpha_0(\partial_t u_{em} + \nabla\cdot\mathbf{s}) + \epsilon_0[(\nabla\cdot\mathbf{e})\mathbf{e} + (\mathbf{e}\cdot\nabla)\mathbf{e}]\\ &\quad - \mu_0^{-1}[(\nabla\wedge\mathbf{B}):\mathbf{B} + (\mathbf{B}\wedge\nabla):\mathbf{B}] - \nabla \mathrm{u}_{em}\\ &\quad - c^{-2}\,\partial_t\mathbf{s},\end{aligned}$$

subtracting a null term involving $\nabla\wedge\mathbf{B}$. Since

(i) $\nabla\cdot(\mathbf{B}\cdot\mathbf{e}) = (\nabla\wedge\mathbf{e}):\mathbf{B} - (\nabla\cdot\mathbf{B})\cdot\mathbf{e} \Rightarrow (\nabla\cdot\mathbf{B})\cdot\mathbf{e} = (\nabla\wedge\mathbf{e}):\mathbf{B} - \nabla\cdot(\mathbf{B}\cdot\mathbf{e}) = -\partial_t\mathbf{B}:\mathbf{B} - \nabla\cdot(\mathbf{B}\cdot\mathbf{e})$, using eq. (19) ($(\nabla\wedge\mathbf{e}):\mathbf{B} + \partial_t\mathbf{B} = 0$),

(ii) $\partial_t(\mathbf{B}\cdot\mathbf{e}) = (\partial_t\mathbf{B})\cdot\mathbf{e} + \mathbf{B}\cdot\partial_t\mathbf{e} \Rightarrow (\partial_t\mathbf{e})\cdot\mathbf{B} = -\partial_t(\mathbf{B}\cdot\mathbf{e}) + (\partial_t\mathbf{B})\cdot\mathbf{e} = -\partial_t(\mathbf{B}\cdot\mathbf{e}) - (\nabla\wedge\mathbf{e})\cdot\mathbf{e} = -\partial_t(\mathbf{B}\cdot\mathbf{e}) - 2^{-1}\nabla\mathbf{e}^2 + (\mathbf{e}\cdot\nabla)\mathbf{e}]$, using eq. (19) and the identity: $2^{-1}\nabla\mathbf{e}^2 = (\mathbf{e}\cdot\nabla))\mathbf{e} + (\nabla\wedge\mathbf{e})\cdot\mathbf{e}$ and

(iii) $2^{-1}\nabla(\mathbf{B}:\mathbf{B}) = (\nabla\cdot\mathbf{B})\cdot\mathbf{B} + (\mathbf{B}\wedge\nabla):\mathbf{B}$. (Note $\mathbf{B}^2 = \mathbf{B}:\mathbf{B} = -\mathbf{b}^2$, $(\mathbf{B}\wedge\nabla):\mathbf{B} = -(\mathbf{b}\cdot\nabla)\mathbf{b}$ and $(\nabla\wedge\mathbf{B}):\mathbf{B} = -(\nabla\cdot\mathbf{b})\mathbf{b}$ in VA).

Appendix III: Expressions for the transformed (reverse) bivector bases

The transformed basis vectors with respect to the space–time 'rotor' $\mathcal{R}$ are given by

$$\begin{aligned}\hat\alpha_0' &= \mathcal{R}\hat\alpha_0\mathcal{R}^\dagger = \exp(\hat{\mathbf{R}}\,\omega/2)\hat\alpha_0\exp(-\hat{\mathbf{R}}\,\omega/2)\\ &= (\cosh\omega/2 + \hat\alpha_0\hat{\mathbf{v}}\sinh\omega/2)\hat\alpha_0(\cosh\omega/2\\ &\quad - \hat\alpha_0\hat{\mathbf{v}}\sinh\omega/2)\\ &= (\cosh\omega/2 + \hat\alpha_0\hat{\mathbf{v}}\sinh\omega/2)(\hat\alpha_0\cosh\omega/2\\ &\quad + \hat{\mathbf{v}}\sinh\omega/2)\\ &= \hat\alpha_0(\cosh^2\omega/2 + \sinh^2\omega/2)\\ &\quad + \hat{\mathbf{v}}(\sinh\omega/2\cosh\omega/2 + \sinh\omega/2\cosh\omega/2)\\ &= \hat\alpha_0\cosh\omega + \hat{\mathbf{v}}\sinh\omega\\ &= \gamma(\hat\alpha_0 + \beta\hat{\mathbf{v}}) \Rightarrow \hat\alpha_0' = c^{-1}\gamma\bar{\mathbf{v}}\end{aligned} \tag{52}$$

and obviously, $\hat\alpha_0'\cdot\hat\alpha_0' = -1$. In STA, therefore, the inertial rest frame of the observer is defined by the future-pointing unit time-like basis vector ($\hat\alpha_0'$) along its space–time velocity $\bar{\mathbf{v}}$, i.e. along the tangent to the world line of the observer. Putting $\mathbf{v} = \hat\alpha_k$ we get directly, $\hat\alpha_0' = \gamma(\hat\alpha_0 + \beta\hat\alpha_k)$. Similarly,

$$\begin{aligned}\hat\alpha_j' &= \mathcal{R}\hat\alpha_j\mathcal{R}^\dagger = \exp(\hat{\mathbf{R}}\,\omega/2)\hat\alpha_j\exp(-\hat{\mathbf{R}}\,\omega/2)\\ &= (\cosh\omega/2 + \hat{\mathbf{R}}\sinh\omega/2)(\hat\alpha_j\cosh\omega/2\\ &\quad - \hat\alpha_j\hat{\mathbf{R}}\sinh\omega/2)\\ &= \hat\alpha_j\cosh^2\omega/2 - \hat\alpha_j\hat{\mathbf{R}}\cosh\omega/2\,\sinh\omega/2\\ &\quad + \hat{\mathbf{R}}\hat\alpha_j\cosh\omega/2\,\sinh\omega/2\\ &\quad - \hat{\mathbf{R}}(\hat\alpha_j\hat{\mathbf{R}})\sinh^2\omega/2\\ &= \hat\alpha_j\cosh^2\omega/2 + (\hat{\mathbf{R}}\hat\alpha_j - \hat\alpha_j\hat{\mathbf{R}})\cosh\omega/2\,\sinh\omega/2\\ &\quad - \hat{\mathbf{R}}(\hat\alpha_j\cdot\hat{\mathbf{R}} + \hat\alpha_j\wedge\hat{\mathbf{R}})\sinh^2\omega/2\\ &= \hat\alpha_j + \hat{\mathbf{R}}\cdot\hat\alpha_j\sinh\omega - \hat{\mathbf{R}}(\hat\alpha_j\cdot\hat{\mathbf{R}})(\cosh\omega - 1).\end{aligned}$$

Since $\hat{\mathbf{R}}\hat\alpha_j - \hat\alpha_j\hat{\mathbf{R}} = 2\hat{\mathbf{R}}\cdot\hat\alpha_j$ and for the unit simple bivector $\hat{\mathbf{R}}$ we can write

$$\begin{aligned}\hat\alpha_j &= (\hat{\mathbf{R}}\hat{\mathbf{R}})\hat\alpha_j \equiv \hat{\mathbf{R}}(\hat{\mathbf{R}}\hat\alpha_j) = \hat{\mathbf{R}}(\hat{\mathbf{R}}.\hat\alpha_j + \hat{\mathbf{R}}\wedge\hat\alpha_j)\\ &= \hat{\mathbf{R}}(-\hat\alpha_j.\hat{\mathbf{R}} + \hat\alpha_j\wedge\hat{\mathbf{R}})\\ \Rightarrow &\ \hat{\mathbf{R}}(\hat\alpha_j\cdot\hat{\mathbf{R}} + \hat\alpha_j\wedge\hat{\mathbf{R}}) = \hat\alpha_j + 2\hat{\mathbf{R}}(\hat\alpha_j\cdot\hat{\mathbf{R}}).\end{aligned}$$

Finally, we get

$$\begin{aligned}\hat\alpha_j' &= \hat\alpha_j + \gamma\,\beta(\hat\alpha_0\,\hat{\mathbf{v}})\cdot\hat\alpha_j\\ &\quad - (\gamma - 1)(\hat\alpha_0\,\hat{\mathbf{v}})\cdot\{\hat\alpha_j\cdot(\hat\alpha_0\,\hat{\mathbf{v}})\}\\ &= \hat\alpha_j + \gamma\,\beta|\mathbf{v}|^{-1}v_j\hat\alpha_0 + (\gamma - 1)|\mathbf{v}|^{-1}v_j\,\hat{\mathbf{v}}.\end{aligned} \tag{53}$$

Note that, with $\hat{\mathbf{v}} = \hat{\alpha}_k$; $|\mathbf{v}|^{-1} v_k = 1$ and $v_j = 0$, $j \neq k \Rightarrow \hat{\alpha}'_k = \gamma(\hat{\alpha}_k + \beta\hat{\alpha}_0)$ and for all $j \neq k$, $\hat{\alpha}'_j = \hat{\alpha}_j$.

$$\begin{aligned}
&\Rightarrow \hat{\alpha}'_j \cdot \hat{\alpha}'_k = [|\mathbf{v}|^{-1} v_j\{\gamma(\hat{\mathbf{v}} + \beta\hat{\alpha}_0) - \hat{\mathbf{v}}\} + \hat{\alpha}_j] \\
&\cdot[|\mathbf{v}|^{-1} v_k\{\gamma(\hat{\mathbf{v}} + \beta\hat{\alpha}_0) - \hat{\mathbf{v}}\} + \hat{\alpha}_k] \\
&= |\mathbf{v}|^{-2} v_j v_k\{\gamma^2(1-\beta^2) + 1 - 2\gamma\} \\
&\quad +2|\mathbf{v}|^{-2} v_j v_k(\gamma - 1) + \delta_{jk} \\
&= |\mathbf{v}|^{-2} v_j v_k\{1 + 1 - 2\gamma + 2(\gamma-1)\} + \delta_{jk} \\
&= \delta_{jk}.
\end{aligned}$$

Also,

$$\begin{aligned}
\hat{\alpha}'_0 \cdot \hat{\alpha}'_j = \gamma(\hat{\alpha}_0 &+\beta\hat{\mathbf{v}}) \cdot [|\mathbf{v}|^{-1} v_j\{\gamma(\hat{\mathbf{v}} + \beta\hat{\alpha}_0) - \hat{\mathbf{v}}\} \\
&+\hat{\alpha}_j] = |\mathbf{v}|^{-1} v_j\{-\gamma^2\beta \\
&+\gamma^2\beta - \gamma\beta + \gamma\beta\} = 0.
\end{aligned}$$

Thus, $\hat{\alpha}'_0$ and $\hat{\alpha}'_j$ are the new orthogonal basis vectors under the Lorentz boost along any arbitrary direction of $\hat{\mathbf{v}}$. Now, for the same electromagnetic field bivector $\mathbf{F}$ expressed in two different coordinate systems, we can write,

$$\begin{aligned}
\mathbf{F} &= F^{\mu\nu}\,\hat{\alpha}_\mu\,\hat{\alpha}_\nu = F'^{\mu\nu}\,\hat{\alpha}'_\mu\,\hat{\alpha}'_\nu \\
&= F'^{\mu\nu}\,\mathcal{R}\hat{\alpha}_\mu\mathcal{R}^\dagger\mathcal{R}\hat{\alpha}_\nu\mathcal{R}^\dagger \\
&= F'^{\mu\nu}\,\mathcal{R}\hat{\alpha}_\mu\,\hat{\alpha}_\nu\mathcal{R}^\dagger
\end{aligned} \tag{54}$$

and using the reverse transformation of $\mathbf{F}$, we get

$$\begin{aligned}
\mathcal{R}^\dagger\mathbf{F}\mathcal{R} &= F^{\mu\nu}\,\mathcal{R}^\dagger\hat{\alpha}_\mu\,\hat{\alpha}_\nu\mathcal{R} = F'^{\mu\nu}\,\hat{\alpha}_\mu\,\hat{\alpha}_\nu \\
&= \mathbf{F}'(\text{say}).
\end{aligned} \tag{55}$$

Now, the reverse transformations of the basis vectors produce $\mathcal{R}^\dagger\hat{\alpha}_0\mathcal{R} = \gamma(\hat{\alpha}_0 - \beta\hat{\mathbf{v}})$ and $\mathcal{R}^\dagger\hat{\alpha}_j\mathcal{R} = \hat{\alpha}_j - \gamma\,\beta|\mathbf{v}|^{-1}v_j\hat{\alpha}_0 + (\gamma-1)|\mathbf{v}|^{-1}v_j\,\hat{\mathbf{v}} \Rightarrow \mathcal{R}^\dagger\hat{\alpha}_j\,\hat{\alpha}_0\mathcal{R} = \{\hat{\alpha}_j - \gamma\,\beta|\mathbf{v}|^{-1}v_j\hat{\alpha}_0 + (\gamma-1)|\mathbf{v}|^{-1}v_j\,\hat{\mathbf{v}}\} \wedge \gamma(\hat{\alpha}_0 - \beta\hat{\mathbf{v}}) = \gamma\hat{\alpha}_j \wedge \hat{\alpha}_0 - \gamma\beta|\mathbf{v}|^{-1}v_k\hat{\alpha}_j \wedge \hat{\alpha}_k + (1-\gamma)|\mathbf{v}|^{-1}v_j\hat{\mathbf{v}} \wedge \hat{\alpha}_0$, $j \neq k$, from which we get the explicit expressions for $\mathcal{R}^\dagger\hat{\alpha}_1\,\hat{\alpha}_0\mathcal{R}$, $\mathcal{R}^\dagger\hat{\alpha}_2\,\hat{\alpha}_0\mathcal{R}$ and $\mathcal{R}^\dagger\hat{\alpha}_3\,\hat{\alpha}_0\mathcal{R}$. Also, $\mathcal{R}^\dagger\hat{\alpha}_j\,\hat{\alpha}_k\mathcal{R} = \{\hat{\alpha}_j - \gamma\,\beta|\mathbf{v}|^{-1}v_j\hat{\alpha}_0 + (\gamma-1)|\mathbf{v}|^{-1}v_j\,\hat{\mathbf{v}}\} \wedge \{\hat{\alpha}_k - \gamma\,\beta|\mathbf{v}|^{-1}v_k\hat{\alpha}_0 + (\gamma-1)|\mathbf{v}|^{-1}v_k\,\hat{\mathbf{v}}\} = \gamma\hat{\alpha}_j \wedge \hat{\alpha}_k + \gamma\beta|\mathbf{v}|^{-1}(v_j\hat{\alpha}_k \wedge \hat{\alpha}_0 - v_k\hat{\alpha}_j \wedge \hat{\alpha}_0) + (1-\gamma)|\mathbf{v}|^{-2}\{v_l v_l\hat{\alpha}_j \wedge \hat{\alpha}_k + v_j v_l\hat{\alpha}_k \wedge \hat{\alpha}_l - v_k v_l\hat{\alpha}_j \wedge \hat{\alpha}_l\}$, which gives expressions for $\mathcal{R}^\dagger\hat{\alpha}_2\,\hat{\alpha}_3\mathcal{R}$, $\mathcal{R}^\dagger\hat{\alpha}_3\,\hat{\alpha}_1\mathcal{R}$ and $\mathcal{R}^\dagger\hat{\alpha}_1\,\hat{\alpha}_2\mathcal{R}$. Using these expressions in eq. (44) one easily gets all the expressions from eqs (45) to (50).

Alternatively, we can also get the above expressions by resolving $\mathbf{F}$ with respect to the space–time unit bivector $\hat{\mathbf{R}} = \hat{\alpha}_0\,\hat{\mathbf{v}}$ ($\hat{\mathbf{v}} = |\mathbf{v}|^{-1}\mathbf{v}$ and $\mathbf{v} = v_l\,\hat{\alpha}_l$) as follows:

$$\begin{aligned}
\mathbf{F} = \mathbf{F}(\hat{\mathbf{R}}\,\hat{\mathbf{R}}) &= (\mathbf{F}\,\hat{\mathbf{R}})\hat{\mathbf{R}} \text{ (since } \hat{\mathbf{R}}\,\hat{\mathbf{R}} = 1) \\
&= (\mathbf{F} : \hat{\mathbf{R}} + \mathbf{F}\cdot\hat{\mathbf{R}} + \mathbf{F}\wedge\hat{\mathbf{R}})\hat{\mathbf{R}} \\
&= (\mathbf{F} : \hat{\mathbf{R}})\hat{\mathbf{R}} + (\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}} + (\mathbf{F}\wedge\hat{\mathbf{R}}) : \hat{\mathbf{R}} \\
&= \mathbf{F}_{\parallel\hat{\mathbf{R}}} + \mathbf{F}_{\perp\hat{\mathbf{R}}}.
\end{aligned}$$

The first term $(\mathbf{F} : \hat{\mathbf{R}})\hat{\mathbf{R}}$ on the right-hand side lies in the plane of $\hat{\mathbf{R}}$ (denoted by $\mathbf{F}_{\parallel\hat{\mathbf{R}}}$) and the second term consists of parts of $\mathbf{F}_{\perp\hat{\mathbf{R}}}$ orthogonal to the plane of $\hat{\mathbf{R}}$ and also having a common basis vector with $\hat{\mathbf{R}}$. The final term $(\mathbf{F}\wedge\hat{\mathbf{R}}) : \hat{\mathbf{R}}$ (also a part of $\mathbf{F}_{\perp\hat{\mathbf{R}}}$) exists for dimensions higher than 3 and lies in plane touching the orthogonal plane of $\hat{\mathbf{R}}$ only at a single point. The inner products of $\hat{\mathbf{R}}$ with both the first and the final terms vanish and simple working out with geometrical calculus produce

$$\begin{aligned}
&(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}} \\
&= \mathbf{F} - |\mathbf{v}|^{-2}[\{v_1^2\,\hat{\alpha}_1\hat{\alpha}_0 + (v_1v_2\,\hat{\alpha}_2\hat{\alpha}_0 \\
&\quad +v_1v_3\,\hat{\alpha}_3\hat{\alpha}_0)\}F^{10} \\
&\quad +\{v_2^2\,\hat{\alpha}_2\hat{\alpha}_0 + (v_1v_2\,\hat{\alpha}_1\hat{\alpha}_0 + v_2v_3\,\hat{\alpha}_3\hat{\alpha}_0)\}F^{20} \\
&\quad +\{v_3^2\,\hat{\alpha}_3\hat{\alpha}_0 + (v_2v_3\,\hat{\alpha}_2\hat{\alpha}_0 \\
&\quad +v_1v_3\,\hat{\alpha}_1\hat{\alpha}_0)\}F^{30} + \{v_1^2\,\hat{\alpha}_2\,\hat{\alpha}_3 + (v_1\,v_2\,\hat{\alpha}_3\,\hat{\alpha}_1 \\
&\quad +v_1\,v_3\,\hat{\alpha}_1\,\hat{\alpha}_2)\}F^{23} \\
&\quad +\{v_2^2\,\hat{\alpha}_3\,\hat{\alpha}_1 + (v_2\,v_3\,\hat{\alpha}_1\,\hat{\alpha}_2 + v_1\,v_2\,\hat{\alpha}_2\,\hat{\alpha}_3)\}F^{31} \\
&\quad +\{v_3^2\,\hat{\alpha}_1\,\hat{\alpha}_2 + (v_1\,v_3\,\hat{\alpha}_2\,\hat{\alpha}_3 + v_2\,v_3\,\hat{\alpha}_3\,\hat{\alpha}_1)\}F^{12}].
\end{aligned}$$

Now, for example, if we consider the velocity boost along the *x*-axis, i.e. $\hat{\mathbf{v}} = \hat{\alpha}_1$, we simply get:

$$\begin{aligned}
&\mathbf{F}_{\parallel\hat{\mathbf{R}}} = (\mathbf{F} : \hat{\mathbf{R}})\hat{\mathbf{R}} = F^{10}\,\hat{\alpha}_1\,\hat{\alpha}_0, \\
&(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}} = F^{20}\,\hat{\alpha}_2\,\hat{\alpha}_0 + F^{30}\,\hat{\alpha}_3\,\hat{\alpha}_0 \\
&\qquad\qquad +F^{31}\,\hat{\alpha}_3\,\hat{\alpha}_1 + F^{12}\,\hat{\alpha}_1\,\hat{\alpha}_2 \\
&\text{and} \\
&(\mathbf{F}\wedge\hat{\mathbf{R}}) : \hat{\mathbf{R}} = F^{23}\,\hat{\alpha}_2\,\hat{\alpha}_3.
\end{aligned}$$

The boost, generated by $\hat{\mathbf{R}}$, operates only on the second term $(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}}$, leaving $(\mathbf{F} : \hat{\mathbf{R}})\hat{\mathbf{R}}$ and $(\mathbf{F}\wedge\hat{\mathbf{R}}) : \hat{\mathbf{R}}$ unchanged.

$$\begin{aligned}
\Rightarrow \mathbf{F}' &= \mathcal{R}^\dagger\mathbf{F}\mathcal{R} \\
&= \mathbf{F} - (\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}} + \mathcal{R}^\dagger\{(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}}\}\mathcal{R}.
\end{aligned} \tag{56}$$

Noting that, $\mathcal{R}^\dagger\{(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}}\}\mathcal{R} = (\mathcal{R}^\dagger)^2\{(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}}\} = \{(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}}\}\mathcal{R}^2 = \{(\mathbf{F}\cdot\hat{\mathbf{R}})\cdot\hat{\mathbf{R}}\}\,\gamma(1 + \beta\,\hat{\alpha}_0\,\hat{\mathbf{v}})$, we get all the expressions, eqs (45)–(50) from eqs (42) and (51).

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