Pramana – J. Phys. (2022) 96:162
https://doi.org/10.1007/s12043-022-02410-2

Phase function method for Hulthén-distorted separable non-local potentials in all partial waves

B SWAIN, U LAHA*, P SAHOO and A K BEHERA

Department of Physics, National Institute of Technology, Jamshedpur 831 014, India
*Corresponding author. E-mail: ujjwal.laha@gmail.com

MS received 5 February 2022; revised 6 April 2022; accepted 6 April 2022

Abstract. The scattering phase parameters for short-range local potentials can be evaluated by using the phase function method (PFM), regarded as an efficient approach for computing scattering phase shifts for quantum mechanial problems, without solving the Schrödinger equation. We adapt PFM to deal with the Hulthén-distorted separable non-local potentials and derive a closed form expression for the phase function with rigorous inclusion of electromagnetic effect. We demonstrate the usefulness of our constructed expression by calculating elastic scattering phase parameters for proton–deuteron (p–d) system which agree quite well with the previous results.

Keywords. Phase function method; Hulthén-distorted separable potential; scattering phase shift; nucleon–nucleus system.

PACS Nos **03.65.Nk; 21.30.Fe; 13.75.Cs; 24.10.-i**

1. Introduction

The phase function method (PFM) [1] represents an alternative approach to traditional quantum scattering theory for computing scattering phase shifts and is mostly useful for those potentials for which the Schrödinger equation does not admit exact analytical solutions. It is well known in the theory of ordinary differential equation that any linear second-order homogeneous differential equation, like the Schrödinger equation, can be converted to a first-order nonlinear equation of the Riccati-type. The function that satisfies the Riccati-type equation is termed as the phase function and reproduces the scattering phase shift at each point by the potential truncated at that point. Thus, this procedure provides the desired method for direct determination of scattering phase shifts by a local potential. However, the PFM needs modification for scattering by non-local or local plus non-local interactions. One of us (UL) [2–4] prescribed a modification in the traditional PFM for treating Coulomb/Hulthén-distorted separable non-local potential and advocated the efficiency of this alternative approach to potential scattering. Laha *et al* [4] studied the S-wave nucleon–nucleon scattering by constructing a compact expression for the phase shift for motion in Hulthén plus separable non-local potential. The Hulthén potential [5] serves as a model for the screened or cut-off Coulomb interactions and is exactly solvable for S-wave only. However, higher partial wave solutions have been derived by several groups [6–9] with certain approximation techniques. The most common one is the use of screened centrifugal barrier [9–14]. Due to the non-existence of the pure Coulomb potential in reality, it is important to study the screened or cut-off Coulomb interaction to realise the effect of screening in charged particle scattering which invariably affects the theory and the interpretation of data. Many standard results in non-relativistic scattering theory have to be customised for charged particle scattering theory as the particle never behaves like free particle. Therefore, in many situations, the electromagnetic part of the interaction is described by a relatively short-range screened Coulomb potential. This is the main motivation for considering screened Coulomb potential. The Hulthén potential [5] is a legendary example of the exponentially screened Coulomb potential. The present text is devoted to deal with Hulthén-distorted Graz [15–17] separable potential for treating nucleon–nucleus elastic scattering. In §2, we derive closed form expression for the phase shift for Hulthén plus separable non-local potential. In §3, we make some checks on our expressions with respect to its Coulomb limit. Section 4 is

Published online: 30 August 2022

devoted to the case study and discussion of our results. Finally, we present concluding remarks in §5.

2. Phase function method for Hulthén-distorted separable potential

The Hulthén [5] potential is defined as

$$V_H(r) = V_0 \frac{e^{-r/a}}{1-e^{-r/a}}, \quad a > 0 \tag{1}$$

and the Graz separable potential is defined by

$$V(r,r') = \lambda_\ell g_\ell(r) g_\ell(r'), \tag{2}$$

where

$$g_\ell(r) = 2^{-\ell}(\ell!)^{-1} r^\ell e^{-\beta_\ell r} \tag{3}$$

and

$$g_\ell(r') = 2^{-\ell}(\ell!)^{-1} r'^\ell e^{-\beta_\ell r'}. \tag{4}$$

The radial Schrödinger wave equation for regular boundary condition under the motion in Hulthén plus Graz separable potential is written as

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - V_0 \frac{e^{-r/a}}{1-e^{-r/a}}\right]\phi_{HG}(k,r) = \lambda_\ell 2^{-2\ell}(\ell!)^{-2} r^\ell e^{-\beta_\ell r} d_{H\ell}(k), \tag{5}$$

where

$$d_{H\ell}(k) = \int_0^\infty dr'\, r'^\ell e^{-\beta_\ell r'} \phi_{HG}(k,r'). \tag{6}$$

The regular solution of eq. (5) is given by

$$\phi_{HG}(k,r) = k\phi_{H\ell}(k,r) + \lambda_\ell 2^{-2\ell}(\ell!)^{-2} d_{H\ell}(k) \times \int_0^r dr'\, e^{-\beta_\ell r'} r'^\ell G_\ell^{H(R)}(r,r'). \tag{7}$$

Multiplying eq. (7) with $r^\ell e^{-\beta_\ell r}$ on both sides and integrating from 0 to ∞ one can get

$$d_{H\ell}(k) = \frac{k\int_0^\infty dr r^\ell\, e^{-\beta_\ell r}\phi_{H\ell}(k,r)}{D_{H\ell}(k)}, \tag{8}$$

where $D_{H\ell}(k)$ is the Fredholm determinant for regular boundary condition and is given by

$$D_{H\ell}(k) = 1 - \lambda_\ell 2^{-2\ell}(\ell!)^{-2} \times \int_0^\infty \int_0^r dr\, dr' r^\ell\, r'^\ell\, e^{-\beta_\ell r}\, e^{-\beta_\ell r'}\, G_\ell^{H(R)}(r,r'). \tag{9}$$

Here $G_\ell^{H(R)}(r,r')$ is the regular Green's function for motion in Hulthén plus Graz separable potential and is written as [18]

$$G_\ell^{H(R)}(r,r') = \frac{1}{f_{H\ell}(k)}\big[\phi_{H\ell}(k,r) f_{H\ell}(k,r') - \phi_{H\ell}(k,r') f_{H\ell}(k,r)\big]. \tag{10}$$

Further, eq. (10) can be modified as [2–4]

$$G_\ell^{H(R)}(r,r') = \frac{1}{|f_{H\ell}(k)|^2}\, [\phi_{H\ell}(k,r) \times Re\left\{f_{H\ell}(-k) f_{H\ell}(k,r')\right\} - \phi_{H\ell}(k,r') Re\left\{f_{H\ell}(-k) f_{H\ell}(k,r)\right\}], \text{ for } r' < r$$
$$= 0, \text{ for } r' > r, \tag{11}$$

where the regular solution $\phi_{H\ell}(k,r)$ and the irregular solution $f_{H\ell}(k,r)$ for Hulthén potential in all partial waves [9] are given by

$$\phi_{H\ell}(k,r) = e^{ikr} a^{\ell+1}(1-e^{-r/a})^{\ell+1} \times {}_2F_1(A,B;C;1-e^{-r/a}) \tag{12}$$

and

$$f_{H\ell}(k,r) = (1-e^{-r/a})^{-\ell} e^{ikr} \times {}_2F_1(A-2\ell-1, B-2\ell-1; 1-2iak, e^{-r/a}) \tag{13}$$

respectively whereas the Jost function $f_{H\ell}(k)$ for the same reads as [9]

$$f_{H\ell}(k) = \frac{a^\ell\, \Gamma(1-2iak)\, \Gamma(2\ell+2)}{\Gamma(A)\, \Gamma(B)} \tag{14}$$

with

$$A = \ell + 1 - ika + ia(V_0+k^2)^{1/2}, \tag{15}$$

$$B = \ell + 1 - ika - ia(V_0+k^2)^{1/2} \tag{16}$$

and

$$C = 2\ell + 2. \tag{17}$$

With the help of eq. (11), eq. (7) becomes

$$\phi_{HG}(k,r) = k\phi_{H\ell}(k,r)\Bigg[1 + \lambda_\ell 2^{-2\ell}(\ell!)^{-2} d_{H\ell}(k) \times \frac{1}{k\,|f_{H\ell}(k)|^2}\int_0^r dr'\, e^{-\beta_\ell r'} \times r'^\ell Re\{f_{H\ell}(-k) f_{H\ell}(k,r')\}\Bigg] - \lambda_\ell 2^{-2\ell}(\ell!)^{-2} d_{H\ell}(k) \frac{1}{|f_{H\ell}(k)|^2} \times Re\left\{f_{H\ell}(-k) f_{H\ell}(k,r)\right\} \times \int_0^r dr'\, e^{-\beta_\ell r'} r'^\ell \phi_{H\ell}(k,r'). \tag{18}$$

We introduce the phase and amplitude functions $\delta_{HG}(k,r)$ and $\alpha_{HG}(k,r)$ as follows:

$$\alpha_{HG}(k,r)\cos\delta_{HG}(k,r) = 1+\lambda_\ell 2^{-2\ell}(\ell!)^{-2}d_{H\ell}(k) \times \frac{1}{k\,|f_{H\ell}(k)|^2}\int_0^r \mathrm{d}r' \mathrm{e}^{-\beta_\ell r'} r'^\ell \times Re\left\{f_{H\ell}(-k)f_{H\ell}(k,r')\right\} \quad (19)$$

and

$$\alpha_{HG}(k,r)\sin\delta_{HG}(k,r) = -\lambda_\ell 2^{-2\ell}(\ell!)^{-2}d_{H\ell}(k) \times \frac{1}{|f_{H\ell}(k)|^2}\int_0^r \mathrm{d}r'\, \mathrm{e}^{-\beta_\ell r'} r'^\ell \phi_{H\ell}(k,r'). \quad (20)$$

So eq. (18) can be expressed as

$$\phi_{HG}(k,r) = \alpha_{HG}(k,r)[k\,\phi_{H\ell}(k,r)\cos\delta_{HG}(k,r) + \sin\delta_{HG}(k,r) Re\{f_{H\ell}(-k)f_{H\ell}(k,r)\}]. \quad (21)$$

Now, from eqs (19) and (20) the phase equation is obtained in the following form:

$$\tan\delta_{HG}(k,r) = \frac{-\lambda_\ell 2^{-2\ell}(\ell!)^{-2}d_{H\ell}(k)\frac{1}{|f_{H\ell}(k)|^2}I_1(\beta_\ell,k,r)}{1+\lambda_\ell 2^{-2\ell}(\ell!)^{-2}d_{H\ell}(k)\frac{1}{k|f_{H\ell}(k)|^2}I_2(\beta_\ell,k,r)}, \quad (22)$$

where

$$I_1(\beta_\ell,k,r) = \int_0^r \mathrm{d}r'\, \mathrm{e}^{-\beta_\ell r'} r'^\ell \phi_{H\ell}(k,r') \quad (23)$$

and

$$I_2(\beta_\ell,k,r) = \int_0^r \mathrm{d}r' \mathrm{e}^{-\beta_\ell r'} r'^\ell Re\left\{f_{H\ell}(-k)f_{H\ell}(k,r')\right\}. \quad (24)$$

With the limiting behaviour of $r\to\infty$, from eq. (22) the desired equation for scattering phase shift can be conveniently expressed as

$$\tan\delta_{HG}(k) = \frac{-\lambda_\ell 2^{-2\ell}(\ell!)^{-2}d_{H\ell}(k)\frac{1}{|f_{H\ell}(k)|^2}I_1(\beta_\ell,k)}{1+\lambda_\ell 2^{-2\ell}(\ell!)^{-2}d_{H\ell}(k)\frac{1}{k|f_{H\ell}(k)|^2}I_2(\beta_\ell,k)}. \quad (25)$$

The two definite integral involved in eq. (25) can be easily solved by using the following standard integral [19–21]:

$$\int_0^1 \mathrm{d}x\, x^{\rho-1}(1-x)^{\sigma-1}\,{}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\,{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;1) \quad (26)$$

with $Re\rho>0$, $Re\sigma>0$, $Re(\gamma+\sigma-\alpha-\beta)>0$ to get

$$I_1(\beta_\ell,k) = \int_0^\infty \mathrm{d}r' \mathrm{e}^{-\beta_\ell r'} r'^\ell \phi_{H\ell}(k,r')$$

$$= a^{2\ell+2}\frac{\Gamma(2\ell+2)\Gamma\{(\beta_\ell-ik)a\}}{\Gamma\{(2\ell+2)+(\beta_\ell-ik)a\}} \times {}_2F_1(A,B;2\ell+2+(\beta_\ell-ik)a;1) \quad (27)$$

and

$$I_2(\beta_\ell,k) = \int_0^\infty \mathrm{d}r' \mathrm{e}^{-\beta_\ell r'} r'^\ell Re\left\{f_{H\ell}(-k)f_{H\ell}(k,r')\right\} = Re\Big\{f_{H\ell}(-k)a^\ell\frac{1}{(\beta_\ell-ik)} \times {}_3F_2(A-2\ell-1,B-2\ell-1, (\beta_\ell-ik)a;(1-2iak),(\beta_\ell-ik)a+1;1)\Big\}. \quad (28)$$

To calculate the scattering phase shift from eq. (25), one needs an analytical expression for $d_{H\ell}(k)$ which involves the integral $I_1(\beta_\ell,k)$ and the quantity $D_{H\ell}(k)$. The Fredholm determinant $D_{H\ell}(k)$ is the double transform of the Hulthén Green's function by the form factors of the separable potential. To that end, we have to find the integral

$$I_3(\beta_\ell,r) = \int_0^r \mathrm{d}r' r'^\ell \mathrm{e}^{-\beta_\ell r'} G_\ell^{H(R)}(r,r').$$

From eqs (10), (12) and (13), the integral $I_3(\beta_\ell,r)$ takes the form

$$I_3(\beta_\ell,r) = \int_0^r \mathrm{d}r' r'^\ell \mathrm{e}^{-\beta_\ell r'} G_\ell^{H(R)}(r,r') = \frac{-a^{2\ell+1}\mathrm{e}^{ikr}(1-\mathrm{e}^{-r/a})^{-\ell}}{f_{H\ell}(k)} \times\Big[{}_2F_1(A-2\ell-1,B-2\ell-1;1-2iak;\mathrm{e}^{-r/a}) \times\int_0^r \mathrm{d}r'(1-\mathrm{e}^{-r'/a})^{2\ell+1} \times \mathrm{e}^{-(\beta_\ell-ik)r'}{}_2F_1(A,B;C;1-\mathrm{e}^{-r'/a}) -(1-\mathrm{e}^{-r/a})^{2\ell+1}{}_2F_1(A,B;C;1-\mathrm{e}^{-r/a}) \times\int_0^r \mathrm{d}r'\mathrm{e}^{-(\beta_\ell-ik)r'}{}_2F_1(A-2\ell-1,B-2\ell-1; 1-2iak;\mathrm{e}^{-r'/a})\Big]. \quad (29)$$

Using the following analytic continuation [19,20]

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \times{}_2F_1(a,b;a+b-c+1;1-z)$$

$$+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\times{}_2F_1(c-a,c-b;c-a-b+1;1-z) \quad (30)$$

along with the standard relations

$$(1-s)^{(\beta_\ell+ik)a-1}=\sum_{n=0}^{\infty}\frac{\Gamma[n+1-(\beta_\ell+ik)a]}{\Gamma[1-(\beta_\ell+ik)a]}\frac{s^n}{n!} \quad (31)$$

and

$$f_\sigma(a,b;c;z)=\frac{1}{c-1}\Big[{}_2F_1(a,b;c;z)\times\int_0^z \mathrm{d}s\, s^{\sigma-1}(1-s)^{a+b-c}{}_2F_1(a-c+1,b-c+1;2-c;s)-z^{1-c}\,{}_2F_1(a-c+1,b-c+1;2-c;z)\times\int_0^z \mathrm{d}s\, s^{\sigma+c-2}(1-s)^{a+b-c}\,{}_2F_1(a,b;c;s)\Big] \quad (32)$$

in eq. (29), we get the expression for $D_{H\ell}(k)$ after using an approximation $r^\ell = a^\ell(1-\mathrm{e}^{-r/a})^\ell$ in eq. (9) as follows:

$$D_{H\ell}(k)=1-\lambda_\ell 2^{-2\ell}(\ell!)^{-2}a^{2\ell+2}\times\sum_{n=0}^{\infty}\frac{\Gamma[n+1-(\beta_\ell+ik)a]}{\Gamma[1-(\beta_\ell+ik)a]}\frac{1}{n!}\times\int_0^\infty \mathrm{d}r(1-\mathrm{e}^{-r/a})^{2\ell+1}\mathrm{e}^{-(\beta_\ell-ik)r}\times f_{n+1}(A,B;C;1-\mathrm{e}^{-r/a}). \quad (33)$$

Now, one can easily evaluate the integral present in eq. (33) by using the following standard integral relation [22]:

$$\int_0^1 \mathrm{d}z(1-z)^{\nu-1}z^{c-1}f_\sigma(a,b;c;pz)=\frac{\Gamma(\sigma+c-1)\Gamma(\nu)}{\Gamma(\sigma+c+\nu-1)}f_\sigma(a,b;c+\nu;p) \quad (34)$$

to get the compact expression for $D_{H\ell}(k)$ as

$$D_{H\ell}(k)=1-\lambda_\ell\, 2^{-2\ell}(\ell!)^{-2}a^{2\ell+3}\times\sum_{n=0}^{\infty}\frac{\Gamma[n+1-(\beta_\ell+ik)a]}{\Gamma[1-(\beta_\ell+ik)a]}\frac{1}{n!}\times\frac{\Gamma(n+2\ell+2)\Gamma((\beta_\ell-ik)a)}{\Gamma(n+2\ell+(\beta_\ell-ik)a+2)}\times f_{n+1}(A,B;(\beta_\ell-ik)a+2\ell+2;1). \quad (35)$$

3. The Coulomb limit

Now we are interested to check the limiting behaviours of our constructed expressions. For $a\to\infty$ and $V_0\to 0$, we have $aV_0 = 2k\eta$, where η is the Sommerfeld parameter.

In all partial wave analysis with the limit $a\to\infty$ we get $A\approx \ell+1+i\eta$, $B\approx\ell+1-i\eta-2iak$ and $C\approx -2iak$. Using the standard limiting conditions and relations [19,20,22,23],

$$\lim_{z\to\infty}\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=z^{\alpha-\beta}[1+O(z^{-1})];\ |\arg(z)|<\pi, \quad (36)$$

$$\lim_{a\to\infty}(2ak)^{i\eta+\ell}f_{H\ell}(-k)=f_{C\ell}(-k), \quad (37)$$

$${}_3F_2(a,b,c;e,f;1)=\frac{\Gamma(e)\Gamma(s)}{\Gamma(e-a)\Gamma(s+a)}\times{}_3F_2(a,f-b,f-c;f,s+a;1)\ \text{with } s=e+f-a-b-c \quad (38)$$

$$f_m(a,b;c;z)=\frac{(m-1)!\Gamma(m+c-1)\Gamma(a)\Gamma(b)}{\Gamma(m+a)\Gamma(m+b)\Gamma(c)}\times\sum_{n=0}^{\infty}\frac{\Gamma(m+n+a)\Gamma(m+n+b)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(m+n+c)}\times\frac{z^{n+m}}{(n+m)!}, \quad (39)$$

$$\sum_{r=0}^{n}\frac{(a)_r(b)_r}{r!(c)_r}=\frac{\Gamma(a+n+1)\Gamma(b+n+1)}{n!\Gamma(a+b+n+1)}\times{}_3F_2(a,b,c+n;c,a+b+n+1;1), \quad (40)$$

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (41)$$

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1(a,c-b;c;z/1-z), \quad (42)$$

$${}_2F_1(a,b;c;z)=(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z) \quad (43)$$

and the integral representations of Gaussian hypergeometric function

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\times(1-tz)^{-a}\mathrm{d}t, \quad (44)$$

we obtain

$$\lim_{a\to\infty}I_1(\beta_\ell,k)=\Gamma(2\ell+2)(\beta_\ell^2+k^2)^{-\ell-1}\left(\frac{\beta_\ell-ik}{\beta_\ell+ik}\right)^{i\eta}, \quad (45)$$

$$\lim_{a\to\infty}I_2(\beta_\ell,k)$$

$$= Re\left\{\frac{f_{C\ell}(-k)}{(2k)^{\ell}(\beta_{\ell}-ik)}e^{\frac{\pi\eta}{2}+\frac{i\ell\pi}{2}}\frac{\Gamma(2\ell+2)}{\Gamma(\ell+2+i\eta)}\right.$$
$$\left.\times{}_2F_1\left(1, i\eta-\ell; \ell+2+i\eta; \frac{\beta_{\ell}+ik}{\beta_{\ell}-ik}\right)\right\} \quad (46)$$

and

$$\lim_{a\to\infty} D_{H\ell}(k) = 1 - \frac{\lambda_{\ell}2^{-2\ell}(\ell!)^{-2}\Gamma(2\ell+2)}{(\ell+1+i\eta)(\beta_{\ell}-ik)}$$
$$\times\left[(\beta_{\ell}^2+k^2)^{-\ell-1}\left(\frac{\beta_{\ell}-ik}{\beta_{\ell}+ik}\right)^{i\eta}\right.$$
$$\times{}_2F_1\left(1, i\eta-\ell; \ell+2+i\eta; \frac{\beta_{\ell}+ik}{\beta_{\ell}+ik}\right)$$
$$-\frac{(2\beta_{\ell})^{-2\ell-1}}{(\beta_{\ell}-ik)}{}_2F_1\left(1, i\eta-\ell; \ell+2+i\eta;\right.$$
$$\left.\left.\left(\frac{\beta_{\ell}+ik}{\beta_{\ell}-ik}\right)^2\right)\right] = D_{C\ell}(k). \quad (47)$$

Van Haeringen [24] derived a number of limiting relations for the Hulthén–Coulomb pair of potentials for S-wave only. In eq. (37) we have generalised it for all partial waves. Our expression in eq. (47) is in exact agreement with Laha *et al* [2,25,26]. For $\ell = 0$, eqs (45)–(47) exactly coincide with those of Laha *et al* [27–29].

Table 1. Parameters for the (p–d) system.

States	β_{ℓ} (fm^{-1})	λ_{ℓ}(fm^{-2l})$^{-3}$	a (fm)
$^2S_{1/2}$	3.97	−162.795	10
$^2P_{1/2}$	1.05	−43.000	35
$^2P_{3/2}$	1.05	−56.000	35
$^2D_{3/2}$	1.0336	−68.688	50

4. Results and discussion

To compute the scattering phase shifts for the proton–deuteron (p–d) system, we have applied the modified phase function method. Scattering phase shifts are obtained from eq. (25) in conjunction with eqs (8), (14), (27), (28) and (35). For this purpose, we have used $\hbar^2/2\mu = 31.1025$ MeV fm^2 and $V_0a = 0.04629$ fm^{-1}.

With the parameters given in table 1, we have calculated the phase parameters for the corresponding states of (p–d) system that are plotted in figure 1. The parameters for $^2S_{1/2}$ state fit the correct binding energy ($E_B = 7.718$ MeV) of the (p–d) system. The other states of the (p–d) system are unbound. Therefore, we have given free running to our parameters in numerical routine to fit the correct phase shifts. We have computed the scattering phase shifts for the (p–d) elastic scattering by the modified phase function method for electromagnetic

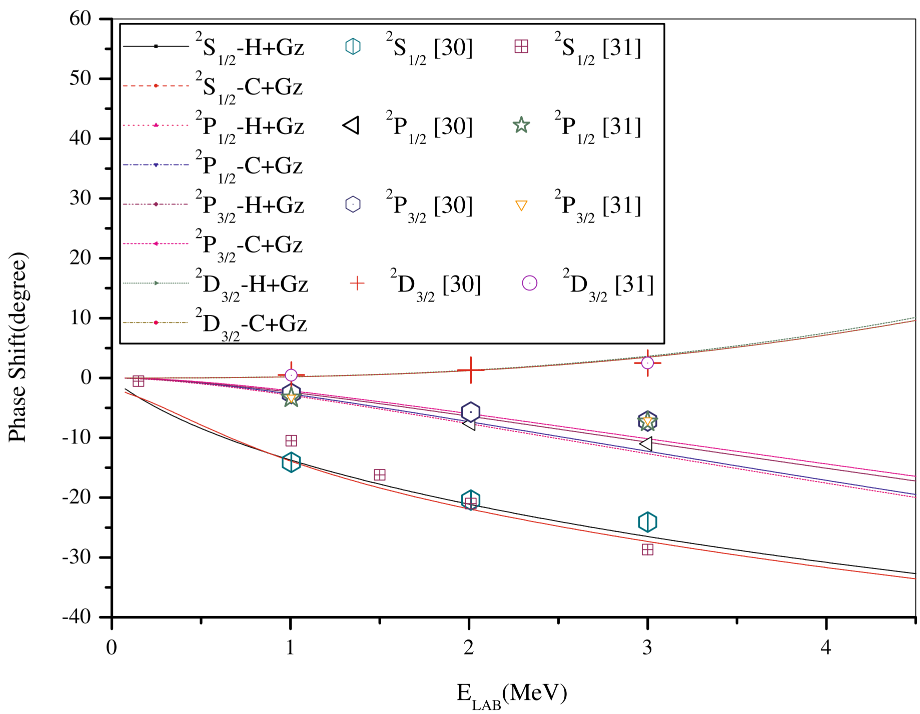

Figure 1. Proton–deuteron scattering phase shift as a function of E_{LAB}

distorted Graz separable potential. Hereafter, the phase parameters are denoted by (H+Gz) and (C+Gz) for Hulthén-distorted and Coulomb-distorted Graz potentials respectively. The Graz separable potential has the ability to fit the nucleon–nucleon scattering data quite accurately up to partial wave $\ell = 2$. We have portrayed the phase parameters along with the previous results [30,31] in figure 1. Huttel *et al* [30] analysed the (p–d) elastic scattering based on the measurements of differential scattering cross-sections and (p–d) analysing powers below the breakup threshold (3.3 MeV) which compares well with the Faddeev calculations for the S-wave NN potentials involving the Coulomb interaction. This phase shift analysis in the energy range 0–3 MeV appears to be reasonable for S- and P-phases including the channel-spin mixing and SD tensor coupling. Ishikawa [31] studied the (p–d) elastic scattering at energies below the three-body breakup threshold with a modified Faddeev formalism with rigorous inclusion of the Coulomb interaction. The modification includes some auxiliary realistic NN potential models without or with three-nucleon forces to cancel the long-range nature of the Coulomb potential. The phase-shift parameters with these potential models reproduce quite accurate values for the (p–d) elastic scattering. Looking at figure 1, it can be seen that our calculated phase shifts are in reasonable agreement with those of refs [30] and [31] except at very low energies for $^2S_{1/2}$ channel. At 3 MeV energy, our phase shift values (H+Gz) for Hulthén potential model (solid line) differ by 3% from the values of both Huttel *et al* [30] and Ishikawa [31] while for Coulomb-distorted Graz separable potential (C+Gz) (dashed line) these variations are about 4% and 2% from those of refs [30,31] respectively. On the other hand, the $^2P_{1/2}$ phase parameters are in good agreement up to 2 MeV but show differences of about 2% and 5% in their numerical values with the results of Huttel *et al* [30] and Ishikawa [31] at 3 MeV. The same nature is noticed for $^2P_{3/2}$ state results with the exception that they discern by 3% only from that of Ishikawa [31]. For the partial wave $\ell = 2$ ($^2D_{3/2}$ state) both the phase parameters match well with the reliable data [30,31] over the entire energy range under consideration. Although our phase values numerically show slight differences in different energy regions, they reproduce correct trends of the experimental phase shifts. Our phase parameters (H+Gz) and (C+Gz) also reproduce sensible quantitative agreement with the works of other groups [32–37]. These works give almost the same phase parameters although their approaches to the problem are different. Thus, one can easily rely on these data.

The discrepancy in $^2S_{1/2}$ phase shifts below 0.5 MeV is undoubtedly related to the fact that the potential employed in the calculation does not reproduce the experimental electromagnetic effect. It is noticed that in the low energy range, within 1 MeV, both the Coulomb- and Hulthén-distorted interactions reproduce almost the same results while those for large values of energies differ slightly due to relatively short-ranged electromagnetic character of the Hulthén potential compared to the Coulomb one. Thus, the effect of screening plays a significant role at higher sides of energy. Also the differences between the phase parameters of the potentials under consideration diminish as ℓ increases.

5. Conclusions

In the present study, we have derived a closed form expression for Hulthén-distorted separable potential by absorbing the Hulthén potential in the comparison functions of the phase method. Our expression for $\tan \delta_{HG}(k)$ reproduces the correct Coulomb limit and enables us to make a comparison between the Hulthén-distorted and the Coulomb-distorted phase shifts for the system under consideration. It is well known that for large values of screening parameter a, the Hulthén potentials goes over to the Coulomb potential. We have used different screening parameters for 'different' partial wave states to fit correct binding energies and phase shifts. From our figure, it is well established that the Hulthén potential gives correct Coulomb behaviour for large a values. Experimental results for n–d and p–d elastic scattering also have the potential to provide new information regarding the nature of the primary interactions in complex nuclear systems. Therefore, it is expected that nucleon–deuteron measurements will eventually play a significant role in realising the nature and role of nuclear three-body forces. The weak proton capture on protons is the most fundamental process in stellar energy synthesis. This reaction first converts hydrogen into helium in main sequence stars like the Sun. After this initial step, the processes like nuclear reaction (fusion), radiative capture of proton by deuteron ($p + d \rightarrow \mathrm{He}^3 + \gamma$) begins to start. This nuclear reaction is important for understanding the theory of Big Bang Nucleosynthesis. The (p–d) reaction strongly affects the primordial deuterium abundance as the deuterons are mainly destroyed through such processes. The relatively small differences observed in the phase parameters are a common feature of all the fits we have done. The observed differences in the phase parameters do not indicate any serious deficiency in the calculations. On the contrary, the overall agreement between our results and the previous calculations [30,31] is remarkable.

References

[1] F Calogero, *Variable phase approach to potential scattering* (Academic Press, New York, 1967)
[2] G C Sett, U Laha and B Talukdar, *J. Phys. A: Math. Gen.* **21**, 3643 (1988)
[3] U Laha, N Haque, T K Nandi and G C Sett, *Z. Phys. A: Atom. Nuclei* **332**, 305 (1989)
[4] U Laha, A K Jana and T K Nandi, *Pramana – J. Phys.* **37**, 387 (1991)
[5] L Hulthén, *Ark. Mat. Astron. Fys. B* **29**, 1 (1942)
[6] C S Jia, J Y Liu and P Q Wang, *Phys. Lett. A* **372**, 4779 (2008)
[7] W C Qiang, W L Chen, K Li and H P Zhang, *Int. J. Mod. Phys. A* **24**, 5523 (2009)
[8] W G Fang, C W Li, W H Ying and L Y Yuan, *Chin. Phys. B* **18**, 3663 (2009)
[9] J Bhoi, A K Behera and U Laha, *J. Math. Phys.* **60**, 083502 (2019)
[10] R L Greene and C Aldrich, *Phys. Rev. A* **14**, 2363 (1976)
[11] B Khirali, A K Behera, J Bhoi and U Laha, *Ann. Phys.* **412**, 168004 (2020)
[12] A K Behera, P Sahoo, B Khirali and U Laha, *Pramana – J. Phys.* **95**, 100 (2021)
[13] P Sarkar, B Khirali, U Laha and P Sahoo, *Int. J. Mod. Phys. E* **30**, 2150066 (2021)
[14] P Sahoo, U Laha, B Khirali and A K Behera, *Rep. Math. Phys.* **88**, 295 (2021)
[15] L Crepinsek, C B Lang, H Oberhummer, W Plessas and H F K Zingl, *Acta Phys. Austriaca* **42**, 139 (1975)
[16] W Schweiger, W Plessas, L P Kok and H van Haeringen, *Phys. Rev. C* **27**, 515 (1983)
[17] J Haidenbauer and W Plessas, *Phys. Rev. C* **30**, 1822 (1984)
[18] R G Newton, *Scattering theory of waves and particles*, 2nd Edn (McGraw-Hill, New York, 1982)
[19] A Erdeyli, *Higher transcendental functions* (McGraw-Hill, New York, 1953) Vol. 1
[20] L J Slater, *Generalized hypergeometric functions* (Cambridge University Press, London, 1966)
[21] I S Gradshteyn and I M Ryzhik, *Tables of integrals, series and products* (Academic Press, London, 2000)
[22] A W Babister, *Transcendental functions satisfying nonhomogeneous linear differential equations* (MacMillan, New York, 1967)
[23] W N Bailey, *Generalized hypergeometric series* (Cambridge University Press, London, 1935)
[24] H van Haeringen, *Phys. Rev. A* **18**, 56 (1978)
[25] U Laha, B J Roy and B Talukdar, *J. Phys. A: Math. Gen.* **22**, 3597 (1989)
[26] U Laha and B Talukdar, *Pramana – J. Phys.* **36**, 289 (1991)
[27] U Laha, *Phys. Rev. A* **74**, 012710 (2006)
[28] U Laha, *Pramana – J. Phys.* **72**, 457 (2009)
[29] U Laha and J Bhoi, *Few-Body Syst.* **54**, 1973 (2013)
[30] E Huttel, W Arnold, H Baumgart, H Berg and G Clausnitzer, *Nucl. Phys. A* **406**, 443 (1983)
[31] S Ishikawa, *Few-Body Syst.* **32**, 229 (2003)
[32] P A Schmelzbach, W Grüebler, R E White, V König, R Risler and P Marmier, *Nucl. Phys. A* **197**, 273 (1972)
[33] J Arvieux, *Nucl. Phys. A* **221**, 253 (1974)
[34] C R Chen, G L Payne, J L Friar and B F Gibson, *Phys. Rev. C* **39**, 1261 (1989)
[35] L D Knutson, L O Lamm and J E McAninch, *Phys. Rev. Lett.* **71**, 3762 (1993)
[36] A Kievsky, S Rosati, W Tornow and M Viviani, *Nucl. Phys. A* **607**, 402 (1996)
[37] S König and H W Hammer, *Phys. Rev. C* **83**, 064001 (2011)