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Estimation of the size of the solar system and its spatial dynamics using Sundman inequality

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Abstract. The essential development of a new mathematical ansatz for estimating the size of the solar system with help of Sundman inequality via Lagrange–Jacobi relation has been implemented in this research. Mean size R of the solar system has been estimated using Sundman inequality at the expanding scenario of the solar system (with existing planets' configuration) for the obtained partial class of solutions for this type of differential inequality by assuming constant total angular momentum for the entire dynamical configuration. The obvious physically reasonable assumption is that the solar system will increase its size during evolution in the future (due to losses of the total angular momentum via tidal interactions), albeit we can assume the compressing scenario.

Keywords. Moment of inertia of the solar system; Sundman inequality; Lagrange–Jacobi relation; tidal interactions.

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1. Introduction to the problem

Sundman inequality [1] plays a significant role in describing the prescribed dynamics of various configurations of celestial bodies (including dynamics of natural satellites around the planets [2]). It is worth noting that only a few cases of analytical application are known in the history of celestial mechanics, including the trivial case of zero velocity surfaces in the three-body problem [1] as well as the elegant Sheeres's ansatz for minimum energy configurations in the N-bodies problem and the celestial mechanics of granular systems [3].

The problem of stability of motion of large configurations of celestial bodies (here, planets) under the mutual influence of their gravity fields has been studied in the past under various versions. In the current investigation, we shall present a new mathematical technique for differential solving of Sundman inequality for the solar system. So, the actual mean size R of the solar system could be theoretically estimated by analysing Sundman inequality under the additional assumption of constant total angular momentum for the solar system. As for a sufficient introduction to the problem, we recommend [4], where a significant historical investigation has been made properly.

From the very beginning, we should note that the approach, suggested in the current research, quite differs from that was used previously in the estimation of solar system size in [5] for the case of 'zero' total angular momentum of the solar system. The solar system mean size, R, has been estimated here at the expanding scenario of the solar system (with existing planet configuration) for the obtained partial class of solutions for this differential inequality by assuming 'non-zero' total angular momentum for the entire dynamical configuration. Indeed, the approach suggested in the current investigation assumes the approximated solution (with semi-analytical findings) considering the case of constant total angular momentum of the solar system, whereas in [5] a simple case of zero total angular momentum of the solar system has been proposed and investigated accordingly. Both procedures quite differ from each other in the sense of ways of integrating the differential inequalities with appropriate restrictions for this or that case.

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In the remaining part of this research, we shall generalise the theoretical basis of the problem presented in [5] onto the case of the constant total angular momentum of the solar system with only the resulting basic formulae presented in the main text. As in [5], we consider the dynamical configuration of the solar system, consisting of the Sun and 8 planets (M and m_i where $i = \{1, \ldots, 8\}$, $(m_i/M) \ll 1$). According to [5], the dynamics of the solar system could be considered in the Cartesian coordinate system. All the following computations are referred to the inertial reference system located at the centre of mass of the solar system whereas its moment of inertia should be defined as

$$I = \left(\sum m_i \cdot (\vec{r}_i)^2\right) + M \cdot (\vec{r}_M)^2$$

which is associated with R, the mean size of the solar system where

$$R \cong \sqrt{\frac{I}{(\sum m_i) + M}}. \tag{*}$$

It is known that the centre of the Sun is quasi-circulating around the aforementioned centre of mass [6]. It is also worth noting that Sundman inequality is valid for the dynamics of the solar system [4]:

$$I \cdot K - J^2 \geq |\vec{C}|^2 \tag{1}$$

with denotations given in [5] (clarification regarding various types of denotations in the formulation of Sundman inequality is reported in Appendix A). As in [5], K is the kinetic energy of the solar system, J is the scalar product of the radius vector of each object in the solar system onto its velocity and $\vec{C}$ denotes the total angular momentum of the solar system.

According to [5], the Lagrange–Jacobi relation stems from the equations of motions of the solar system (for the Newtonian dynamics of gravitational fields):

$$\dot{J} = \frac{\ddot{I}}{2} = K - U, \tag{2}$$

where the total energy is assumed to be invariant of the system. Eliminating K by using the Lagrange–Jacobi relation (2), one can transform Sundman inequality (1) into the differential inequality

$$I \cdot \ddot{I} - 2I \cdot H - \frac{1}{4}(\dot{I})^2 - |\vec{C}|^2 \geq 0, \tag{3}$$

where the total energy $H =$ const. if we consider the solar system as a closed dynamical configuration in the absence of interchange by the total energy with boundaries of this system along with zero level dissipation of energy inside (ejected in the material of planets and gravitational waves or radiation leaving the solar system, where thermal radiation is the primary means of energy transfer).

2. Solving procedure for inequality (3) with constant total angular momentum

We should calculate the regions within the solar system where inequality (3) is absolutely valid taking into account also the additional assumption of constant total angular momentum for the solar system.

Let us transform (3) by the special change of variables which means the change of variables in differential inequality:

$$I \cdot \ddot{I} - 2I \cdot H - \frac{1}{4}(\dot{I})^2 - |\vec{C}|^2 \geq 0$$

$$\left\{ \left(\frac{\mathrm{d}I}{\mathrm{d}t}\right) = p(I) \rightarrow \left(\frac{\mathrm{d}^2 I}{\mathrm{d}t^2}\right) = \left(\frac{\mathrm{d}p}{\mathrm{d}I}\right) \cdot p \right\}$$

(as for appropriate clarifications, see Appendix A). All hard algebraical manipulations have been moved to the Appendix, with only formulae (A.4) and (A.5) presented in the main text (which correspond to (4) and (5) respectively).

$$\sqrt{I + \left(\frac{C_1}{8H}\right) \cdot \sqrt{I} - \frac{|\vec{C}|^2}{2H}} - \left(\frac{C_1}{16H}\right) \cdot \ln\left|2\sqrt{I} + \left(\frac{C_1}{8H}\right) + 2\sqrt{I + \left(\frac{C_1}{8H}\right) \cdot \sqrt{I} - \frac{|\vec{C}|^2}{2H}}\right| - C_2 \geq \pm\sqrt{2H} \cdot t. \tag{4}$$

(some of the physical values in (4) are given in Appendix B), $\{|\vec{C}|^2, \ C_1\} \neq 0$:

$$\sqrt{I} \in \left[\left(\frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}} - \left(\frac{C_1}{16H}\right)\right), +\infty\right), \quad \forall C_1. \tag{5}$$

Obviously, the physically reasonable assumption is that the solar system will increase its size during evolution in the future (due to losses of the angular momentum via tidal interactions). By using transformations (A.1)–(A.3) in Appendix, the end result (4) leaves open both possible solutions: compressing or expanding the solar system. When the size of the solar system decreases in future, we should choose the sign "−" in inequality (4).

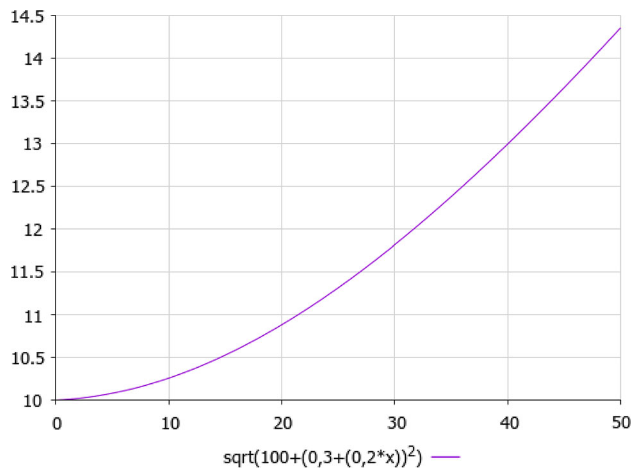

Figure 1. Plot of the lower bound for $R(t)$, the size of the solar system (6), corresponding to the case $R \geq \frac{1}{\sqrt{(\sum m_i) + M}} \sqrt{\frac{|\vec{C}|^2}{2H} + (C_2 + \sqrt{2H} \cdot t)^2}$.

If we consider $C_1 = 0$ as well, inequality (4) can be simplified as follows, using expression (*) for R, the size of the solar system:

$$\sqrt{I - \frac{|\vec{C}|^2}{2H}} \geq (C_2 \pm \sqrt{2H} \cdot t)$$

$$\Rightarrow I \geq \frac{|\vec{C}|^2}{2H} + (C_2 \pm \sqrt{2H} \cdot t)^2$$

$$R \geq \frac{\sqrt{\frac{|\vec{C}|^2}{2H} + (C_2 \pm \sqrt{2H} \cdot t)^2}}{\sqrt{(\sum m_i) + M}}. \tag{6}$$

Figures 1 and 2 show the lower bound of the estimation for R, which corresponds to the aforementioned inequality (6) (we designate $x = t$ just for presenting the plot of the solution).

We should especially clarify in advance that the interval chosen for t for figures 1, 2 (and figure 3) is scale-compatible by assuming that C_2, $|\vec{C}|^2$ and H are constants given in Appendix B.

Furthermore, taking into account expression (*) for R, we obtain from inequality (4) in the general case for $C_1 \neq 0$ (here we have taken into account that $(m_i/M) \ll 1$):

$$\sqrt{R^2 + \left(\frac{C_1}{8H\sqrt{M}}\right)R - \frac{|\vec{C}|^2}{2HM}} - \left(\frac{C_1}{16H\sqrt{M}}\right) \cdot \ln\left(\sqrt{M} \cdot \left|2R + \left(\frac{C_1}{8H\sqrt{M}}\right)\right.\right.$$

$$\left.\left. +2\sqrt{R^2 + \left(\frac{C_1}{8H\sqrt{M}}\right)R - \frac{|\vec{C}|^2}{2HM}}\right|\right) - \frac{C_2}{\sqrt{M}} \geq \pm\sqrt{\frac{2H}{M}}t. \tag{7}$$

(The approximation of inequality (7) is considered in Appendix C).

The last but important note: though the aforementioned increase in the size of the solar system during the evolution in the future is the most expected scenario, in the case of the compressing scenario, the solar system's size should allow exceeding the level of minimal distances [5] for all the planets and Sun in our analysis

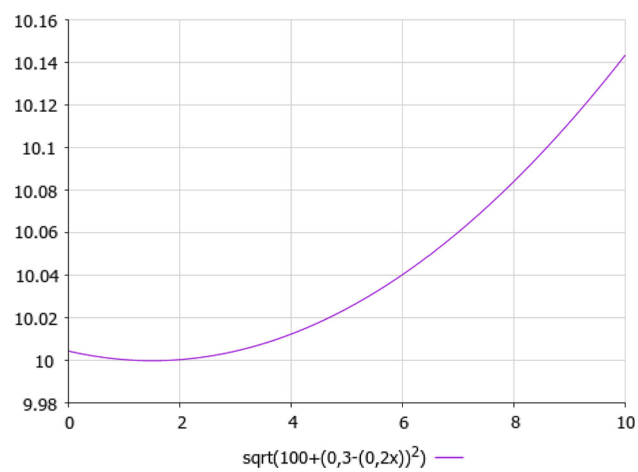

Figure 2. Plot of the lower bound for $R(t)$, the size of the solar system (6), corresponding to the case $R \geq \frac{1}{\sqrt{(\sum m_i) + M}} \sqrt{\frac{|\vec{C}|^2}{2H} + (C_2 - \sqrt{2H} \cdot t)^2}$.

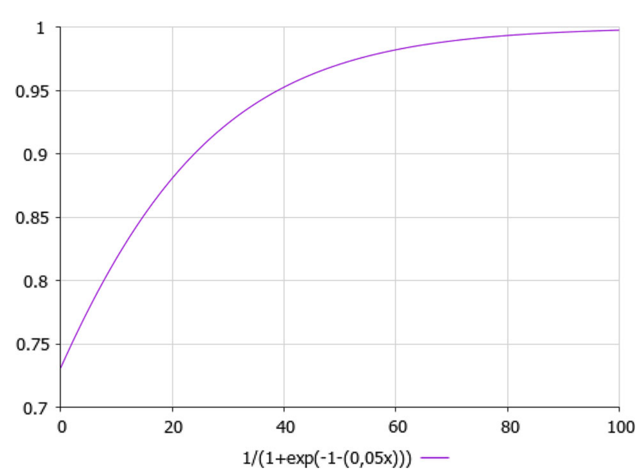

Figure 3. Plot of the lower bound of the estimation for the solar system's size $R(t)$ (C.4) (depending on time t).

(to avoid situations of collisions of planets in the solar system).

3. Discussion and conclusion

The paper presents an application to the solar system of the so-called Sundman inequality, one of the very few exact results available concerning the N-body problem. As reported in [7], the total angular momentum of the solar system is preferably conserved (we should note that tidal interactions in the solar system allow the transfer of an infinitesimal amount of angular momentum from axial rotations to orbital revolutions due to tidal friction, and *vice-versa*).

The development of a new mathematical ansatz for estimating the size of the solar system with the help of Sundman inequality via Lagrange–Jacobi relation has been implemented in this research. The solar system mean size R (*) has been estimated for the expanding scenario of the solar system by assuming constant total angular momentum for the entire dynamical configuration. As the main findings, we have found among a new class of partial solutions for the Sundman inequality that the solar system should be eventually expanding according to the approximately linear dependence of size of the solar system $R(t)$ (6) on time t. The obvious physically reasonable assumption is that the solar system will be increasing its size during its evolution in future (due to losses of the total angular momentum), even though we can assume the compressing scenario.

We can clearly conclude that we can use the results of this investigation to determine the extent of the size of the Oort cloud. The basic purpose of the work is that the actual mean size R of the solar system could be theoretically estimated via the analysis of the Sundman inequality (4), (5) and (7). Obviously, we should take into account our forthcoming hunting for the ninth planet in the solar system [8] that the orbit of the ninth planet is to be located within the aforementioned estimation (7) for the mean size of the solar system (where the mass of the ninth planet $m_9(\ll \sum m_i) + M$).

In addition, it is worth discussing that now, while planetary migration in the protoplanetary disk is a well understood dynamical phenomenon, it is also true that the existence of life shows that (for example) the semi-major axis of the Earth's orbit is not significantly changed in the last 3.5 billion years. So we should clarify that the terms "expansion"/"contraction" (or compressing scenario with respect to the solar system) mainly refers to four outer giant planets which are known to be the most massive planets in the solar system. As for the possible physical mechanism explaining the expansion or the contraction of the solar system itself due to losses of the total angular momentum (taking into account the tidal interactions), such a natural algorithm appears to work similarly to the phenomenon of the Moon which is known to be orbiting or moving in its mean motion spirally outwards the Earth during centuries with respect to the mutual centre of mass of 'Earth–Moon' system. Indeed, when Laplace (and, also, Lagrange) formulated his conclusion regarding the stability of the solar system, he had not taken into account the losses of the total angular momentum (they supposed the conservation of angular momentum). In addition, Laplace (together with Lagrange) did not take into consideration the resonance spin–spin coupling interaction phenomena, assuming low eccentricity of the planet orbits (while Venus is self-rotating in the opposite direction with respect to other planets, Uranus is tilted sideways, nearly into the plane of its solar orbit. Its north and south poles, therefore, lie where most other planets have their equators).

There are some remarkable articles which discuss the problem under consideration [9–26]. Among them, modern criticism (with respect to the Laplace formalism regarding the stability of the solar system) was made in [25]. Transferring an infinitesimal amount of angular momentum from axial rotations to orbital revolutions due to tidal friction, and *vice-versa* has been used in [26].

Appendix A (the heavy algebraical manipulations from §2)

First, we should clarify (as mentioned in §1) the various types of denotations in the formulation of Sundman inequality. That is, if one-half of the moment of inertia is used (instead of full expression (*)), the classical formulation of the kinetic energy of the system of celestial bodies will be required (as the sum of the quantities for all of them, which is assumed to be proportional to the mass of the body multiplied by one-half of the square of its velocity). If the full moment of inertia is used (as in expression (*)), the formulation of kinetic energy of the entire dynamical configuration in a form presented in (1) should be used for another formulation of Sundman inequality, as it was used in (1)–(3) in §1.

Secondly, let us proceed here to explain our solving procedure. Let us transform inequality (3) by special change in variables

$$\Rightarrow \quad I \cdot \frac{1}{2}\left(\frac{\mathrm{d}\left(p^2\right)}{\mathrm{d}I}\right) - 2I \cdot H \; - \; \frac{1}{4}p^2 - |\vec{C}|^2 \; \geq 0$$

$$\left\{p^2 = Y(I)\right\}$$

$$\Rightarrow \quad \frac{\mathrm{d}Y(I)}{\mathrm{d}I} - \frac{1}{2}\left(\frac{Y(I)}{I}\right) - 4 \cdot H - \frac{2|\vec{C}|^2}{I} \geq 0, \tag{A.1}$$

where $H = \text{const.}$ should be determined or estimated additionally. Let us consider first the case of equality in (A.1) ($C_1 = \text{const}$):

$$\frac{\mathrm{d}Y(I)}{\mathrm{d}I} - \frac{1}{2}\left(\frac{Y(I)}{I}\right) - 4 \cdot H - \frac{2|\vec{C}|^2}{I} = 0 \tag{A.2}$$

$$Y(I) = \mathrm{e}^{-\int -\frac{1}{2}\left(\frac{\mathrm{d}I}{I}\right)} \cdot \left[\int\left(\left(4 \cdot H + \frac{2|\vec{C}|^2}{I}\right) \cdot \mathrm{e}^{\int -\frac{1}{2}\left(\frac{\mathrm{d}I}{I}\right)}\right)\mathrm{d}I + C_1\right]$$
$$= 8H \cdot I + C_1 \cdot \sqrt{I} - 4|\vec{C}|^2.$$

Equation (A.2) yields

$$Y(I) \geq 8H \cdot I + C_1 \cdot \sqrt{I} - 4|\vec{C}|^2 \tag{A.3}$$

which should be used in (A.1) to transform it to (A.3) again (besides, let us note that the process of integration for the inequality below should be by definition the proper summation of the chosen set of inequalities). These transformations should prove the validity of inequality (A.3)

$$\begin{cases} (\mathrm{A.1}) \Rightarrow \frac{\mathrm{d}Y(I)}{\mathrm{d}I} - 4 \cdot H - \frac{2|\vec{C}|^2}{I} \geq \frac{1}{2}\left(\frac{Y(I)}{I}\right), \\ (\mathrm{A.3}) \Rightarrow \frac{1}{2}\left(\frac{Y(I)}{I}\right) \geq 4H + C_1 \cdot \frac{1}{2\sqrt{I}} - \frac{2|\vec{C}|^2}{I}, \end{cases} \Rightarrow$$

$$\frac{\mathrm{d}Y(I)}{\mathrm{d}I} - 4 \cdot H - \frac{2|\vec{C}|^2}{I} \geq \frac{1}{2}\left(\frac{Y(I)}{I}\right) \geq 4H + C_1 \cdot \frac{1}{2\sqrt{I}} - \frac{2|\vec{C}|^2}{I} \Rightarrow$$

$$\int \mathrm{d}Y \geq \int\left(8H + C_1 \cdot \frac{1}{2\sqrt{I}}\right)\mathrm{d}I \Rightarrow Y(I) \geq 8H \cdot I + C_1 \cdot \sqrt{I} + C_0 \quad \Leftrightarrow$$

$$(\mathrm{A.3})\{C_0 = -4|\vec{C}|^2\}.$$

Thus, we obtain further from the differential inequality (A.3) as below ($C_2 = \text{const}$):

$$p^2 \geq 8H \cdot I + C_1 \cdot \sqrt{I} - 4\,|\vec{C}|^2 \Rightarrow \frac{\mathrm{d}I}{\mathrm{d}t} \geq \pm\sqrt{8H \cdot I + C_1 \cdot \sqrt{I} - 4|\vec{C}|^2}$$

$$\{(8H \cdot I + C_1 \cdot \sqrt{I} - 4|\vec{C}|^2) \geq 0\}$$

$$\Rightarrow \int \frac{\mathrm{d}I}{\sqrt{8H \cdot I + C_1 \cdot \sqrt{I} - 4|\vec{C}|^2}} \geq \pm \int \mathrm{d}t$$

$$\{\sqrt{I} = u\}$$

$$\Rightarrow \int \frac{u\mathrm{d}u}{\sqrt{u^2 + \left(\frac{C_1}{8H}\right) \cdot u - \frac{|\vec{C}|^2}{2H}}} \geq \pm\sqrt{2H} \cdot \int \mathrm{d}t \quad \Rightarrow$$

$$\int \frac{u\mathrm{d}u}{\sqrt{u^2 + \left(\frac{C_1}{8H}\right) \cdot u - \frac{|\vec{C}|^2}{2H}}} = \sqrt{u^2 + \left(\frac{C_1}{8H}\right) \cdot u - \frac{|\vec{C}|^2}{2H}} - \left(\frac{C_1}{16H}\right) \cdot \int \frac{\mathrm{d}u}{\sqrt{u^2 + \left(\frac{C_1}{8H}\right) \cdot u - \frac{|\vec{C}|^2}{2H}}}$$

$$\left\{\int \frac{\mathrm{d}u}{\sqrt{u^2 + B \cdot u + D}} = \ln\left|2u + B + 2\sqrt{u^2 + B \cdot u + D}\right|,\ B^2 \neq 4D\right\}$$

$$\sqrt{I + \left(\frac{C_1}{8H}\right) \cdot \sqrt{I} - \frac{|\vec{C}|^2}{2H}} - \left(\frac{C_1}{16H}\right) \cdot \ln\left|2\sqrt{I} + \left(\frac{C_1}{8H}\right) + 2\sqrt{I + \left(\frac{C_1}{8H}\right) \cdot \sqrt{I} - \frac{|\vec{C}|^2}{2H}}\right| - C_2 \geq \pm\sqrt{2H} \cdot t \tag{A.4}$$

Analysing the expression

$$\left(I + \left(\frac{C_1}{8H}\right) \cdot \sqrt{I} - \frac{|\vec{C}|^2}{2H}\right) \geq 0$$

in (A.4), we obtain in case $\{|\vec{C}|^2, \ C_1\} \neq 0$ as below (some of the physical values in (A.4) are given in Appendix B):

$$\left(I + \left(\frac{C_1}{8H}\right) \cdot \sqrt{I} - \frac{|\vec{C}|^2}{2H}\right) \geq 0$$
$$\left\{\sqrt{I}_{1,2} = -\left(\frac{C_1}{16H}\right) \pm \frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}}\right\}$$
$$\Rightarrow \left(\sqrt{I} + \left(\frac{C_1}{16H}\right) + \frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}}\right) \cdot \left(\sqrt{I} + \left(\frac{C_1}{16H}\right) - \frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}}\right) \geq 0 \quad \Rightarrow$$

Appendix B (useful data for some physical values in inequality (4))

Let us write the appropriate data for some physical values in inequalities (4), (7) and in expression (*):

(1) $(\sum m_i) + M = 1.014\ M \cong 2.02 \cdot 10^{30}$kg.
(2) We can conclude that the potential energy of the Sun [11] plays a crucial role in estimating the magnitude of the total energy H of the solar system.
(3) As for the absolute magnitude of the total angular momentum of the solar system [12], we can estimate it as $|\vec{C}| \cong 3.32 \cdot 10^{45}$ kg · m²/s.
(4) Taking into account the current age of the solar system ($t \sim 4.6$ billion years), let us assume that by comparing the right part of inequality (4) to the constant C_2 (as well as comparing constant C_1 in combination with respect to other terms in the left part

$$\sqrt{I + \left(\frac{C_1}{8H}\right) \cdot \sqrt{I} - \frac{|\vec{C}|^2}{2H})},$$

the aforesaid constants can be chosen as follows:

$$C_2 \sim \{1, \ldots, 9\} \cdot 10^{37}\ \mathrm{kg}^{\frac{1}{2}} \cdot \mathrm{m},$$

as for its mutual commensurability with other terms at both parts of inequality (4), and

$$\left(\frac{|C_1|}{16H}\right) \sim \sqrt{I} \sim \frac{|\vec{C}|}{\sqrt{2H}} \quad \Rightarrow \quad |C_1| \sim 8\sqrt{2H} \cdot |\vec{C}|$$
$$\cong 8\sqrt{2 \cdot 2.275918 \cdot 10^{41} \left[\mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}^2}\right]} \cdot 3.3212 \cdot 10^{45}\ \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}$$

$$A)\begin{cases} \sqrt{I} + \left(\frac{C_1}{16H}\right) + \frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}} \geq 0 \\ \sqrt{I} + \left(\frac{C_1}{16H}\right) - \frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}} \geq 0 \end{cases} \quad B)\begin{cases} \sqrt{I} + \left(\frac{C_1}{16H}\right) + \frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}} \leq 0 \\ \sqrt{I} + \left(\frac{C_1}{16H}\right) - \frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{\|\vec{C}|^2}{H}} \leq 0 \end{cases}$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\underbrace{\sqrt{I} \geq \frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}} - \left(\frac{C_1}{16H}\right), \qquad \sqrt{I} \leq -\left(\frac{C_1}{16H}\right) - \frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}}}$$
$$\Downarrow$$
$$\sqrt{I} \in \left[\left(\frac{1}{2}\sqrt{\left(\frac{C_1}{8H}\right)^2 + 2\frac{|\vec{C}|^2}{H}} - \left(\frac{C_1}{16H}\right)\right), +\infty\right), \quad \forall\, C_1. \tag{A.5}$$

$$\Rightarrow \quad |C_1| \sim 8\sqrt{2\cdot 22.75918\cdot 10^{40}} \cdot 3.3212\cdot 10^{45} \cong 1.7926\cdot 10^{67}\text{kg}^{\frac{3}{2}}\cdot \text{m}^3/\text{s}^2.$$

Regarding the estimation of the total angular momentum of the solar system, it is quite obvious that the value reported in [12] (which includes calculating the total sum of orbital angular momenta along with angular momenta due to self-rotation for all the planets, as it was done earlier in [13]) is significantly larger than each value due to the self-rotation for Jupiter/Saturn/Uranus/Neptune (not less than 7 orders). See the cases of Jupiter and Saturn:

(1) Jupiter, value from $|\vec{C}_J| = 6.9\cdot 10^{38}$ kg · m^2/s or $4.14\cdot 10^{38}$ kg · m^2/s in [14] to value $|\vec{C}_J| \cong (1.9\cdot 10^{27}$ kg) $\times$ $(2.3\cdot 10^{11}$ m^2/s^2) $\cong 4\cdot 10^{38}$ kg · m^2/s in [15] (see formula (70) there) or value $|\vec{C}_J| \cong 1.3\cdot 10^{40}$ kg · m^2/s in [16] (see formula (35) there);
(2) Saturn, $|\vec{C}_S| \cong (5.68\cdot 10^{26}$kg) $\times$ $(1.3\cdot 10^{11}$m^2/s^2) $\cong 0.74\cdot 10^{38}$ kg · m^2/s in [15] (see formula (69) there) or value $|\vec{C}_S| \cong 3.6\cdot 10^{39}$ kg · m^2/s in [16] (see formula (36) there).

Meanwhile, in [17] the absolute magnitude of the total angular momentum of the solar system was estimated as $|\vec{C}| \cong 3\cdot 10^{43}$ kg · m^2/s (in [15], $|\vec{C}\,| \cong 180 \times (1.99\cdot 10^{30}$ kg) $\times$ $(8.6\cdot 10^{30}$ m^2/s^2) $\cong 3.08\cdot 10^{43}$ kg · m^2/s except the angular momentum of Sun itself, see formulae (71) and (77) there, which is not less than 100 times or up to 2 orders less than the total angular momentum of the solar system, reported in [12].

In fact, it is stated in [12] that the number given above also considers Oort Cloud and Kuiper Belt Objects.

Appendix C (approximation of inequality (7))

Let us assume that the constant C_2 is given by the proper expression according to the definition below (R_0 = const.):

$$\frac{C_2}{\sqrt{M}} = \sqrt{R_0^2 + \left(\frac{C_1}{8H\sqrt{M}}\right)R_0 - \frac{|\vec{C}|^2}{2HM}} - \left(\frac{C_1}{16H\sqrt{M}}\right)\cdot \ln\left(\sqrt{M}\left|2R_0 + \left(\frac{C_1}{8H\sqrt{M}}\right) + 2\sqrt{R_0^2 + \left(\frac{C_1}{8H\sqrt{M}}\right)R_0 - \frac{|\vec{C}|^2}{2HM}}\right|\right).$$

Then we obtain from (7)

$$\sqrt{R^2 + \left(\frac{C_1}{8H\sqrt{M}}\right)R - \frac{|\vec{C}|^2}{2HM}} - \left(\frac{C_1}{16H\sqrt{M}}\right) \cdot \ln\left(\sqrt{M}\left|2R_0 + \left(\frac{C_1}{8H\sqrt{M}}\right) + 2\sqrt{R_0^2 + \left(\frac{C_1}{8H\sqrt{M}}\right)R_0 - \frac{|\vec{C}|^2}{2HM}}\right|\right) - \sqrt{R_0^2 + \left(\frac{C_1}{8H\sqrt{M}}\right)R_0 - \frac{|\vec{C}|^2}{2HM}} + \left(\frac{C_1}{16H\sqrt{M}}\right)\cdot \ln\left(\sqrt{M}\left|2R_0 + \left(\frac{C_1}{8H\sqrt{M}}\right) + 2\sqrt{R_0^2 + \left(\frac{C_1}{8H\sqrt{M}}\right)R_0 - \frac{|\vec{C}|^2}{2HM}}\right|\right) \geq \pm\sqrt{\frac{2H}{M}}t. \tag{C.1}$$

Meanwhile, we can assume that

$$\frac{|\vec{C}|^2}{2HM} \ll R^2$$

in (C.1), and also let us assume that

$$\left(\frac{C_1}{8H\sqrt{M}}\right)R \ll R^2.$$

In this case, we can reduce inequality (C.1) to the following equation (C.2):

$$\left(\frac{C_1}{16H\sqrt{M}}\right)\cdot \ln\left(\frac{\left|4R_0 + \left(\frac{C_1}{8H\sqrt{M}}\right)\right|}{\left|4R + \left(\frac{C_1}{8H\sqrt{M}}\right)\right|}\right) \geq R_0 - R \pm \sqrt{\frac{2H}{M}}t$$
$$\Rightarrow \left(\frac{C_1}{16H\sqrt{M}}\right)\cdot \ln\left(\frac{R_0}{R}\right) \geq R_0 - R \pm \sqrt{\frac{2H}{M}}t \tag{C.2}$$
$$\text{sgn}(C_1)\cdot \ln\left(\frac{R_0}{R}\right) \geq \frac{16H\sqrt{M}}{|C_1|} \times \left(R_0 - R \pm \sqrt{\frac{2H}{M}}t\right)$$
$$\Rightarrow \frac{\exp(\text{sgn}(C_1)\cdot \ln R_0)}{\exp\left(\frac{16H\sqrt{M}}{|C_1|}\left(R_0 \pm \sqrt{\frac{2H}{M}}t\right)\right)}$$

$$\geq R^{\mathrm{sgn}(C_1)} \cdot \exp\left(-\frac{16H\sqrt{M}}{|C_1|}R\right),$$

where the right part of the last inequality can be approximated by the Taylor series expansion as follows (as first approximation):

$$\frac{\exp(\mathrm{sgn}(C_1)\cdot \ln R_0)}{\exp\left(\frac{16H\sqrt{M}}{|C_1|}\left(R_0 \pm \sqrt{\frac{2H}{M}t}\right)\right)} \geq R^{\mathrm{sgn}(C_1)} \cdot \left(1 - \frac{16H\sqrt{M}}{|C_1|}R\right). \quad \text{(C.3)}$$

If we choose $\mathrm{sgn}(C_1) = -$in (C.3), we should obtain accordingly

$$R \geq R_0 \cdot \left(\frac{16H\sqrt{M}}{|C_1|}R_0 + \exp\left(-\frac{16H\sqrt{M}}{|C_1|}\left(R_0 \pm \sqrt{\frac{2H}{M}t}\right)\right)\right)^{-1}. \quad \text{(C.4)}$$

Figure 3 shows schematically the lower bound of the estimation for the solar system's size R (depending on time t), which corresponds to inequality (C.4) where we choose the sign "+".

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