Pramana – J. Phys. (2022) 96:153
https://doi.org/10.1007/s12043-022-02399-8

Effects of $f(\mathfrak{R}, \mathbb{T}^2)$ gravity on the stability of anisotropic perturbed Einstein Universe

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MS received 17 January 2022; revised 3 March 2022; accepted 31 March 2022

Abstract. In this paper, we study the stability of the Einstein static Universe against anisotropic homogeneous perturbations in energy–momentum squared gravity. For this purpose, we consider Bianchi IX space–time with isotopic matter distribution and use small perturbations on the fluid variables as well as scale factors. We consider particular models of this theory to explore stable modes of the Einstein cosmos for both conserved as well as non-conserved energy–momentum tensor. It is found that stable modes of the Einstein Universe appear in both cases for specific values of the equation of state parameter. The stable solutions increase for positive values of the model parameter. It is worth mentioning here that stable regions exist in contrast to general relativity. We conclude that stable solutions increase here compared to other alternative gravitational theories.

Keywords. Curvature–matter coupled gravity; Einstein Universe; stability analysis.

PACS Nos **04.50.Kd; 04.20.Jb; 98.80.-k; 98.80.Jk**

1. Introduction

Cosmological observations reveal that the Universe is created by the massive expansion of matter and energy. This expansion raises many questions for the scientific community about the formation, evolution and structure of the cosmos. One of the most fundamental questions is about the origin of the Universe that provided fascinating results. In the past decades, this question has led to thoughtful discussions based on general relativity (GR) and cosmology. According to the fundamental physical circumstances on cosmic matter configuration, GR equations implied that the current expanding Universe must be preceded by a singularity, where physical quantities such as energy density and space–time curvature diverge. To resolve this problem, different approaches have been proposed to develop non-singular cosmic models. Accordingly, the emergent Universe has been established which resolves the primordial singularity, i.e., the Universe starts from a static phase and then progresses to an expansion era that gives an inflationary phase [1]. Thus, the initial cosmic state is the Einstein Universe (EU) instead of Big-Bang singularity in the emergent cosmic scenario.

The emergent Universe model has interesting characteristics like there is no primordial singularity, the Universe is eternal and shows static cosmic behaviour in infinite past. The main purpose for constructing this scenario is the existence of stable Einstein static solutions. The EU solution must be stable against all types of isotropic as well as anisotropic perturbations to ensure that the Universe can remain in the static state. However, the emergent cosmic hypothesis is not successful in the framework of GR, because isotropic and anisotropic homogeneous perturbations destroy the stability of the Universe in early times [2]. It is also analysed that EU remains stable against inhomogeneous perturbations when the speed of sound (c_s) satisfies the condition $c_s^2 > 1/5$ [3]. Barrow and Yamamoto [4] examined the stability of the EU corresponding to various matter configurations and obtained unstable regions against homogeneous perturbations.

Several cosmic experiments like Planck satellites and Wilkinson Microwave Anisotropy Probe show that the present cosmos is homogeneous and isotropic at large scales. This stage of the Universe is described by the FRW model which ignores all structures of the Universe. The discovery of cosmic microwave background

Published online: 16 August 2022

radiation (CMBR) reveals that the early Universe was spatially homogeneous as well as anisotropic while this anisotropy still exists in terms of CMB temperature in the present Universe [5,6]. Bianchi-type cosmic models are assumed to be the most important and fascinating mathematical models that describe the effect of anisotropy in the early Universe [7]. These anisotropic models show that the initial anisotropy has an impact on the cosmic expansion. The minor anisotropy in the early times will stop the expansion of the Universe and yields a highly isotropic Universe [8]. Barrow *et al* [9] analysed the stability of EU using the Bianchi IX model and found unstable solutions against homogeneous perturbations.

The current cosmic expansion has been the most fascinating development for the researchers in recent years. This cosmic expansion is assumed to be the consequence of a mysterious force known as dark energy which has repulsive behaviour. Many researchers have been working to uncover the hidden aspects of this mysterious force. The cosmological constant is the first candidate which explains the ambiguous characteristics of dark energy, but it has two main problems, fine-tuning and coincidence. To resolve these issues, several modifications of GR known as modified gravitational theories have been established. These theories are the key factors to unveil dark aspects of the cosmos. The simplest modified theory is $f(\mathfrak{R})$ gravity which has significant literature [10] to comprehend its physical features. The concept of curvature–matter coupling generalised this modified theory. These are non-conserved proposals that contain an additional force, and consequently, the path of the particle is changed. These modified approaches are quite useful to understand cosmic mysteries. The minimal coupling theory is discussed in [11] while the non-minimal coupled theory is named as $f(\mathfrak{R}, \mathcal{T}, \mathfrak{R}_{\mu\nu}\mathcal{T}^{\mu\nu})$ theory [12].

A new modification of GR has been developed by including a non-linear term ($\mathbb{T}^2 = \mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}$) in the functional action known as $f(\mathfrak{R}, \mathbb{T}^2)$ theory which is also named as energy–momentum squared gravity (EMSG) [13]. This newly developed theory establishes a particular connection between geometry and matter. It also includes an additional force that yields a more comprehensive description to unveil the cosmic mysteries. This proposal contains squared and product terms of matter contents which are useful to examine different fascinating cosmological results. However, the density profile supports the inflationary cosmological models which resolve important cosmological problems such as horizon and flatness issues. It is noteworthy that this approach represents the entire cosmic history as well as the evolutionary picture of the Universe.

Roshan and Shojai [14] showed that EMSG has a bounce at the early Universe and they claimed that the cosmological constant plays a crucial role to resolve singularity. Board and Barrow [15] discussed the behaviour of analytic solutions through cosmic evolution, existence and absence of singularities with a specific EMSG model. Akarsu *et al* [16] introduced a scale-independent EMSG that provides various gravitational couplings corresponding to distinct sources, which yields significant applications in the field of cosmology. Morares and Sahoo [17] investigated traversable wormholes in this framework. The physically realistic and stable stellar objects through a polytropic equation of state (EoS) and Tolman–Oppenheimer-Volkoff equation have been examined in [18]. Bahamonde *et al* [19] studied various models of this theory and found that these models manifest the current accelerated evolution of the cosmos. The thermodynamic aspects of the compact objects have been investigated in [20]. Ranjit *et al* [21] found possible solutions for matter density from the proposed model of EMSG and examined their cosmological applications. Recently, we have studied exact solutions through the Noether symmetry strategy in this framework [22]. We have also discussed the dynamics of self-gravitating objects and concluded that EMSG reduces the collapse rate compared to GR [23].

Modified theories are considered the most elegant and propitious approaches to examine the stable zones of the EU. The stable solutions of EU by employing homogeneous perturbations in $f(\mathfrak{R})$ theory has been studied in [24]. Bohmer and Lobo [25] investigated the stable modes of the EU through an EoS parameter in $f(\mathcal{G})$ gravity. Li *et al* [26] studied the stability of the EU in the framework of teleparallel theory and found that stable regions appear for both open/closed cosmic models. Huang *et al* [27] used different perturbations to analyse the stable zones of EU in modified Gauss–Bonnet gravity. Bohmer *et al* [28] investigated the stability of the perturbed EU in scalar-field theories. Shabani and Ziaie [29] explored stable regions of EU in $f(\mathfrak{R}, \mathcal{T})$ framework which were not stable in $f(\mathfrak{R})$ theory. Sharif and Waseem [30] analysed the stability of EU against homogeneous/inhomogeneous perturbations with perfect fluid configurations in $f(\mathfrak{R}, \mathcal{T}, \mathfrak{R}_{\mu\nu}\mathcal{T}^{\mu\nu})$ theory. They also studied the stable zones of the EU with specific $f(\mathfrak{R}, \mathcal{T})$ models and obtained stable solutions for specific values of EoS parameters [31]. Recently, we have explored the stable modes of EU against homogeneous perturbations in $f(\mathfrak{R}, \mathbb{T}^2)$ theory and analysed that stable solutions exist for all values of the EoS parameter [32].

This paper investigates the stability of the EU against anisotropic homogeneous perturbations in the background of EMSG. The paper is planned as follows: Section 2 establishes the static as well as perturbed field equations. Section 3 explores the stable modes of static EU for both conserved as well as non-conserved energy–momentum tensor (EMT). The summary of the obtained results is given in §4.

2. Einstein Universe

This section establishes the static and perturbed field equations of the homogeneous and anisotropic Universe model in the context of EMSG. The action of this modified theory is expressed as [33]

$$\mathcal{S}=\int\frac{\sqrt{-g}}{2\kappa^2}f(\mathfrak{R},\mathbb{T}^2)\mathrm{d}^4x+\int\sqrt{-g}\mathbb{L}_m\mathrm{d}^4x, \qquad (1)$$

where g stands for the determinant of the line element, $\kappa^2=1$ is the coupling constant and $\mathbb{L}_m$ defines matter-Lagrangian density. Here, $\mathbb{T}^2=\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}$, where $\mathcal{T}_{\mu\nu}$ is the EMT defined as

$$\mathcal{T}_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathbb{L}_m)}{\delta g^{\mu\nu}}.$$

The corresponding field equations are obtained as

$$\mathfrak{R}_{\mu\nu}f_{\mathfrak{R}}+g_{\mu\nu}\Box f_{\mathfrak{R}}-\nabla_\mu\nabla_\nu f_{\mathfrak{R}}-\frac{1}{2}g_{\mu\nu}f \\ =\mathcal{T}_{\mu\nu}-\Theta_{\mu\nu}f_{\mathbb{T}^2}, \qquad (2)$$

where

$$\Box=\nabla_\xi\nabla^\xi,\ f\equiv f(\mathfrak{R},\mathbb{T}^2),\ f_{\mathbb{T}^2}=\frac{\partial f}{\partial\mathbb{T}^2},\ f_{\mathfrak{R}}=\frac{\partial f}{\partial\mathfrak{R}}$$

and

$$\Theta_{\mu\nu}=-2\mathbb{L}_m\left(\mathcal{T}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\mathcal{T}\right)-\frac{4\partial^2\mathbb{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}\mathcal{T}^{\alpha\beta} \\ -\mathcal{T}\mathcal{T}_{\mu\nu}+2\mathcal{T}_\mu^\xi\mathcal{T}_{\nu\xi}. \qquad (3)$$

The EMT determines the distribution of matter and energy in the system and every non-vanishing element yields dynamical variables with some physical attributes. We assume the perfect matter configuration as

$$\mathcal{T}_{\mu\nu}=(\varrho+\mathbb{P})\chi_\mu\chi_\nu-\mathbb{P}g_{\mu\nu}, \qquad (4)$$

where ϱ, $\mathbb{P}$ and χ_μ represent the energy density, pressure and four-velocity of the fluid, respectively. Solving eq. (3), we have

$$\Theta_{\mu\nu}=-\left(3\mathbb{P}^2+\varrho^2+4\mathbb{P}\varrho\right)\chi_\mu\chi_\nu.$$

Rearranging eq. (2), we obtain

$$G_{\mu\nu}=\frac{1}{f_{\mathfrak{R}}}\left(\mathcal{T}_{\mu\nu}+\mathbf{T}_{\mu\nu}\right), \qquad (5)$$

where $\mathbf{T}_{\mu\nu}$ defines the additional effects of EMSG, represented as

$$\mathbf{T}_{\mu\nu}=\frac{1}{2}g_{\mu\nu}(f-\mathfrak{R}f_{\mathfrak{R}})-\Theta_{\mu\nu}f_{\mathbb{T}^2} \\ +\left(\nabla_\mu\nabla_\nu-g_{\mu\nu}\nabla_\xi\nabla^\xi\right)f_{\mathfrak{R}}.$$

This modified theory is non-conserved which gives the existence of an additional force and results in the non-geodesic motion of particles, given by

$$\nabla^\mu\mathcal{T}_{\mu\nu}=-\frac{1}{2}(f_{\mathbb{T}^2}g_{\mu\nu}\nabla^\mu\mathbb{T}^2-2\nabla^\mu(f_{\mathbb{T}^2}\Theta_{\mu\nu})). \qquad (6)$$

In order to examine the homogeneous and anisotropic space–time, we assume Bianchi IX Universe as [34]

$$\mathrm{d}s^2=\mathrm{d}t^2-a^2(t)(\mathrm{d}x-\cos(y)\mathrm{d}z)^2 \\ -b^2(t)(\mathrm{d}y^2-\sin^2(y)\mathrm{d}z^2), \qquad (7)$$

where the scale parameters are denoted by $a(t)$ and $b(t)$, respectively. The resulting field equations are

$$\varrho+\frac{1}{2}f-\left(\frac{2\dot{a}\dot{b}}{ab}+\frac{1}{b^2}+\frac{\dot{b}^2}{b^2}-\frac{a^2}{4b^4}+\frac{\mathfrak{R}}{2}\right)f_{\mathfrak{R}} \\ -\left(\frac{\dot{a}}{a}-\frac{2\dot{b}}{b}\right)\dot{f}_{\mathfrak{R}} \\ +\left(\varrho^2+3\mathbb{P}^2+4\varrho\mathbb{P}\right)f_{\mathbb{T}^2}=0, \qquad (8)$$

$$\mathbb{P}-\frac{1}{2}f-\left(\frac{3a^2}{4b^4}-\frac{2\ddot{b}}{b}-\frac{\dot{b}^2}{b^2}-\frac{1}{b^2}-\frac{\mathfrak{R}}{2}\right)f_{\mathfrak{R}} \\ +\frac{2\dot{b}}{b}\dot{f}_{\mathfrak{R}}+\ddot{f}_{\mathfrak{R}}=0, \qquad (9)$$

$$\mathbb{P}-\frac{1}{2}f-\left(\frac{\ddot{a}}{a}+\frac{\dot{a}\dot{b}}{ab}+\frac{\ddot{b}}{b}+\frac{a^2}{4b^4}-\frac{\mathfrak{R}}{2}\right)f_{\mathfrak{R}} \\ +\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)\dot{f}_{\mathfrak{R}}+\ddot{f}_{\mathfrak{R}}, \qquad (10)$$

where dot is the rate of change with respect to time while the corresponding values of $\mathfrak{R}$ and $\mathbb{T}^2$ are

$$\mathfrak{R}=-\left(\frac{4\ddot{b}}{b}+\frac{2\ddot{a}}{a}+\frac{2\dot{b}^2}{b^2}+\frac{4\dot{a}\dot{b}}{ab}-\frac{a^2}{2b^4}+\frac{2}{b^2}\right), \\ \mathbb{T}^2=\varrho^2+3\mathbb{P}^2. \qquad (11)$$

The question about the origin of the Universe has provided interesting outcomes in the past decades. The field equations of GR indicate that current cosmic expansion must be preceded by a singularity, where the physical characteristics diverge. To address this problem, the emergent Universe strategy was established which resolves the Big-Bang singularity [35]. This cosmos has fascinating characteristics, i.e., no primordial singularity, ever-existing cosmos and static cosmic behaviour in the infinite past. Thus, the primary motivation for establishing this technique is to analyse the presence of a stable EU. The EU solutions must be stable against all sorts of perturbations to ensure that the cosmos can remain in a static state. We consider

$$a(t) = b(t) = \frac{a_*}{2}$$

to analyse the Einstein static Universe [36]. The corresponding field equations become

$$\varrho_* - \left(\frac{3}{a_*^2} + \frac{\mathfrak{R}_*}{2}\right) f_{\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2) + \left(\varrho_*^2 + 3\mathbb{P}_*^2 + 4\varrho_*\mathbb{P}_*\right) f_{\mathbb{T}^2}(\mathfrak{R}_*, \mathbb{T}_*^2) + \frac{1}{2} f(\mathfrak{R}_*, \mathbb{T}_*^2) = 0, \quad (12)$$

$$\mathbb{P}_* - \frac{1}{2} f(\mathfrak{R}_*, \mathbb{T}_*^2) + \frac{1}{a_*^2} f_{\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2) + \frac{1}{2}\mathfrak{R} f_{\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2) = 0, \quad (13)$$

where

$$\mathfrak{R}_* = \mathfrak{R}(a_*) = -\frac{6}{a_*^2} \quad \text{and} \quad \mathbb{T}_*^2 = 3\mathbb{P}_*^2 + \varrho_*^2.$$

2.1 *Anisotropic perturbations*

The perturbation technique is a very promising method that simplifies the complex physical system. There are several types of perturbations such as homogeneous, inhomogeneous, isotropic and anisotropic perturbations. These perturbations have been used by several researchers to examine the stable modes of the EU [37]. Here, we investigate the stable zones of EU against anisotropic perturbations in $f(\mathfrak{R}, \mathbb{T}^2)$ theory and parameterise this analysis by EoS parameter (ψ). Here, we have considered spatially homogeneous space–time. Therefore, we use anisotropic perturbations only in scale factors and fluid parameters, defined as

$$\varrho = \varrho_* + \delta\varrho(\varepsilon_1, \varepsilon_2), \quad \mathbb{P} = \mathbb{P}_* + \delta\mathbb{P}(\varepsilon_1, \varepsilon_2), \quad (14)$$

$$a(t) = \frac{a_*}{2}(1 + \varepsilon_1(t)), \quad b(t) = \frac{b_*}{2}(1 + \varepsilon_2(t)), \quad (15)$$

where ε_1, ε_2, $\delta\varrho$ and $\delta\mathbb{P}$ are perturbed quantities. The Taylor series expansion on $f(\mathfrak{R}, \mathbb{T}^2)$ up to first order yields

$$f(\mathfrak{R}, \mathbb{T}^2) = f(\mathfrak{R}_*, \mathbb{T}_*^2) + \delta\mathfrak{R} f_{\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2) + \delta\mathbb{T}^2 f_{\mathbb{T}^2}(\mathfrak{R}_*, \mathbb{T}_*^2), \quad (16)$$

where

$$\delta\mathfrak{R} = \frac{4}{a_*^2}(\varepsilon_1 + 2\varepsilon_2) - 4\ddot{\varepsilon}_2 - 2\ddot{\varepsilon}_1, \quad \delta\mathbb{T}^2 = \mathbb{T}_*^2\delta\varrho. \quad (17)$$

By using eqs (14)–(17), the perturbed field equations (8)–(10) become

$$\left(\frac{2\varepsilon_1}{a_*^2} + \frac{4\varepsilon_2}{a_*^2}\right) f_{\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2) + \varrho_*^2(3\psi^2 + 4\psi + 1) f_{\mathfrak{R}\mathbb{T}^2}(\mathfrak{R}_*, \mathbb{T}_*^2) \times \left[\frac{4}{a_*^2}(\varepsilon_1 + 2\varepsilon_2) - 2(\ddot{\varepsilon}_1 + 2\ddot{\varepsilon}_2)\right] + \left[1 + 2\varrho_*\left\{(3\psi^2 + 4\psi + 1) + \frac{\varrho_*}{4}(1 + 3\psi^2)\right\} f_{\mathbb{T}^2}(\mathfrak{R}_*, \mathbb{T}_*^2) + 2\varrho_*^3(3\psi^2 + 4\psi + 1)(1 + 3\psi^2) \times f_{\mathbb{T}^2\mathbb{T}^2}(\mathfrak{R}_*, \mathbb{T}_*^2)\right]\delta\varrho = 0, \quad (18)$$

$$\left(2\ddot{\varepsilon}_2 - \frac{6\varepsilon_1}{a_*^2} + \frac{4\varepsilon_2}{a_*^2}\right) f_{\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2) + \left[\psi - \frac{\varrho_*^2}{2}(1 + 3\psi^2) f_{\mathbb{T}^2}(\mathfrak{R}_*, \mathbb{T}_*^2)\right]\delta\varrho + \frac{4}{a_*^2}(\ddot{\varepsilon}_1 + 2\ddot{\varepsilon}_2) f_{\mathfrak{R}\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2) + \varrho_*^2(1 + 3\psi^2)\delta\ddot{\varrho} f_{\mathfrak{R}\mathbb{T}^2}(\mathfrak{R}_*, \mathbb{T}_*^2) - 2 f_{\mathfrak{R}\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2)(\varepsilon_1^{iv} + 2\varepsilon_2^{iv}) = 0, \quad (19)$$

$$\left(\ddot{\varepsilon}_1 + \ddot{\varepsilon}_2 + \frac{2\varepsilon_1}{a_*^2} - \frac{4\varepsilon_2}{a_*^2}\right) f_{\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2) + \frac{4}{a_*^2}(\ddot{\varepsilon}_1 + 2\ddot{\varepsilon}_2) f_{\mathfrak{R}\mathfrak{R}}(\mathfrak{R}_*, \mathbb{T}_*^2) + \left[\psi - \frac{\varrho_*^2}{2}(1 + 3\psi^2) f_{\mathbb{T}^2}(\mathfrak{R}_*, \mathbb{T}_*^2)\right]\delta\varrho + \varrho_*^2(1 + 3\psi^2)\delta\ddot{\varrho} f_{\mathfrak{R}\mathbb{T}^2}(\mathfrak{R}_*, \mathbb{T}_*^2) - 2(\varepsilon_1^{iv} + 2\varepsilon_2^{iv}) = 0. \quad (20)$$

The solution of these equations helps to examine the stable modes of EU but these are very difficult to solve due to their complex and highly non-linear nature. To address this issue, we assume a particular type of generic function which provides minimal coupling between geometric and matter parts as [38]

$$f(\mathfrak{R}, \mathbb{T}^2) = f_1(\mathfrak{R}) + f_2(\mathbb{T}^2). \quad (21)$$

The corresponding perturbed field equations become

$$\left(\frac{2\varepsilon_1}{a_*^2} + \frac{4\varepsilon_2}{a_*^2}\right) f_1'(\mathfrak{R}_*) + \left[1 + 2\varrho_*\left\{(3\psi^2 + 4\psi + 1) + \frac{\varrho_*}{4}(1 + 3\psi^2)\right\} f_2'(\mathbb{T}_*^2) + 2\varrho_*^3(3\psi^2 + 4\psi + 1) \times (1 + 3\psi^2) f_2''(\mathbb{T}_*^2)\right]\delta\varrho = 0, \quad (22)$$

$$\left(2\ddot{\varepsilon}_2-\frac{6\varepsilon_1}{a_*^2}+\frac{4\varepsilon_2}{a_*^2}\right)f_1'(\Re_*)$$
$$+\left[\frac{4}{a_*^2}(\ddot{\varepsilon}_1+2\ddot{\varepsilon}_2)-2(\varepsilon_1^{iv}+2\varepsilon_2^{iv})\right]f_1''(\Re_*)$$
$$+\left[\psi-\frac{\varrho_*^2}{2}(1+3\psi^2)f_2'(\mathbb{T}_*^2)\right]\delta\varrho=0, \quad (23)$$
$$\left(\ddot{\varepsilon}_1+\ddot{\varepsilon}_2+\frac{2\varepsilon_1}{a_*^2}-\frac{4\varepsilon_2}{a_*^2}\right)f_1'(\Re_*)$$
$$+\left[\frac{4}{a_*^2}(\ddot{\varepsilon}_1+2\ddot{\varepsilon}_2)-2(\varepsilon_1^{iv}+2\varepsilon_2^{iv})\right]f_1''(\Re_*)$$
$$+\left[\psi-\frac{\varrho_*^2}{2}(1+3\psi^2)f_2'(\mathbb{T}_*^2)\right]\delta\varrho=0, \quad (24)$$

where prime denotes the derivative corresponding to $\Re$ or $\mathbb{T}^2$.

The presence of stable regions of EU for particular values of $f_1(\Re)$ and $f_2(\mathbb{T}^2)$ are examined in the following section.

3. Stability analysis

Here, we analyse the solutions of EU for both conserved as well as non-conserved cases with $f_1(\Re)=\Re$. Firstly, we take the conserved case to formulate the specific form of $f_2(\mathbb{T}^2)$ and examine the stable solutions graphically. Secondly, we consider the non-conserved case to investigate the stability of the EU.

3.1 *Conserved EMT*

The conservation law is violated by modified theories that include curvature–matter coupling. The non-conservation equation with isotropic matter distribution is given by

$$\dot{\varrho}+\varrho(1+\psi)(\dot{a}a^{-1}+2\dot{b}b^{-1})$$
$$=(3\psi^2+1+4\psi)\varrho^2\dot{f}_{\mathbb{T}^2}$$
$$-\frac{3\dot{a}}{a}(3\psi^2+1+4\psi)\varrho^2 f_{\mathbb{T}^2}$$
$$-f_{\mathbb{T}^2}(\dot{\varrho}\varrho(3+4\psi)+\dot{\varrho}\psi(9\psi+4)\varrho). \quad (25)$$

We assume that the conservation law holds and the resulting differential equation for the proposed model (21) becomes

$$(3+8\psi+9\psi^2)f_2'(\mathbb{T}^2)$$
$$+\varrho_*^2(1+4\psi+3\psi^2)(2+6\psi^2)f_2''(\mathbb{T}^2)=0. \quad (26)$$

The solution of this equation is

$$f_2(\mathbb{T}^2)=c_1 e^{\frac{-(9\psi^2+8\psi+3)\mathbb{T}^2}{2\varrho_*^2(9\psi^4+12\psi^3+6\psi^2+4\psi+1)}}+c_2 \quad (27)$$

which gives a unique value of $f_2(\mathbb{T}^2)$ for which EMT remains conserved while c_1 and c_2 are integration constants. Inserting eq. (27) with $f_1(\Re)=\Re$ in (22)–(24), the corresponding perturbed equations become

$$\delta\varrho+2\mathcal{A}\varrho_*\left((3\psi^2+4\psi+1)+\frac{\varrho_*}{4}(1+3\psi^2)\right)\delta\varrho$$
$$-\mathcal{A}\varrho_*(9\psi^2+8\psi+3)\delta\varrho$$
$$+\frac{2}{a_*^2}(\varepsilon_1+2\varepsilon_2)=0, \quad (28)$$
$$2\ddot{\varepsilon}_2-\frac{2}{a_*^2}(3\varepsilon_1-2\varepsilon_2)$$
$$+\left(\psi-\frac{1}{2}\mathcal{A}\varrho_*^2(1+3\psi^2)\right)\delta\varrho=0, \quad (29)$$
$$\ddot{\varepsilon}_1+\ddot{\varepsilon}_2+\frac{2}{a_*^2}(\varepsilon_1-2\varepsilon_2)$$
$$+\left(\psi-\frac{1}{2}\mathcal{A}\varrho_*^2(1+3\psi^2)\right)\delta\varrho=0, \quad (30)$$

where

$$\mathcal{A}=-\frac{1}{2}\frac{c_1(9\psi^2+8\psi+3)e^{\frac{-(9\psi^2+8\psi+3)}{2(3\psi+1)(\psi+1)}}}{\varrho_*(3\psi+1)(\psi+1)(3\psi^2+1)}.$$

Eliminating $\delta\varrho$ from eqs (29) and (30), we have

$$\left[2\ddot{\varepsilon}_2-\frac{2}{a_*^2}(3\varepsilon_1-2\varepsilon_2)\right]$$
$$\times\left[1+2\varrho_*\left\{(3\psi^2+4\psi+1)+\frac{\varrho_*}{4}(1+3\psi^2)\right\}\mathcal{A}\right.$$
$$\left.-\varrho_*(9\psi^2+8\psi+3)\mathcal{A}\right]$$
$$-\frac{2}{a_*^2}(\varepsilon_1+2\varepsilon_2)\left[\psi-\frac{1}{2}\varrho_*^2(1+3\psi^2)\mathcal{A}\right]=0, \quad (31)$$
$$\left[\ddot{\varepsilon}_1+\ddot{\varepsilon}_2+\frac{2}{a_*^2}(\varepsilon_1-2\varepsilon_2)\right]$$
$$\times\left[1+2\varrho_*\left\{(3\psi^2+4\psi+1)+\frac{\varrho_*}{4}(1+3\psi^2)\right\}\mathcal{A}\right.$$
$$\left.-\varrho_*(9\psi^2+8\psi+3)\mathcal{A}\right]$$
$$-\frac{2(\varepsilon_1+2\varepsilon_2)}{a_*^2}\left[\psi-\frac{1}{2}\varrho_*^2(1+3\psi^2)\mathcal{A}\right]=0. \quad (32)$$

Adding eqs (12) and (13) and using eqs (31) and (32), we obtain

$$\left[2\ddot{\varepsilon}_2-(\varrho_*(1+\psi)+\varrho_*^2\left(3\psi^2+4\psi+1\right)\mathcal{A})\right.$$
$$\left.\times(3\varepsilon_1-2\varepsilon_2)\right]$$
$$\times\left[1+\mathcal{A}\frac{\varrho_*^2}{2}(1+3\psi^2)\right.$$

$$+2\varrho_*(3\psi^2+4\psi+1)\mathcal{A} -\varrho_*(9\psi^2+8\psi+3)\mathcal{A}\Big] -(\varepsilon_1+2\varepsilon_2)(\varrho_*(1+\psi)+\varrho_*^2(3\psi^2+4\psi+1)\mathcal{A}) \times\left[\psi-\frac{1}{2}\varrho_*^2(1+3\psi^2)\mathcal{A}\right]=0, \quad (33)$$

$$\Big[\ddot{\varepsilon_1}+\ddot{\varepsilon_2}+(\varrho_*(1+\psi) +\varrho_*^2\left(3\psi^2+4\psi+1\right)\mathcal{A})(\varepsilon_1-2\varepsilon_2)\Big] \times\left[1+2\mathcal{A}\varrho_*(3\psi^2+4\psi+1)+\frac{\mathcal{A}\varrho_*^2}{2}(1+3\psi^2) -\varrho_*(9\psi^2+8\psi+3)\mathcal{A}\right]-(\varepsilon_1+2\varepsilon_2) \times(\varrho_*(1+\psi)+\varrho_*^2\left(3\psi^2+4\psi+1\right)\mathcal{A}) \times\left[\psi-\frac{1}{2}\varrho_*^2(1+3\psi^2)\mathcal{A}\right]=0. \quad (34)$$

The corresponding perturbed field equations turn out to be

$$2\Omega_1\ddot{\varepsilon_2}-\Omega_2(3\Omega_1+\Omega_3)\varepsilon_1 +2\Omega_2(\Omega_1-\Omega_3)\varepsilon_2=0, \quad (35)$$

$$\Omega_1(\ddot{\varepsilon_1}+\ddot{\varepsilon_2})-\Omega_2(\Omega_1-\Omega_3)\varepsilon_1 -2\Omega_2(\Omega_1+\Omega_3)\varepsilon_2=0, \quad (36)$$

where

$$\Omega_1=1-\varrho_*(9\psi^2+8\psi+3)\mathcal{A} -2\varrho_*\left((3\psi^2+4\psi+1)+\frac{\varrho_*}{4}(1+3\psi^2)\right)\mathcal{A},$$

$$\Omega_2=\varrho_*(1+\psi)+\mathcal{A}\varrho_*^2(3\psi+1)(\psi+1),$$

$$\Omega_3=\psi-\frac{1}{2}\varrho_*^2(3\psi^2+1)\mathcal{A}.$$

The solution of the above system of equations is

$$\varepsilon_1(t)=k_1\mathrm{e}^{\Upsilon_1 t}+k_2\mathrm{e}^{-\Upsilon_1 t}+k_3\mathrm{e}^{\Upsilon_2 t}+k_4\mathrm{e}^{-\Upsilon_2 t},$$

$$\varepsilon_2(t)=-\frac{1}{2}\tilde{k}_1\mathrm{e}^{\Upsilon_1 t}-\frac{1}{2}\tilde{k}_2\mathrm{e}^{-\Upsilon_1 t}+\tilde{k}_3\mathrm{e}^{\Upsilon_2 t}+\tilde{k}_4\mathrm{e}^{-\Upsilon_2 t},$$

where k_i and $\tilde{k}_i$ $(i=1,\ldots,4)$ are the integration constants while Υ_j $(j=1,2)$ define the frequencies given as

$$\Upsilon_1^2=4\Omega_2,\quad \Upsilon_2^2=\frac{\Omega_2}{2\Omega_1}(\Omega_1+3\Omega_3). \quad (37)$$

For $c_1=0$, these equations reduce to GR as

$$\Upsilon_1^2=4\varrho_*(1+\psi),\quad \Upsilon_2^2=\frac{1}{2}\varrho_*(1+\psi)(1+3\psi).$$

It is known that the behaviour of perturbations gives stable as well as unstable modes of EU. The presence of stable and unstable regions of the EU depends only on the exponential growth of perturbations. The inequality $\Upsilon_j^2<0$ yields stable solutions whereas unstable ones exist for $\Upsilon_j^2>0$. We consider $\varrho_*=0.3$ [39] to investigate the stable modes of EU graphically. Figure 1 shows the stable modes of EU against homogeneous anisotropic perturbations for conserved EMT. The stable region of the EU is defined by the common regions of both frequencies given in the system. It is observed that stable regions exist for $\psi>-1$ and $\psi<-1.3$ corresponding to the positive values of integration constant. Here, we also find stable modes of EU which were unstable in other gravitational theories [40].

3.2 *Non-conserved EMT*

Now, we investigate the effects of non-conserved EMT on the stability of EU. For this purpose, we assume [41]

$$f(\mathfrak{R},\mathbb{T}^2)=\mathfrak{R}+\eta(\mathbb{T}^2)^n, \quad (38)$$

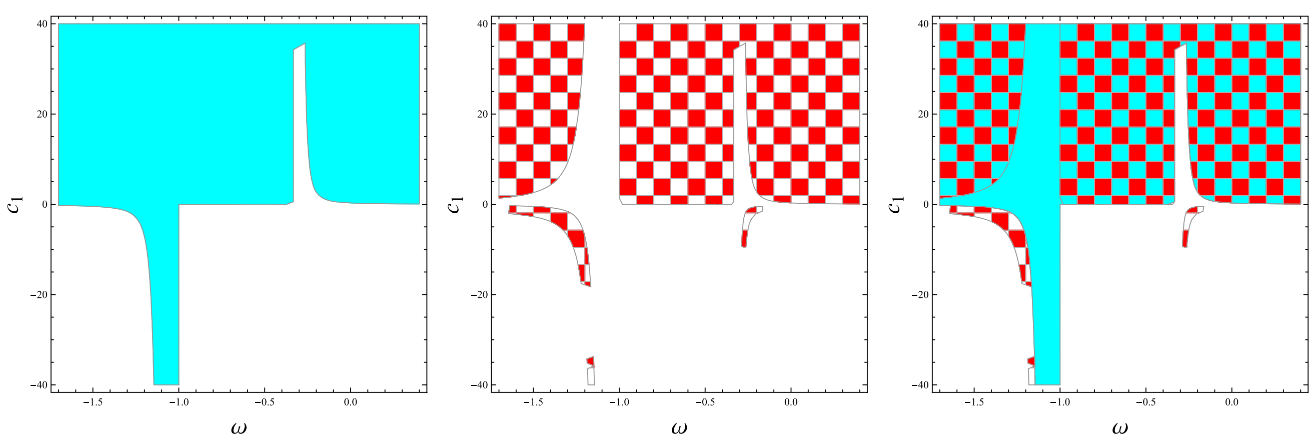

Figure 1. Stable modes of EU for Υ_1^2 (cyan) and Υ_1^2 (red).

where η is an arbitrary constant. The perturbed field equations become

$$\begin{aligned}&\frac{2}{a_*^2}(\varepsilon_1+2\varepsilon_2)\\&\quad+\Bigg[1+2\varrho_*\Bigg\{(3\psi^2+4\psi+1)\\&\quad+\frac{\varrho_*}{4}(1+3\psi^2)\Bigg\}n\eta(\mathbb{T}_*^2)^{n-1}\\&\quad+2\varrho_*^3(3\psi^2+4\psi+1)\\&\quad\times(1+3\psi^2)n(n-1)\eta(\mathbb{T}_*^2)^{n-2}\Bigg]\delta\varrho=0,\end{aligned}\tag{39}$$

$$\begin{aligned}&2\ddot{\varepsilon}_2-\frac{2}{a_*^2}(3\varepsilon_1-2\varepsilon_2)\\&\quad+\left[\psi-\frac{\varrho_*^2}{2}(1+3\psi^2)n\eta(\mathbb{T}_*^2)^{n-1}\right]\delta\varrho=0,\end{aligned}\tag{40}$$

$$\begin{aligned}&\ddot{\varepsilon}_1+\ddot{\varepsilon}_2+\frac{2}{a_*^2}(\varepsilon_1-2\varepsilon_2)\\&\quad+\left[\psi-\frac{\varrho_*^2}{2}(1+3\psi^2)n\eta(\mathbb{T}_*^2)^{n-1}\right]\delta\varrho=0.\end{aligned}\tag{41}$$

Eliminating $\delta\varrho$ from eqs (40) and (41), we have

$$\begin{aligned}&\left[2\ddot{\varepsilon}_2-\frac{2}{a_*^2}(3\varepsilon_1+2\varepsilon_2)\right]\\&\quad\times\Bigg[1+2\varrho_*^3(3\psi^2+4\psi+1)\\&\quad\times n(n-1)\eta(\mathbb{T}_*^2)^{n-2}(1+3\psi^2)\\&\quad+2\eta\varrho_*\Big\{(3\psi^2+4\psi+1)\\&\quad+\frac{\varrho_*}{4}(1+3\psi^2)\Big\}n(\mathbb{T}_*^2)^{n-1}\Bigg]\\&\quad-\frac{2\psi}{a_*^2}(\varepsilon_1+2\varepsilon_2)\\&\quad+\frac{\varrho_*^2}{a_*^2}(\varepsilon_1+2\varepsilon_2)(1+3\psi^2)n\eta(\mathbb{T}_*^2)^{n-1}=0,\end{aligned}\tag{42}$$

$$\begin{aligned}&\left[\ddot{\varepsilon}_1+\ddot{\varepsilon}_2+\frac{2}{a_*^2}(\varepsilon_1-2\varepsilon_2)\right]\\&\quad\times\Bigg[1+2\varrho_*^3n(n-1)\eta(3\psi^2\\&\quad+4\psi+1)(\mathbb{T}_*^2)^{n-2}(1+3\psi^2)\\&\quad+2\varrho_*\Big\{(3\psi^2+4\psi+1)\\&\quad+\frac{\varrho_*}{4}(1+3\psi^2)\Big\}n\eta(\mathbb{T}_*^2)^{n-1}\Bigg]\\&\quad-\frac{2\psi}{a_*^2}(\varepsilon_1+2\varepsilon_2)\\&\quad+\frac{\varrho_*^2}{a_*^2}(\varepsilon_1+2\varepsilon_2)(1+3\psi^2)n\eta(\mathbb{T}_*^2)^{n-1}=0.\end{aligned}\tag{43}$$

Solving eqs (12) and (13) and using eqs (42) and (43), we obtain

$$\begin{aligned}&\Big[2\ddot{\varepsilon}_2-(\varrho_*(1+\psi)+\varrho_*^2\big(3\psi^2\\&\quad+4\psi+1\big)n\eta(\mathbb{T}_*^2)^{n-1})(3\varepsilon_1+2\varepsilon_2)\Big]\\&\quad\times\Bigg[1+2\varrho_*^3(3\psi^2+4\psi+1)n(n-1)\\&\quad\times\eta(\mathbb{T}_*^2)^{n-2}(1+3\psi^2)+2\eta\varrho_*\\&\quad\times\left\{(3\psi^2+4\psi+1)+\frac{\varrho_*}{4}(1+3\psi^2)\right\}n(\mathbb{T}_*^2)^{n-1}\Bigg]\\&\quad-\psi(\varepsilon_1+2\varepsilon_2)(\varrho_*(1+\psi)+\varrho_*^2\big(3\psi^2+4\psi+1\big)\\&\quad\times n\eta(\mathbb{T}_*^2)^{n-1})+\frac{\varrho_*^2}{2}(\varrho_*(1+\psi)\\&\quad+\varrho_*^2\big(3\psi^2+4\psi+1\big)n\eta(\mathbb{T}_*^2)^{n-1})(\varepsilon_1+2\varepsilon_2)\\&\quad\times(1+3\psi^2)n\eta(\mathbb{T}_*^2)^{n-1}=0,\end{aligned}\tag{44}$$

$$\begin{aligned}&\Big[\ddot{\varepsilon}_1+\ddot{\varepsilon}_2+(\varrho_*(1+\psi)\\&\quad+\varrho_*^2\big(3\psi^2+4\psi+1\big)n\eta(\mathbb{T}_*^2)^{n-1})(\varepsilon_1-2\varepsilon_2)\Big]\\&\quad\times\Bigg[1+2\varrho_*^3n(n-1)\eta(3\psi^2\\&\quad+4\psi+1)(\mathbb{T}_*^2)^{n-2}(1+3\psi^2)+2\varrho_*\\&\quad\times\left\{(3\psi^2+4\psi+1)+\frac{\varrho_*}{4}(1+3\psi^2)\right\}n\eta(\mathbb{T}_*^2)^{n-1}\Bigg]\\&\quad-\psi(\varepsilon_1+2\varepsilon_2)(\varrho_*(1+\psi)+\varrho_*^2\big(3\psi^2+4\psi+1\big)\\&\quad\times n\eta(\mathbb{T}_*^2)^{n-1})+\frac{\varrho_*^2}{2}(\varepsilon_1+2\varepsilon_2)(\varrho_*(1+\psi)\\&\quad+\varrho_*^2\big(3\psi^2+4\psi+1\big)n\eta(\mathbb{T}_*^2)^{n-1})\\&\quad\times(1+3\psi^2)n\eta(\mathbb{T}_*^2)^{n-1}=0.\end{aligned}\tag{45}$$

The perturbed field equations (44) and (45) in terms of ε_1 and ε_2 turn out to be

$$\begin{aligned}&2\Delta_1\ddot{\varepsilon}_2-\Delta_2(3\Delta_1+\Delta_3)\varepsilon_1\\&\quad+2\Delta_2(\Delta_1-\Delta_3)\varepsilon_2=0,\end{aligned}\tag{46}$$

$$\begin{aligned}&\Delta_1(\ddot{\varepsilon}_1+\ddot{\varepsilon}_2)-\Delta_2(\Delta_1-\Delta_3)\varepsilon_1\\&\quad-2\Delta_2(\Delta_1+\Delta_3)\varepsilon_2=0,\end{aligned}\tag{47}$$

where

$$\begin{aligned}\Delta_1&=1+2\varrho_*((3\psi^2+4\psi+1)\\&\quad+\frac{\varrho_*}{4}(1+3\psi^2))n\eta(\varrho_*^2(1+3\psi^2))^{n-1}\\&\quad+2\varrho_*^3(3\psi^2+4\psi+1)(1+3\psi^2)\end{aligned}$$

$$\times n(n-1)\eta(\varrho_*^2(1+3\psi^2))^{n-2},$$
$$\Delta_2 = \varrho_*(1+\psi) + \varrho_*^2\left(3\psi^2 + 4\psi + 1\right) n\eta(\varrho_*^2(1+3\psi^2))^{n-1},$$
$$\Delta_3 = \psi - \frac{\varrho_*^2}{2}(1+3\psi^2)n\eta(\varrho_*^2(1+3\psi^2))^{n-1}.$$

The solution of the above equation is

$$\varepsilon_1(t) = h_1 e^{\Pi_1 t} + h_2 e^{-\Pi_1 t} + h_3 e^{\Pi_2 t} + h_4 e^{-\Pi_2 t},$$
$$\varepsilon_2(t) = -\frac{1}{2}\tilde{h}_1 e^{\Pi_1 t} - \frac{1}{2}\tilde{h}_2 e^{-\Pi_1 t} + \tilde{h}_3 e^{\Pi_2 t} + \tilde{h}_4 e^{-\Pi_2 t}.$$

Here integration constants are denoted by h_i and $\tilde{h}_i$ while the frequencies are determined by Π_j given as

$$\Pi_1^2 = 4\Delta_2, \quad \Pi_2^2 = \frac{\Delta_2}{2\Delta_1}(\Delta_1 + 3\Delta_3). \tag{48}$$

The graphical representations of the stable EU regions for various values of η are exhibited in figures 2–5. It can be seen that stable region exists in the range of $\psi < -1$ and $\psi > -0.3$ against positive values of the model parameter while for negative values of the model parameter, stable modes appear only for $-1 < \psi < -0.4$. The analysis shows stable modes which were not obtained in GR [9] and these stable regions increase as η becomes positive.

4. Final remarks

The emergent cosmos has been identified as a viable possibility to avoid the primordial singularity and modified theories are considered successful to study this hypothesis. This paper explores the stability of EU using anisotropic homogeneous perturbations in EMSG. We have used small perturbations on scale factors as well as matter contents and developed static/perturbed field equations which are simplified by the EoS parameter. Different EMSG models have been considered to solve the perturbed equations to investigate the stable zones of the EU. We have used the anisotropic perturbation technique and analysed the conserved/non-conserved EMT

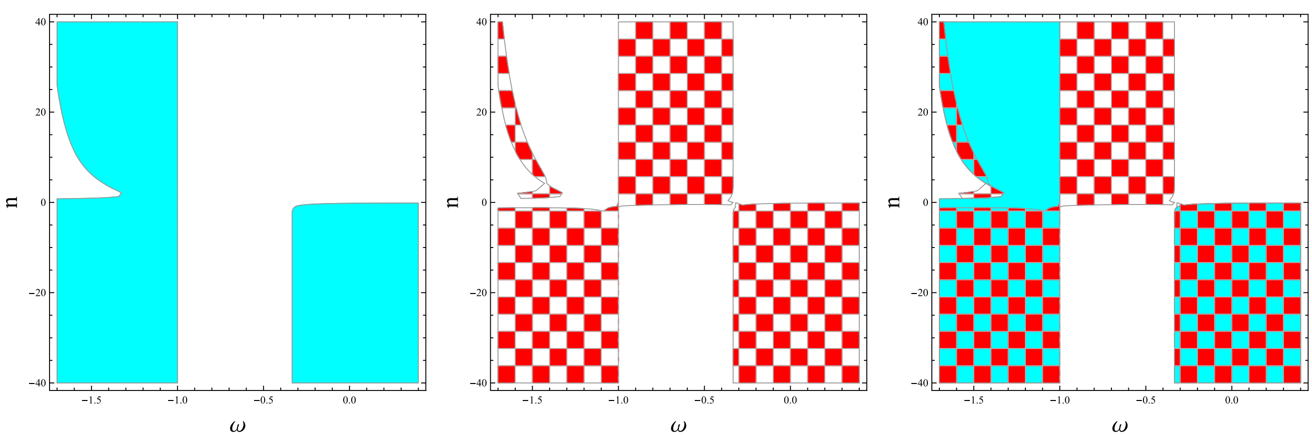

Figure 2. Stable modes of EU with $\eta = 1$ for Π_1^2 (cyan) and Π_2^2 (red).

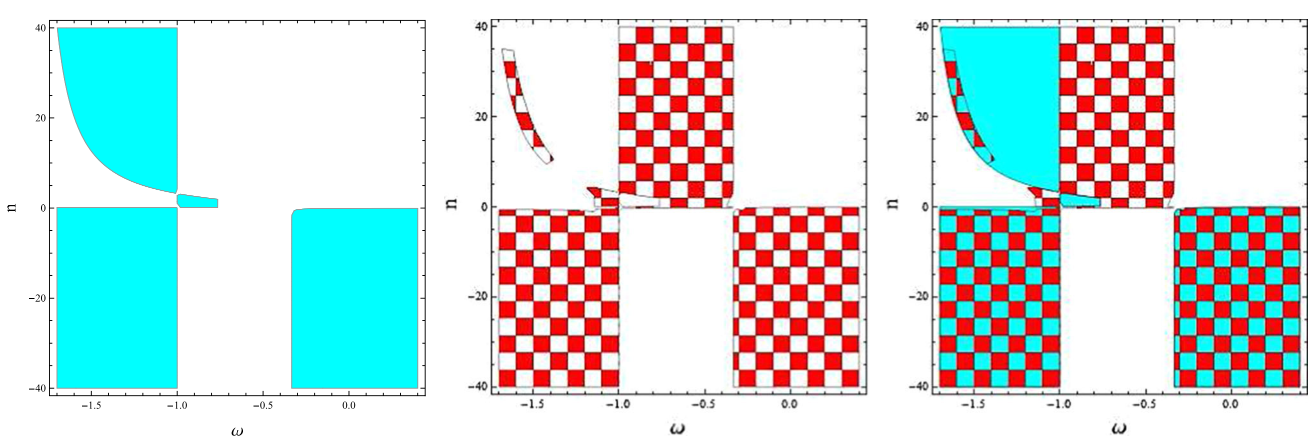

Figure 3. Stable modes of EU with $\eta = 5$ for Π_1^2 (cyan) and Π_2^2 (red).

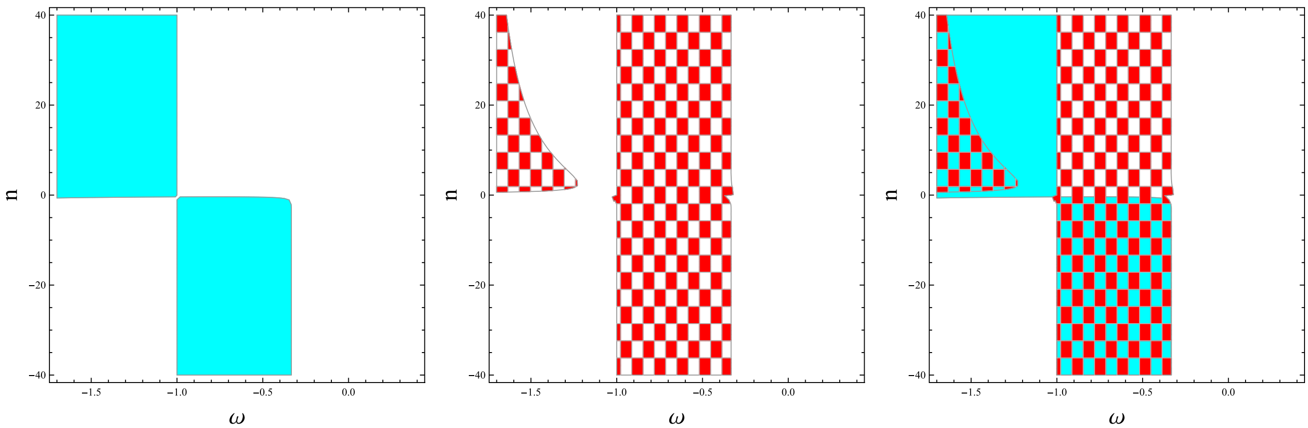

Figure 4. Stable modes of EU with $\eta = -1$ for Π_1^2 (cyan) and Π_2^2 (red).

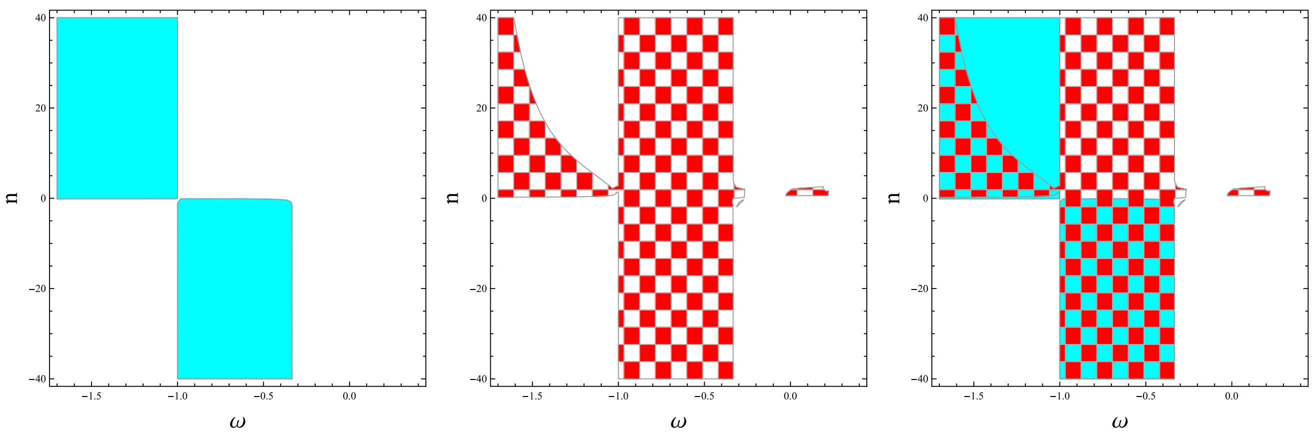

Figure 5. Stable modes of EU with $\eta = -5$ for Π_1^2 (cyan) and Π_2^2 (red).

cases for the proposed models. The main findings are given as follows.

- A unique expression of $f_2(\mathbb{T}^2)$ (for the conserved EMT case) has been developed that satisfies the conservation equation. We have investigated the stable modes of EU against c_1. It is found that the EU is stable for positive values of integration constant. It is noteworthy to mention here that our solutions reduce to GR for $c_1 = 0$.
- A particular value of $f_2(\mathbb{T}^2)$ has been considered to analyse the stability of EU in the non-conserved case. We have found that stable regions exist for $\psi < -1$ and $\psi > -0.3$ corresponding to positive values of η while for negative values, stable modes appear in the range of $-1 < \psi < -0.4$. The stable solutions become larger as the model parameter increases positively.

It is interesting to mention here that the stable regions of EU exist in EMSG against Bianchi IX Universe model which were not obtained in GR [9]. Thus, EMSG gives a feasible framework for a successful emergent cosmic hypothesis based on static EU. Also, our solutions reduce to isotropic homogeneous perturbations in $f_2(\mathbb{T}^2)$ theory when scale parameters are the same [32]. We conclude that the range of EoS parameter in $f_2(\mathbb{T}^2)$ enhances and yields stable regions for both conserved and non-conserved EMTs that are unstable in other alternative gravitational theories [40].

Acknowledgements

This work has been supported by the Pakistan Academy of Sciences Project.

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