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Qualitative and quantitative analysis of high-order nonlinear time–space fractional Schrödinger equation in mono-mode optical fibres

XIANG XIAO

School of Mathematics, Physics and Statistics, Shanghai University of Engineering Science, Shanghai 201620, China
E-mail: xiaoxiang@sues.edu.cn

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Abstract. The dynamic properties and the analytic solutions of high-order nonlinear time–space fractional Schrödinger equation are studied in this article. Based on a conserved Hamiltonian, the topological structure and the existence of the periodic and soliton solutions are studied by classifying the equilibrium points using the bifurcation method. Moreover, all exact travelling wave solutions are constructed to verify the prior estimation in the qualitative analysis by the complete discrimination system for the polynomial method.

Keywords. Time–space fractional Schrödinger equation; qualitative analysis; quantitative analysis; exact travelling wave solution; complete discrimination system for polynomial method.

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1. Introduction

Partial differential equations have important applications in optics, biology, fluid mechanics, chemistry, signal processing, control theory, epidemiology and many other fields [1–6]. For hundred of years, scientists have done many developments and contributions to find exact solutions of partial differential equations, such as first integration method [7,8], Adomian analysis method [9,10], variational iteration method [11,12], exponential expansion method [13,14], (G'/G)-expansion method [15,16], Hirota bilinear method [17–21], Bäcklund transformation method [22,23], canonical-like transformation method [24] and rational sine-Gordon expansion method [25]. However, traditional integer-order partial differential equations cannot meet the needs of building models to analyse some special problems such as anomalous diffusion [26]. So, fractional partial differential equations have been proposed to solve these problems.

In this article, we consider the following high-order nonlinear time–space fractional Schrödinger equation

$$\frac{\partial^\alpha \Omega}{\partial t^\alpha} - i\left(\eta_1 \frac{\partial^{2\beta}\Omega}{\partial x^{2\beta}} + \eta_2(|\Omega|^2\Omega) - \eta_3\frac{\partial^{3\beta}\Omega}{\partial x^{3\beta}} - \eta_4\frac{\partial^\beta(|\Omega|^2\Omega)}{\partial x^\beta} - \eta_5\Omega\frac{\partial^\beta(|\Omega|^2)}{\partial x^\beta}\right) = 0 \tag{1}$$

with conformal fractional derivative [27], where α is the time-fractional order and β is the space-fractional order, $\Omega(x,t)$ is the complex valued function defining wave profile in optical fibres, x expresses the normalized distance along the fibre, t represents the time with the frame of reference moving along the fibre at group velocity. η_1 and η_2 represent the group velocity dispersion and nonlinear term, respectively, η_3 is the intermodal dispersion, η_4 is the self-steepening perturbation term and η_5 is the nonlinear dispersion coefficient. When $\alpha = \beta = 1$, eq. (1) is reduced to the high integer-order nonlinear Schrödinger equation considered in [28]. Especially, if $\alpha = \beta = 1$ and $\eta_3 = \eta_4 = \eta_5 = 0$, eq. (1) is just the classic integrable Schrödinger equation.

Equation (1) and its integer-order form have wide applications in representing the propagation of short light pulses in the mono-mode optical fibres. Furthermore, when the external electric field exists, these equations can be used to solve non-harmonic motion of electrons bound in molecules [29]. So it is of great significance to find their solutions in practice. Recently, some scholars have studied the high integer-order nonlinear Schrödinger equation. For instance, Tariq and

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Seadawy adopted the auxiliary equation method to find a variety of analytical solutions, and new travelling wave solutions are observed to analyse the propagation of short light pulses in an optical fibre [30]. Arshad *et al* implemented a powerful method, namely modified extended mapping method, to successfully find optical soliton solutions which are very helpful in experiment and theory because of their potential applications in the high-speed optical fibre transmission system [31].

Although researchers have explored some methods to study high integer-order nonlinear Schrödinger equation, works related to qualitative analysis and time–space fractional form are very few. In this article, we apply the complete discrimination system for the polynomial method to carry out qualitative and quantitative analysis which allows researchers to comprehend and explicate the physical phenomena of eq. (1). The complete discrimination system for the polynomial method is a powerful method, which was first proposed by Liu [32–37] and further developed by Kai [38–42]. By this method, qualitative and quantitative researches have been done to many complicated nonlinear partial differential equations through travelling wave transformation. For example, Kai *et al* [43–45] discussed the bifurcation, critical condition and topological properties of the nonlinear dynamics by the complete discrimination system for the polynomial method, and some concrete examples such as perturbed Gardner's equation with high-order dispersion, Boussinesq-type equations and Ito-type coupled nonlinear wave equations indicate that this method not only can be used to get quantitative results, but also can conduct qualitative analysis of the nonlinear dynamic system.

The construction of this article is given as follows: The travelling wave transformation is given in §2, qualitative analysis and the existence of the periodic and soliton solutions are analysed in §3. All single travelling wave solutions are obtained to verify the conclusion and concrete examples under concrete parameters are also shown to ensure the existence of each solution in §4. Finally, a brief summary is drawn.

2. Travelling wave transformation

We consider the travelling wave transformation eq. (2) and transform eq. (1) from fractional partial differential equation to ordinary differential equation as

$$\Omega(x,t) = u(\xi)\exp(i\psi), \quad \xi = \tau\left(\rho\frac{t^\alpha}{\alpha} + \frac{x^\beta}{\beta}\right),$$
$$\psi = \omega\frac{t^\alpha}{\alpha} + \kappa\frac{x^\beta}{\beta} + \theta, \tag{2}$$

where ξ represents the amplitude of the wave, ω is the frequency, κ is the wave number and θ is the phase constant. Substituting eq. (2) in (1) yields

$$\frac{\partial^\alpha\Omega}{\partial t^\alpha} = (\tau\rho u' + i\omega u)\exp(i\psi),$$
$$\frac{\partial^\beta\Omega}{\partial x^\beta} = (\tau u' + i\kappa u)\exp(i\psi), \tag{3}$$
$$\frac{\partial^{2\beta}\Omega}{\partial x^{2\beta}} = (\tau^2 u'' + 2i\kappa\tau u' - \kappa^2 u)\exp(i\psi), \tag{4}$$
$$\frac{\partial^{3\beta}\Omega}{\partial x^{3\beta}} = (\tau^3 u''' + 3i\kappa\tau^2 u'' - 3\kappa^2\tau u' - i\kappa^3 u) \times\exp(i\psi), \tag{5}$$
$$\frac{\partial^\beta(|\Omega|^2\Omega)}{\partial x^\beta} = (3\tau u^2 u' + i\kappa u^3)\exp(i\psi),$$
$$\frac{\partial^\beta(|\Omega|^2)}{\partial x^\beta} = 2\tau u u', \tag{6}$$
$$\begin{aligned}&(\tau\rho u' + i\omega u)\exp(i\psi) - i[\eta_1(\tau^2 u'' + 2i\kappa\tau u' \\ &\quad -\kappa^2 u)\exp(i\psi) + \eta_2 u^3\exp(i\psi)] \\ &\quad -\eta_3(\tau^3 u''' + 3i\kappa\tau^2 u'' \\ &\quad -3\kappa^2\tau u' - i\kappa^3 u)\exp(i\psi) \\ &\quad -\eta_4(3\tau u^2 u' + i\kappa u^3)\exp(i\psi) \\ &\quad -2\eta_5\tau u^2 u'\exp(i\psi) = 0.\end{aligned} \tag{7}$$

Separating into real and imaginary parts, we get

$$-\tau(3\eta_4 + 2\eta_5)u^2u' + \tau(\rho + 2\eta_1\kappa + 3\eta_3\kappa^2)u' - \eta_3\tau^3 u''' = 0, \tag{8}$$
$$(\eta_2 - \eta_4\kappa)u^3 + (\omega + \eta_1\kappa^2 + \eta_3\kappa^3)u - \tau^2(\eta_1 + 3\eta_3\kappa)u'' = 0. \tag{9}$$

By integrating eq. (8) once and ignoring the constant of integration, we have

$$-\tau(3\eta_4 + 2\eta_5)u^3 + 3\tau(\rho + 2\eta_1\kappa + 3\eta_3\kappa^2)u - 3\eta_3\tau^3 u'' = 0. \tag{10}$$

Since eqs (9) and (10) are similar, equating their coefficients, we get

$$-\tau(3\eta_4 + 2\eta_5) = \eta_2 - \eta_4\kappa, \tag{11}$$
$$3\tau(\rho + 2\eta_1\kappa + 3\eta_3\kappa^2) = \omega + \eta_1\kappa^2 + \eta_3\kappa^3, \tag{12}$$
$$3\eta_3\tau = \eta_1 + 3\eta_3\kappa. \tag{13}$$

Multiplying u' on both sides of eq. (10), and integrating it on ξ again, we get

$$(u')^2 = a_4u^4 + a_2u^2 + a_0, \tag{14}$$

where

$$a_0 = \frac{2c}{3\eta_3\tau^3}, \quad a_2 = \frac{\rho + 2\eta_1\kappa + 3\eta_3\kappa^2}{\eta_3\tau^2},$$
$$a_4 = \frac{-3\eta_4 - 2\eta_5}{6\eta_3\tau^2},$$

and c is an integral constant.

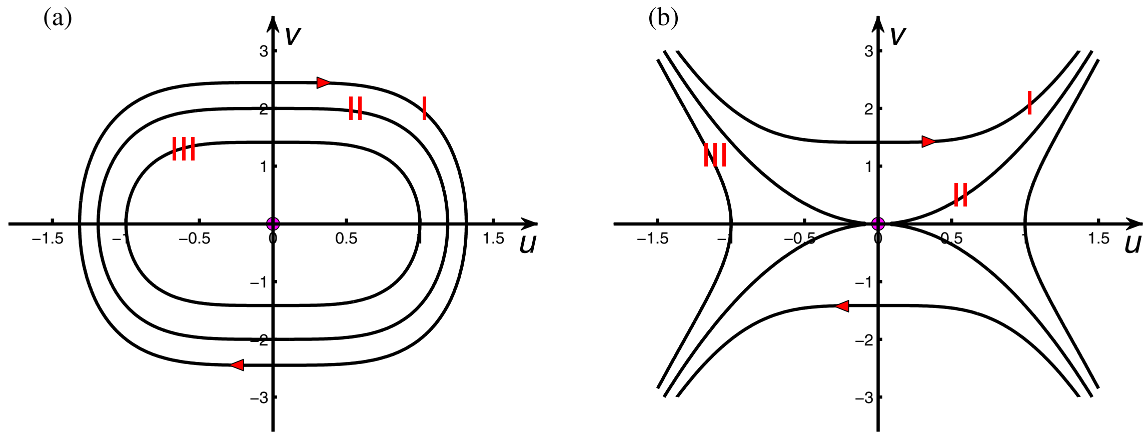

Figure 1. The global phase portraits of (15) in Case 1.1. (**a**) $a_4 = -2$, $a_2 = 0$ and (**b**) $a_4 = 2$, $a_2 = 0$.

3. Dynamic properties

In this section, we analyse eq. (14) qualitatively by transforming it to the equivalent plane dynamic system as follows:

$$\begin{cases} u' = v, \\ v' = 2a_4(u^3 + bu), \end{cases} \tag{15}$$

where $b = a_2/2a_4$. So its Hamilton energy function is expressed as

$$H(u, v) = \frac{v^2}{2} - \frac{a_4u^4 + a_2u^2 + a_0}{2}. \tag{16}$$

It is obvious that the potential energy function

$$U(u) = -\frac{a_4u^4 + a_2u^2 + a_0}{2}$$

in (16) is just the opposite of a fourth-order polynomial. Furthermore, we can deduce

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}\xi} &= \frac{\partial H}{\partial u}\frac{\partial u}{\partial \xi} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial \xi} \\ &= -2a_4(u^3 + bu)u' + vv' = -v'v + vv' = 0. \end{aligned} \tag{17}$$

From (17), it is shown that the Hamilton energy function (16) is an autonomous conservation system. So its contour lines can present the trajectories of the two-dimensional dynamic system (15). By denoting the complete discrimination system for the third-order polynomial

$$\Delta = -\frac{b^3}{27}, \tag{18}$$

and considering the roots of $U'(u) = -2a_4(u^3 + bu)$, we shall analyse qualitatively the dynamic properties of (15) through graphical representation.

Case 1.1. When $b = 0$, then $\Delta = 0$, $U'(u) = 0$ has a real root zero of multiplicity three, which gives

$$U'(u) = -2a_4u^3. \tag{19}$$

For example, when $a_4 = -2$, $a_2 = 0$, the dynamic system (15) has a centre $(0, 0)$, and its global phase portrait can be seen in figure 1a. However, when $a_4 = 2$, $a_2 = 0$, $(0, 0)$ is a cuspidal point which can be seen in figure 1b.

Case 1.2. When $b > 0$, then $\Delta < 0$, $U'(u) = 0$ has one single real root and a pair of conjugate roots, i.e.,

$$U'(u) = -2a_4(u - r_0)[(u - s)^2 + l^2], \tag{20}$$

where $r_0 + 2s = 0$. This situation is very similar to Case 1.1, and the corresponding figure can be seen in figures 2a and 2b.

Case 1.3. When $b < 0$, then $\Delta > 0$, $U'(u) = 0$ has three distinct real roots such that

$$U'(u) = -2a_4(u - r_1)(u - r_2)(u - r_3), \tag{21}$$

where $r_1 + r_2 + r_3 = 0$ and $r_1 > r_2 > r_3$.

When $a_4 = -1$, $a_2 = 2$, figure 3a has two closed orbits (I, II) and two homoclinic orbits (III, IV), which show the existence of periodic and bell-shaped soliton solutions. On the other hand, when $a_4 = 1$, $a_2 = -2$, figure 3b has a closed orbit (I) and two heteroclinic orbits (II, III), which indicate the existence of the periodic and kink soliton solutions.

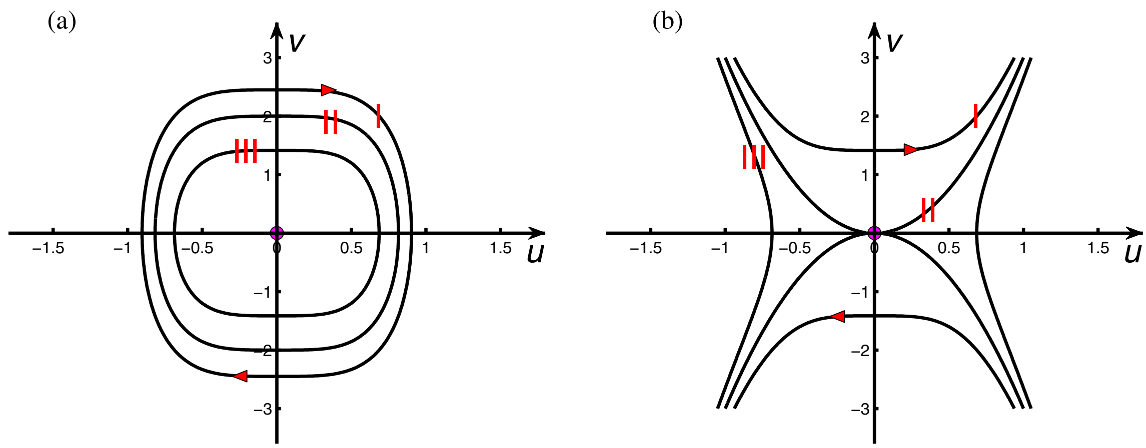

Figure 2. The global phase portraits of (15) in Case 1.2. (**a**) $a_4 = -1$, $a_2 = -8$ and (**b**) $a_4 = 1$, $a_2 = 8$.

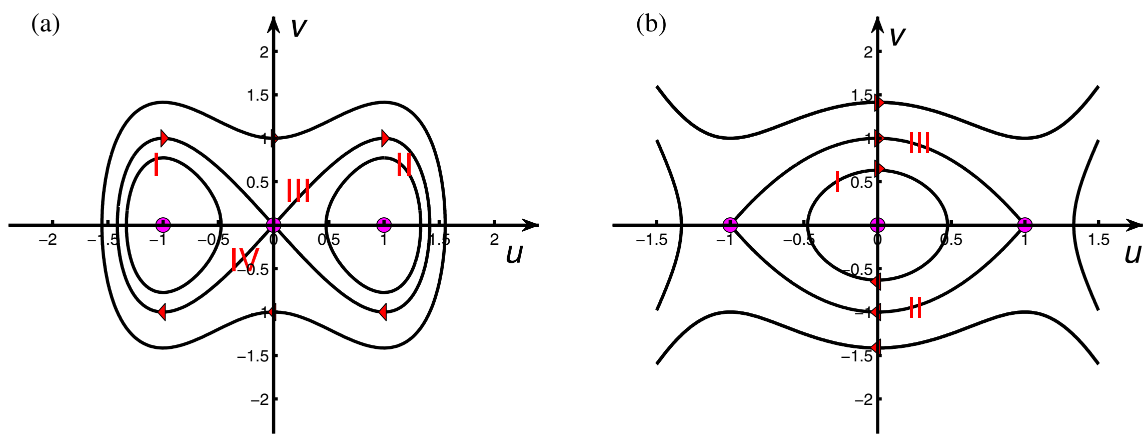

Figure 3. The global phase portraits of (15) in Case 1.3. (**a**) $a_4 = -1$, $a_2 = 2$ and (**b**) $a_4 = 1$, $a_2 = -2$.

4. Exact travelling wave solutions

In this section, we implement further transformation based on eq. (14) as follows:

If $a_4 > 0$, we set

$$w = (a_4)^{\frac{1}{4}} u, \quad \xi_1 = (a_4)^{\frac{1}{4}} \xi. \tag{22}$$

Then eq. (14) becomes

$$(w_{\xi_1})^2 = L(w) = w^4 + p w^2 + r, \tag{23}$$

where

$$p = \frac{a_2}{\sqrt{a_4}}, \quad r = a_0.$$

If $a_4 < 0$, we get

$$w = (-a_4)^{\frac{1}{4}} u, \quad \xi_1 = (-a_4)^{\frac{1}{4}} \xi. \tag{24}$$

Then eq. (14) turns into

$$(w_{\xi_1})^2 = -L(w) = -(w^4 + p w^2 + r), \tag{25}$$

where

$$p = \frac{-a_2}{\sqrt{-a_4}}, \quad r = -a_0.$$

We denote

$$\begin{aligned} &D_1 = 4, \ D_2 = -p, \ D_3 = -2p^3 + 8pr, \\ &D_4 = 4p^4 r - 32 p^2 r^2 + 64 r^3, \\ &E_2 = 9p^2 - 32 pr. \end{aligned} \tag{26}$$

Nine cases will be discussed according to the complete discrimination system for the fourth-order polynomial, respectively, in the following.

Case 2.1. When $D_2 > 0$, $D_3 > 0$ and $D_4 = 0$, $L(w) = 0$ has two single real roots and a real root with multiplicity two,

$$L(w) = (w - r_1)(w - r_2)(w - r_3)^2, \tag{27}$$

where $r_1 > r_2$, and we have

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}w}{(w - r_3)\sqrt{(w - r_1)(w - r_2)}}. \tag{28}$$

If $r_2 < r_3 < r_1$, we get

$$\pm(\xi_1 - \xi_0) = \frac{2}{d_1(r_3 - r_2)} \arctan \frac{\sqrt{\frac{w - r_1}{w - r_2}}}{d_1}, \tag{29}$$

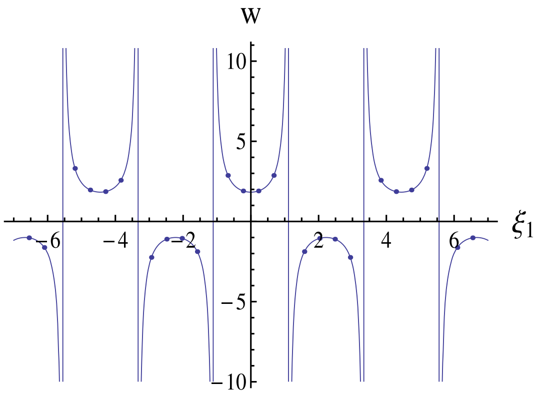

Figure 4. Triangle function periodic solution in Case 2.1 when $\xi_0 = 0$.

where

$$d_1 = \sqrt{\frac{r_1 - r_3}{r_3 - r_2}}.$$

When $w > r_1$ or $w < r_2$, w has the form

$$w = \frac{r_2 - r_1}{d_1^2 \tan^2\left(\frac{d_1(r_3-r_2)(\xi_1-\xi_0)}{2}\right) - 1} + r_2. \tag{30}$$

If $r_2 < r_1 < r_3$ or $r_2 < r_3 < r_1$, we have

$$\pm(\xi_1 - \xi_0) = \frac{1}{d_2(r_3 - r_2)} \ln\left|\frac{\sqrt{\frac{w-r_1}{w-r_2}} - d_2}{\sqrt{\frac{w-r_1}{w-r_2}} + d_2}\right|, \tag{31}$$

where

$$d_2 = \sqrt{\frac{r_1 - r_3}{r_2 - r_3}}.$$

When $w > r_1$ or $w < r_2$, w can be expressed as

$$w = \frac{r_2 - r_1}{d_2^2\left(\frac{2}{1 - \mathrm{e}^{\pm d_2(r_3-r_2)(\xi_1-\xi_0)}} - 1\right)^2 - 1} + r_2. \tag{32}$$

For example, when $\eta_1 = 6, \eta_2 = -4, \eta_3 = 1, \eta_4 = 1$, $\eta_5 = -\frac{5}{2}$, $\tau = 1$, $\kappa = -1$, $\omega = 4$, $\rho = 14$, $\theta = 0$, $c = 0$, we can get $p = -2, r = 0, r_1 = \sqrt{2}, r_2 = -\sqrt{2}$, $r_3 = 0$. If $w > \sqrt{2}$ or $w < -\sqrt{2}$, it yields a triangle function periodic solution

$$w = \frac{2\sqrt{2}}{1 - \tan^2\left(\frac{\sqrt{2}(\xi_1-\xi_0)}{2}\right)} - \sqrt{2} \tag{33}$$

and its corresponding figure is shown in figure 4.

If $\xi_0 = 0$, the exact solution of eq. (1) is

$$\Omega_1(x,t) = \left(\frac{2\sqrt{2}}{1 - \tan^2\left(\frac{7\sqrt{2}t^\alpha}{\alpha} + \frac{\sqrt{2}x^\beta}{2\beta}\right)} - \sqrt{2}\right) \times \exp\left(i\left(\frac{4t^\alpha}{\alpha} - \frac{x^\beta}{\beta}\right)\right). \tag{34}$$

Case 2.2. When $D_2 > 0$, $D_3 = D_4 = 0$ and $E_2 > 0$, $L(w) = 0$ has two double distinct real roots such that

$$L(w) = (w - r_1)^2(w - r_2)^2, \tag{35}$$

where $r_1 > r_2$, and it yields

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}w}{(w - r_1)(w - r_2)} = \frac{1}{r_1 - r_2} \ln\left|\frac{w - r_1}{w - r_2}\right|. \tag{36}$$

When $w > r_1$ or $w < r_2$, we have

$$w = \frac{r_2 - r_1}{\mathrm{e}^{\pm(r_1-r_2)(\xi_1-\xi_0)} - 1} + r_2 = \frac{r_2 - r_1}{2}\left(\coth\frac{\pm(r_1 - r_2)(\xi_1 - \xi_0)}{2} - 1\right) + r_2. \tag{37}$$

When $r_2 < w < r_1$, we get

$$w = \frac{r_2 - r_1}{-\mathrm{e}^{\pm(r_1-r_2)(\xi_1-\xi_0)} - 1} + r_2 = \frac{r_2 - r_1}{2}\left(\tanh\frac{\pm(r_1 - r_2)(\xi_1 - \xi_0)}{2} - 1\right) + r_2. \tag{38}$$

Obviously, eqs (37) and (38) are two solitary wave solutions. When $\eta_1 = 6$, $\eta_2 = -4$, $\eta_3 = 1$, $\eta_4 = 1$, $\eta_5 = -5/2$, $\tau = 1$, $\kappa = -1$, $\omega = 1$, $\rho = 13$, $\theta = 0$, $c = 3/2$, we get $p = -2$, $r = 1$, $r_1 = 1$, $r_2 = -1$. If $-1 < w < 1$, we have

$$w = \pm\tanh(\xi_1 - \xi_0) \tag{39}$$

and its corresponding graphs can be seen in figures 5a and 5b.

If $\xi_0 = 0$, the exact solution of eq. (1) is

$$\Omega_2(x,t) = \pm\tanh\left(\frac{13t^\alpha}{\alpha} + \frac{x^\beta}{\beta}\right) \times \exp\left(i\left(\frac{t^\alpha}{\alpha} - \frac{x^\beta}{\beta}\right)\right). \tag{40}$$

Remark. From figures 4 and 5, when $a_4 = 1$, $a_2 = -2$ which can decide $p = -2$, eqs (33) and (39) can verify the conclusions that eq. (14) has periodic and soliton solutions, and it shows that the qualitative results obtained in Case 1.3 are correct.

Case 2.3. When $D_i > 0$ $(i = 2, 3, 4)$, $L(w) = 0$ has four distinct real roots, namely

$$L(w) = (w - r_1)(w - r_2)(w - r_3)(w - r_4), \tag{41}$$

where $r_1 > r_2 > r_3 > r_4$. Then, we have

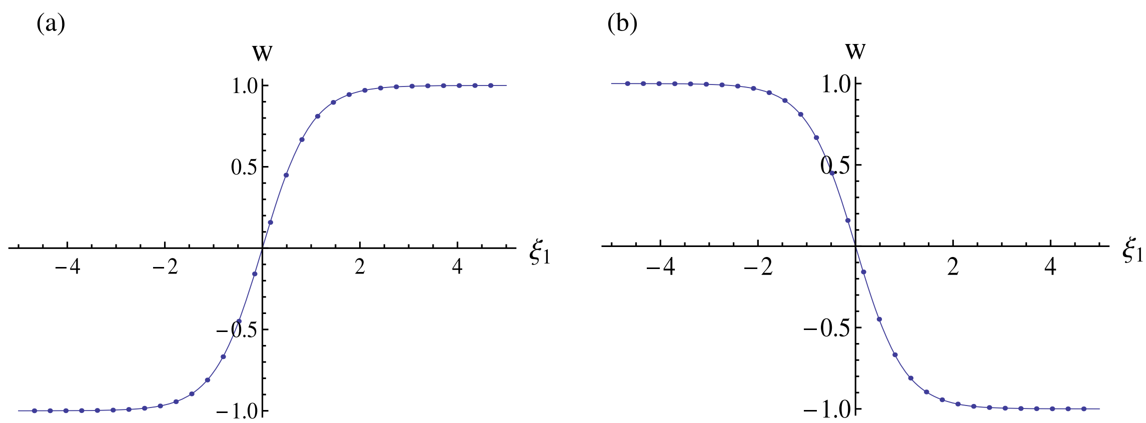

Figure 5. Solitary wave solutions in Case 2.2 when $\xi_0 = 0$. (**a**) Kink soliton solution and (**b**) antikink soliton solution.

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}w}{\sqrt{(w - r_1)(w - r_2)(w - r_3)(w - r_4)}}. \tag{42}$$

When $a_4 > 0$, if $w > r_1$ or $w < r_4$, by using eq. (42), we obtain

$$w = \frac{r_2(r_1 - r_4)\mathrm{sn}^2\left(\frac{\sqrt{(r_1-r_3)(r_2-r_4)}}{2}(\xi_1 - \xi_0), m\right) - r_1(r_2 - r_4)}{(r_1 - r_4)\mathrm{sn}^2\left(\frac{\sqrt{(r_1-r_3)(r_2-r_4)}}{2}(\xi_1 - \xi_0), m\right) - (r_2 - r_4)} \tag{43}$$

and if $r_3 < w < r_2$, then we have

$$w = \frac{r_4(r_2 - r_3)\mathrm{sn}^2\left(\frac{\sqrt{(r_1-r_3)(r_2-r_4)}}{2}(\xi_1 - \xi_0), m\right) - r_3(r_2 - r_4)}{(r_2 - r_3)\mathrm{sn}^2\left(\frac{\sqrt{(r_1-r_3)(r_2-r_4)}}{2}(\xi_1 - \xi_0), m\right) - (r_2 - r_4)}, \tag{44}$$

where

$$m = \sqrt{\frac{(r_1 - r_4)(r_2 - r_3)}{(r_1 - r_3)(r_2 - r_4)}}.$$

For $a_4 < 0$, if $r_2 < w < r_1$, we get

$$w = \frac{r_3(r_1 - r_2)\mathrm{sn}^2\left(\frac{\sqrt{(r_1-r_3)(r_2-r_4)}}{2}(\xi_1 - \xi_0), m\right) - r_2(r_1 - r_3)}{(r_1 - r_2)\mathrm{sn}^2\left(\frac{\sqrt{(r_1-r_3)(r_2-r_4)}}{2}(\xi_1 - \xi_0), m\right) - (r_1 - r_3)} \tag{45}$$

and if $r_4 < w < r_3$, then we have

$$w = \frac{r_1(r_3 - r_4)\mathrm{sn}^2\left(\frac{\sqrt{(r_1-r_3)(r_2-r_4)}}{2}(\xi_1 - \xi_0), m\right) - r_4(r_3 - r_1)}{(r_3 - r_4)\mathrm{sn}^2\left(\frac{\sqrt{(r_1-r_3)(r_2-r_4)}}{2}(\xi_1 - \xi_0), m\right) - (r_3 - r_1)}, \tag{46}$$

where

$$m = \sqrt{\frac{(r_1 - r_2)(r_3 - r_4)}{(r_1 - r_3)(r_2 - r_4)}}.$$

Expressions (43)–(46) are elliptic function double periodic solutions.

For example, when $\eta_1 = 6, \eta_2 = -4, \eta_3 = 1, \eta_4 = 1, \eta_5 = -5/2, \tau = 1, \kappa = -1, \omega = -8, \rho = 10, \theta = 0, c = 6$, we can get $p = -5, r = 4, r_1 = 2, r_2 = 1, r_3 = -1, r_4 = -2$, and when $w > 2$ or $w < -2$, w can be shown as

$$w = \frac{4\mathrm{sn}^2\left(\frac{3}{2}(\xi_1 - \xi_0), \frac{8}{9}\right) - 6}{4\mathrm{sn}^2\left(\frac{3}{2}(\xi_1 - \xi_0), \frac{8}{9}\right) - 3}. \tag{47}$$

If $\xi_0 = 0$, the exact solution of eq. (1) is

$$\Omega_3(x,t) = \frac{4\mathrm{sn}^2\left(\frac{3}{2}\left(\frac{10t^\alpha}{\alpha} + \frac{x^\beta}{\beta}\right), \frac{8}{9}\right) - 6}{4\mathrm{sn}^2\left(\frac{3}{2}\left(\frac{10t^\alpha}{\alpha} + \frac{x^\beta}{\beta}\right), \frac{8}{9}\right) - 3} \times \exp\left(-i\left(\frac{8t^\alpha}{\alpha} + \frac{x^\beta}{\beta}\right)\right). \tag{48}$$

Case 2.4. When $D_2 > 0$, $D_3 = D_4 = E_2 = 0$, $L(w) = 0$ has a real root of multiplicity three and a real root of multipicity one in the following form:

$$L(w) = (w - r_1)^3(w - r_2). \tag{49}$$

For $a_4 > 0$, then

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}w}{(w - r_1)\sqrt{(w - r_1)(w - r_2)}} = \frac{2}{r_2 - r_1}\sqrt{\frac{w - r_2}{w - r_1}}. \tag{50}$$

When $w > r_1$, $w > r_2$ or $w < r_1$, $w < r_2$, we have

$$w = \frac{4(r_1 - r_2)}{(r_2 - r_1)^2(\xi_1 - \xi_0)^2 - 4} + r_1. \tag{51}$$

For $a_4 < 0$, then

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}w}{(w - r_1)\sqrt{(w - r_1)(r_2 - w)}} = \frac{2}{r_2 - r_1}\sqrt{\frac{r_2 - w}{w - r_1}}. \tag{52}$$

When $w > r_1$, $w < r_2$ or $w < r_1$, $w > r_2$, we get

$$w = \frac{4(r_1 - r_2)}{-(r_2 - r_1)^2(\xi_1 - \xi_0)^2 - 4} + r_1. \tag{53}$$

Equations (51) and (53) are rational solutions, but the condition cannot be satisfied in practice, and so eq. (1) has no solution in this case.

Case 2.5. When $D_2 = D_3 = D_4 = 0$, $L(w) = 0$ has a real root zero of multiplicity four, which gives

$$L(w) = w^4 \tag{54}$$

and we have

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}w}{w^2} = -\frac{1}{w}. \tag{55}$$

It is very easy to derive the solution of eq. (55) such that

$$w = \mp(\xi_1 - \xi_0)^{-1}, \tag{56}$$

which is also a rational solution.

For example, when $\eta_1 = 6, \eta_2 = -4, \eta_3 = 1, \eta_4 = 1, \eta_5 = -5/2, \tau = 1, \kappa = -1, \omega = 7, \rho = 15, \theta = 0, c = 0$, we get $p = 0, r = 0$, which yields

$$w = -(\xi_1 - \xi_0)^{-1}. \tag{57}$$

If $\xi_0 = 0$, the exact solution of eq. (1) is

$$\Omega_5(x,t) = -\left(\frac{15t^\alpha}{\alpha} + \frac{x^\beta}{\beta}\right)^{-1} \times \exp\left(i\left(\frac{7t^\alpha}{\alpha} - \frac{x^\beta}{\beta}\right)\right). \tag{58}$$

Case 2.6. When $D_2 < 0$, $D_3 = D_4 = 0$, $L(w) = 0$ has a pair of double conjugate complex roots, namely

$$L(w) = \left((w - s)^2 + l^2\right)^2, \tag{59}$$

where $l > 0$. We get

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}w}{(w - s)^2 + l^2} = \frac{1}{l}\arctan\frac{w - s}{l}. \tag{60}$$

The solution is

$$w = \pm l\tan\left(l(\xi_1 - \xi_0)\right) + s, \tag{61}$$

which is a triangular function periodic solution.

For example, when $\eta_1 = 6, \eta_2 = -4, \eta_3 = 1, \eta_4 = 1, \eta_5 = -5/2, \tau = 1, \kappa = -1, \omega = 13, \rho = 17, \theta = 0, c = 3/2$, we get $p = 2, r = 1, s = 0, l = 1$ and eq. (61) becomes

$$w = \tan(\xi_1 - \xi_0). \tag{62}$$

If $\xi_0 = 0$, the exact solution of eq. (1) is

$$\Omega_6(x,t) = \tan\left(\frac{17t^\alpha}{\alpha} + \frac{x^\beta}{\beta}\right) \times \exp\left(i\left(\frac{13t^\alpha}{\alpha} - \frac{x^\beta}{\beta}\right)\right). \tag{63}$$

Case 2.7. When $D_2D_3 \leq 0$ and $D_4 > 0$, $L(w) = 0$ has two pairs of conjugate complex roots, i.e.

$$L(w) = \left((w - s_1)^2 + l_1^2\right)\left((w - s_2)^2 + l_2^2\right), \tag{64}$$

where $l_1 \geq l_2 > 0$. So we have

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}w}{\sqrt{\left((w - s_1)^2 + l_1^2\right)\left((w - s_2)^2 + l_2^2\right)}}. \tag{65}$$

Let

$$a=s_1c+l_1d,\ b=s_1d-l_1c,\ c=-l_1-\frac{l_2}{m_1},$$
$$d=s_1-s_2,\ e=\frac{(s_1-s_2)^2+l_1^2+l_2^2}{2l_1l_2},$$
$$m_1=e+\sqrt{e^2-1}. \tag{66}$$

Then we have

$$w=\frac{a\mathrm{sn}(\pm\eta(\xi_1-\xi_0),m)+b\mathrm{cn}(\pm\eta(\xi_1-\xi_0),m)}{c\mathrm{sn}(\pm\eta(\xi_1-\xi_0),m)+d\mathrm{cn}(\pm\eta(\xi_1-\xi_0),m)}, \tag{67}$$

where

$$m=\sqrt{\frac{m_1^2-1}{m_1^2}}$$

and

$$\eta=\frac{l_2\sqrt{(c^2+d^2)(m_1^2c^2+d^2)}}{c^2+d^2}$$

and eq. (67) is an elliptic function solution.

For example, when $\eta_1=6, \eta_2=-4, \eta_3=1, \eta_4=1, \eta_5=-5/2, \tau=1, \kappa=-1, \omega=13, \rho=1, \theta=0, c=27/2$, we get $p=2, r=9, s_1=1, s_2=-1$ and $l_1=l_2=\sqrt{3}$. Then, we get

$$a=\frac{2\sqrt{3}}{3},\quad b=6,\quad c=-\frac{4\sqrt{3}}{3},$$
$$d=2,\quad \eta=\frac{3\sqrt{91}}{7},\quad m=\frac{2\sqrt{2}}{3}. \tag{68}$$

So, w can be expressed as

$$w=\frac{\frac{2\sqrt{3}}{3}\mathrm{sn}(\frac{3\sqrt{91}}{7}(\xi_1-\xi_0),\frac{2\sqrt{2}}{3})+6\mathrm{cn}(\frac{3\sqrt{91}}{7}(\xi_1-\xi_0),\frac{2\sqrt{2}}{3})}{-\frac{4\sqrt{3}}{3}\mathrm{sn}(\frac{3\sqrt{91}}{7}(\xi_1-\xi_0),\frac{2\sqrt{2}}{3})+2\mathrm{cn}(\frac{3\sqrt{91}}{7}(\xi_1-\xi_0),\frac{2\sqrt{2}}{3})}. \tag{69}$$

If $\xi_0=0$, the exact solution of eq. (1) is

$$\Omega_7(x,t)=\frac{\frac{2\sqrt{3}}{3}\mathrm{sn}(\frac{3\sqrt{91}}{7}\left(\frac{t^\alpha}{\alpha}+\frac{x^\beta}{\beta}\right),\frac{2\sqrt{2}}{3})+6\mathrm{cn}(\frac{3\sqrt{91}}{7}\left(\frac{t^\alpha}{\alpha}+\frac{x^\beta}{\beta}\right),\frac{2\sqrt{2}}{3})}{-\frac{4\sqrt{3}}{3}\mathrm{sn}(\frac{3\sqrt{91}}{7}\left(\frac{t^\alpha}{\alpha}+\frac{x^\beta}{\beta}\right),\frac{2\sqrt{2}}{3})+2\mathrm{cn}(\frac{3\sqrt{91}}{7}\left(\frac{t^\alpha}{\alpha}+\frac{x^\beta}{\beta}\right),\frac{2\sqrt{2}}{3})}\exp\left(i\left(\frac{13t^\alpha}{\alpha}-\frac{x^\beta}{\beta}\right)\right). \tag{70}$$

Case 2.8. When $D_2D_3\geq 0$ and $D_4<0$, $L(w)=0$ has two distinct real roots and a pair of conjugate complex roots which is given by

$$L(w)=(w-r_1)(w-r_2)\left((w-s)^2+l^2\right), \tag{71}$$

where $r_1>r_2$ and $l>0$. Then, the following equation can be obtained:

$$\pm(\xi_1-\xi_0)=\int\frac{\mathrm{d}w}{\sqrt{(w-r_1)(w-r_2)\left((w-s)^2+l^2\right)}}. \tag{72}$$

Let

$$a=\frac{1}{2}(r_1+r_2)c-\frac{1}{2}(r_1-r_2)d,$$
$$b=\frac{1}{2}(r_1+r_2)d-\frac{1}{2}(r_1-r_2)c,$$
$$c=r_1-s-\frac{l}{m_1},$$
$$d=r_1-s-lm_1,$$
$$e=\frac{l^2+(r_1-s)(r_2-s)}{l(r_1-r_2)},$$
$$m_1=e+\sqrt{e^2+1}. \tag{73}$$

Then, we get the solution as

$$w=\frac{a\,\mathrm{cn}(\eta(\xi_1-\xi_0),m)+b}{c\,\mathrm{cn}(\eta(\xi_1-\xi_0),m)+d}, \tag{74}$$

where

$$\eta=\frac{\sqrt{\mp 2lm_1(r_1-r_2)}}{2mm_1},$$
$$m=\sqrt{\frac{1}{1+m_1^2}}.$$

Equation (74) is also an elliptic function solution.

For example, when $\eta_1=6, \eta_2=-4, \eta_3=1, \eta_4=1, \eta_5=-5/2, \tau=1, \kappa=-1, \omega=16, \rho=18, \theta=0, c=-6$, we get $p=3, r=-4, r_1=1, r_2=-1, s=0, l=2$, and we have

$$a=3,\quad b=c=0,\quad d=-3,\quad e=\frac{3}{4},\quad m_1=2,$$
$$m=\frac{\sqrt{5}}{5},\quad \eta=\sqrt{5}. \tag{75}$$

So, the solution is

$$w = -\mathrm{cn}\left(\sqrt{5}(\xi_1 - \xi_0), \frac{\sqrt{5}}{5}\right). \tag{76}$$

If $\xi_0 = 0$, the exact solution of eq. (1) is

$$\Omega_8(x,t) = -\mathrm{cn}\left(\sqrt{5}\left(\frac{18t^\alpha}{\alpha} + \frac{x^\beta}{\beta}\right), \frac{\sqrt{5}}{5}\right) \times \exp\left(i\left(\frac{16t^\alpha}{\alpha} - \frac{x^\beta}{\beta}\right)\right). \tag{77}$$

Case 2.9. When $D_2D_3 < 0$ and $D_4 = 0$, $L(w) = 0$ has a real root of multiplicity two and a pair of conjugate complex roots, which can be expressed as

$$L(w) = (w - r_0)^2\left((w - s)^2 + l^2\right), \tag{78}$$

where $l > 0$. We have

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}w}{(w - r_0)\sqrt{(w-s)^2 + l^2}} = \frac{1}{\delta}\ln\left|\frac{\delta\sqrt{(w-s)^2+l^2} - l^2 + (s-r_0)(w-s)}{l(w - r_0)}\right|, \tag{79}$$

where

$$\delta = \sqrt{(r_0 - s)^2 + l^2}.$$

Therefore, we obtain

$$w = \frac{2l(\mu + \delta)(\eta \pm \sqrt{\mu^2 + \eta^2 - \delta^2})}{(\mu + \delta)^2 - (\eta \pm \sqrt{\mu^2 + \eta^2 - \delta^2})^2} + s, \tag{80}$$

where

$$\mu = (s - r_0)\mathrm{e}^{\pm\delta(\xi_1 - \xi_0) + l}$$

and

$$\eta = l\mathrm{e}^{\pm\delta(\xi_1 - \xi_0)} - (s - r_0).$$

Equation (80) is a solitary wave solution, but the exact solution of eq. (1) does not exist for the same reason as for Case 2.4.

5. Conclusion

This paper studies the high-order nonlinear time-space fractional Schrödinger equation arising from monomode optical fibres. By the basic theories of qualitative analysis and the complete discrimination system for the polynomial method, the topological structure is established and the existence of the periodic and soliton solutions is predicted. Each type of travelling wave solutions obtained by the complete discrimination system for the polynomial method is used to prove the conclusion explicitly in qualitative analysis. The results indicate that the method adopted here is efficient and powerful and it will be widely used.

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