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PT-symmetric potentials with imaginary asymptotic saturation

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Abstract. We point out that PT-symmetric potentials $V_{PT}(x)$ having imaginary asymptotic saturation, $V_{PT}(x = \pm\infty) = \pm i V_1$, $V_1 \in \mathbb{R}$ are devoid of scattering states and spectral singularity. We show the existence of real (positive and negative) discrete spectrum both with and without complex conjugate pair(s) of eigenvalues (CCPEs). If the eigenstates are arranged in the ascending order of the real part of the discrete eigenvalues, the initial states have few nodes but latter ones oscillate fast. Both real and imaginary parts of $\psi_n(x)$ vanish asymptotically, and $|\psi_n(x)|$ are nodeless. For the CCPEs, these are asymmetric and peaking on the left (right) and for real energies these are symmetric and peaking at the origin. For CCPEs $E_\pm$, the eigenstates $\psi_\pm$ follow the interesting property, $|\psi_+(x)| = N|\psi_-(-x)|$, $N \in \mathbb{R}^+$.

1. Introduction

Complex PT-symmetric potentials $V_{PT}(x)$ which are divergent functions of x are known to have real discrete spectrum below or above exceptional points g_n [1–4]. Complex PT-symmetric potentials which are asymptotically convergent, are not exceptional in this regard [5–7]. The PT-symmetric quantum mechanics has been reformulated in terms of pseudo-Hermiticity [8] and biorthogonal basis [9].

The first $V_{PT}(x)$ with imaginary asymptotic saturation was proposed and studied in the form of complex Rosen–Morse potential [6,10]. In this case, a real discrete spectrum was revealed which is null in the absence of the real attractive part of the potential. More recently, the Dirac delta potential with a simple imaginary step added to it, is found to have one real discrete eigenvalue. A new model, a square well potential with imaginary saturation, has been solved and shown to degenerate to the result of Dirac delta model in limit $a \to 0$ [11]. However, two contradictory ideas have been professed: in Rosen–Morse case the real discrete spectrum comes from the poles [10] of transmission amplitude $t(E)$, but in the square well and the Dirac well cases the discrete

eigenvalue is obtained as a zero [11] of the reflection amplitude $r(E)$.

In this work, we bring attention to the fact that when a potential has imaginary asymptotic saturation $V_{PT}(\pm\infty) = i V_1$, $V_1 \in \mathbb{R}$, the propagating plane waves $e^{\pm ikx}$, $k = \sqrt{2mE}/\hbar$ can no more be solutions of the Schrödinger equation even in the asymptotic region. One will instead have $\psi(x) \sim e^{(\pm\alpha + i\beta)x}$ on the left and right sides of the potential. Their current densities do not equilibrate as they become local (function of x). In other words, current density is not preserved because of the lack of unitarity. Consequently, it is not really possible to define reflection $r(E)$ and transmission $t(E)$ amplitudes. Seeing the discrete spectrum sometime as poles of $t(E)$ and other times as zeros of $r(E)$ has been ad-hoc [10,11]. Moreover, reflection amplitude $t(E)$ being non-reciprocal (unequal) [12] for the left and right injection, one may also get two sets of zeros to give two discrete spectra for such PT-symmetric potentials.

For simplicity, scattering potentials are the ones that vanish asymptotically or saturate. In various fields, to account for a loss of flux (absorption), a negative definite potential $V_i(x)$ is added which vanishes

asymptotically. In this case, unitarity ($R + T \neq 1$) is lost. Instead, we get $R + T + A = 1$ (A stands for absorption probability) [12]. The presence of imaginary part of the potential results in non-unitarity. This is standard and well known, but here our topic of discussion is that when imaginary potential saturates to $\pm i V_1$ asymptotically the asymptotic solutions are of the type $\psi(x) = N e^{(\pm k_1 + \pm i k_2)x}$ (see eq. (2)). These give rise to current densities $J(x)$ which either vanish or diverge when $x \to \pm\infty$. With these current densities, we cannot define reflection $R(E)$ or transmission $T(E)$ coefficients. Also, see §7, for a discussion on the Rosen–Morse potential.

Fundamental theorem of algebra asserts that if a polynomial equation $f(x) = 0$ is real on real line ($\forall x \in \mathbb{R}$) and if it has roots, they will either be real or complex conjugate pairs (non-real). For example, roots of $f(E) = (E^2 - 4)(E^2 - vE + 9) = 0$ will be two real and two CCPEs, if $v < 6$. If $v = 6$, there are three distinct real roots and if $v > 6$, then there are four real roots. Here $v = 6$ is the critical value of v, $v_c = 6$. This theorem can as well be extended to other (non-polynomial) equations. We conjecture this as the essence of Hermitian and complex PT-symmetric quantum mechanics. In the former, we have only real eigenvalues and in the latter, in addition to real, we can also have CCPEs. This can be seen in the models discussed in [1–7,10,11] earlier. In the following models too, we shall see that the eigenvalue equations $f(E)$ in eqs (4), (8), (13) and (17) are real on real line.

For over two decades, PT-symmetric quantum mechanics [1] has been successful yet there are unproven conjectures which keep coming out true, time and again. Even real eigenvalues and real energy-band structure remain to be proved in general in a model-independent way. According to conjecture [13] the complex eigenstates in PT-symmetric potentials have $|\psi_n(x)|$ as a node-less function of x. In the present work, this inspires us to study $|\psi_n(x)|$, which are node-less and vanish asymptotically irrespective of whether the eigenvalue is real or non-real. We shall find two new and additional features of these eigenstates.

In this paper, we revisit the earlier model of stepped Dirac delta potential [11] in our terms. We examine three new PT-symmetric potentials (see eqs (6), (10) and (15)) with imaginary asymptotic saturation to bring out their new crucial features which have been missed out earlier. We also discuss exactly solvable complex PT-symmetric Rosen–Morse [6,10] potential which has purely real discrete spectrum.

2. Imaginary asymptotic saturation of a potential

The Schrödinger equation (SE) for a general potential is written as

$$\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}[E - V(x)]\psi(x) = 0. \tag{1}$$

Here, energy E is in general non-real (complex) and so are the potentials which are non-Hermitian and PT-symmetric. In the following, we solve (1) for various PT-symmetric potentials with imaginary asymptotic saturation to obtain two kinds of eigenvalues: complex conjugate pairs or real (positive and negative) or both. In general, the boundary condition to be imposed over eigenfunctions is

$$\psi(x \sim -\infty) = A e^{K_1 x}, \ \psi(x \sim \infty) = B e^{-K_2 x},$$
$$K_1 = \sqrt{-\frac{2m}{\hbar^2}(E + i V_1)}, \ K_2 = \sqrt{-\frac{2m}{\hbar^2}(E - i V_1)}. \tag{2}$$

Alternatively, if we define $K_1 = k_1 + i k_2$ and $K_2 = k_1 - i k_2$, where $k_1, k_2 > 0$, we see asymptotically converging solutions on both sides, on the left $\psi(x \sim -\infty) \sim A e^{(k_1 + i k_2)x}$ and on the right $\psi(x \sim \infty) \sim B e^{-(k_1 - i k_2)x}$. Interestingly, at any energy, all the solutions will be converging asymptotically and so these potentials cannot have scattering states. Only at special discrete energies, they will be continuous and differentiable everywhere and specially at $x = 0$. The definition of complex conjugate K_1 and K_2 (2) will remain the same for all the potentials to be discussed afterwards.

A one-dimensional Hermitian potential may have resonances which are called Gamow–Sigert states, where $E = E_n - i \Gamma_n/2$ or $k = a_n - i b_n$, $a_n, b_n > 0$. At a resonance, the asymptotic solutions of SE become outgoing waves on the left and the right like $e^{\mp ikx}$. A spectral singularity (SS) [14] in a non-Hermitian potential is a resonance at a real discrete energy embedded in the continuum of positive energies. It is also called 'zero-width resonance'. As discussed above, the non-Hermitian potentials with imaginary asymptotic saturation have asymptotic convergence instead of asymptotically outgoing waves. Hence, they cannot support any spectral singularity. Also, it appears that in a non-Hermitian potential there is a scarcity of SSs. Usually, one or two such zero-width resonances [15] are found in a fixed potential. Surprisingly, Ghatak *et al* [16] mistook the bound states of a variant of non-Hermitian Rosen–Morse potential as multiple spectral singularities, just because its eigenvalues are real and positive.

Recently, it has been conjectured [17] that such a fixed potential has at most one SS as discussed in §2, and

if it exists, it sets the upper (or rough upper) bound to the real part of complex conjugate pairs of eigenvalues. Real discrete eigenstates of the present class of potentials oscillate but they vanish asymptotically and so they cannot represent the spectral singularity state [14]. Recently, a different non-Hermitian parametrisation of non-PT-symmetric version of complex Rosen–Morse potential: $V(x) = iV_0 \operatorname{sech}^2 x + V_1 \tanh x$, $V_0, V_1 \in \mathbb{R}$ [18] has been employed to investigate the phenomena of SS and coherent perfect absorption (CPA). Existence of single SS at $E = E_*$ and CPA in the time-reversed potential at this energy has been observed.

3. Simple stepped dirac delta potential

First, we see the absence of spectrum in the simple imaginary step potential (see the solid back line in figure 1a)

$$V(x \le 0) = -iV_1, \quad V(x > 0) = iV_1, \qquad (3)$$

as the solutions of (1) for (3) which vanish at $x = \mp\infty$, respectively. Matching the wave functions (2) and their derivative at $x = 0$, we get $K_1(E) = -K_2(E) \implies 2k_1(E) = 0$ which is an equation of E not having any real or complex root for $V_1 \ne 0$. It is not difficult to see the absence of any root E for it. Hence, no (discrete or continuous) energy state is possible in this complex PT-symmetric potential which is two piece. Similarly, for the simple diffused step $(V(x) = 2ic \tanh x, c \in \mathbb{R})$ the discrete spectrum will be null. This can be seen in the model of Rosen–Morse potential when $s = 0$ in eqs (18)–(20).

For various eigenvalue equations $f(E) = 0$ (see eqs (4), (8), (13) and (17)) which are forthcoming, we device a method for finding complex roots. For instance, for potential (2), we write $f(E) = K_1(E) + K_2(E) = 2k_1(E)$ and $E = \mathcal{E} + i\gamma$. We plot $\mathbb{R}f(\mathcal{E}, \gamma) = 0$ (blue contours) and $\mathbb{I}f(\mathcal{E}, \gamma) = 0$ (red contours) and note their coordinates in the $(\mathcal{E}, \gamma)$ plane where they intersect each other. It must be remarked that only when blue and red contours intersect, it indicates the correct root of $f(E)$ in a given case. This method does not yield any eigenvalue in model (3). In this regard, see the contour plots in figures 2–5, for eigenvalue equations (8), (13) and (17) which are implicit functions of E. The eigenvalues of the type $\mathcal{E}_n \pm i\gamma_n$ are called complex conjugate pair of eigenvalues (CCPEs: $E_\pm = \mathcal{E} \pm i\gamma$).

For a simple demonstration, let us perturb the Dirac delta potential (well) $V(x) = -V_0\delta(x)$, $V_0 > 0$ by the simple imaginary step (3). Then, by taking the solution of Schrödinger equation (1) for this case as $\psi_<(x) = Ae^{K_2 x}$ and $\psi_>(x) = Be^{-K_1 x}$, we match them at $x = 0$ and mismatch their derivative at $x = 0$ taking care of delta singularity and we get $F(E) = K_1(E) + K_2(E) -$

$V_0 = 0$, $V_0 > 0$ or more transparently

$$f(E) = \sqrt{-E - iV_1} + \sqrt{-E + iV_1}$$
$$- \sqrt{\frac{2m}{\hbar^2}} V_0 (= U_0) = 0. \qquad (4)$$

Strictly, when $V_0, U_0 > 0$, we can square (4) for rationalisation and after a simple manipulation, thankfully, we get an explicit expression for the eigenvalue as

$$E = \frac{4V_1^2 - U_0^4}{4U_0^2}, \quad U_0 > 0. \qquad (5)$$

As it is well known, the delta well has single eigenvalue at $E = -U_0^2/4$. By the perturbation due to (3), it remains single, positive or negative.

We depict the scenario of the single eigenvalue in the Argand plane $(\mathcal{E}, \gamma)$. In all the calculations we propose to use $2m = 1 = \hbar^2$. We take $V_0 = 4$ and $V_1 = 0$ in figure 2a to see that the blue and red contours cut only once and we get $E = -4$, but in figure 2b when $V_1 = 5$, we get a single intersection at $E = -2.4375$ (consistent with (5)). Remarkably, no other intersecting or non-intersecting contours are present here, and any other real or CCPEs are ruled out for this model. The delta well perturbed by imaginary step (3) has been studied as a scattering problem in [11] and the eigenvalue equation (5) has been obtained from the zero of the reflection amplitude. We remark that this method is ad-hoc and arbitrary, since these potentials discussed in §2, do not admit scattering states. In figure 2c, note that $|\psi|$ is node-less and vanishes asymptotically. However, $\mathbb{R}\psi$ and $\mathbb{I}\psi$ may cut x-axis (like a node does).

4. A linear three-piece step

Linear three-piece PT-symmetric step potential (see the red-dotted line in figure 1a) with imaginary asymptotic saturation can be created as

$$V(x \le -a) = -iV_1, \quad V(-a < x < a) = iV_1 x/a,$$
$$V(x \ge a) = iV_1, V_1, a \in \mathbb{R}. \qquad (6)$$

The solution of Schrödinger equation (1) for this potential can be written as

$$\psi(x \le -a) = Ae^{K_1 x},$$
$$\psi(-a < x < a) = CAi(h(x)) + DBi(h(x)),$$
$$\psi(x \ge a) = Be^{-K_2 x}. \qquad (7)$$

Here

$$h(x) = \frac{2m}{g\hbar^2}\left(E - iV_1\frac{x}{a}\right), \quad g = \left(\frac{2mV_1}{\hbar^2 a}\right)^{2/3}.$$
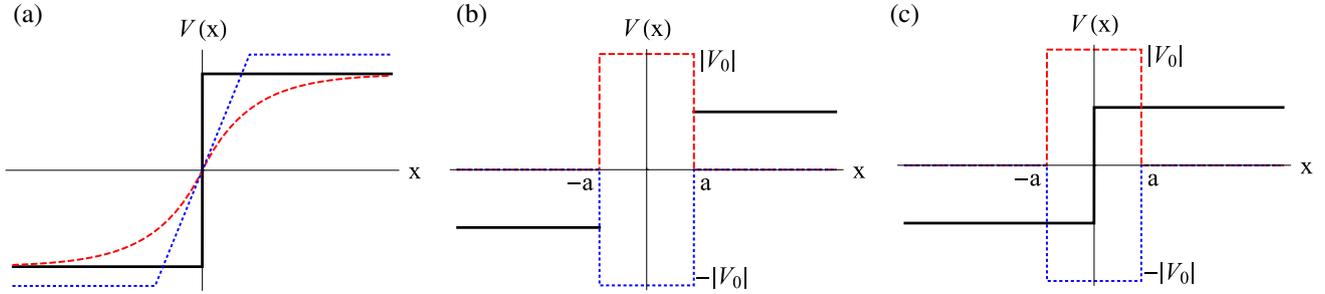
(a)

(b)

(c)

Figure 1. (**a**) Purely imaginary simple step (black solid line), eq. (3); linear step (blue-dotted line), eq. (6) and tanh x step (red-dashed line). (**b**) Winged rectangular potential, eq. (10). (**c**) Stepped rectangular potential, eq. (15). The black lines represent the strength $\pm i V_1$ on the right/left.
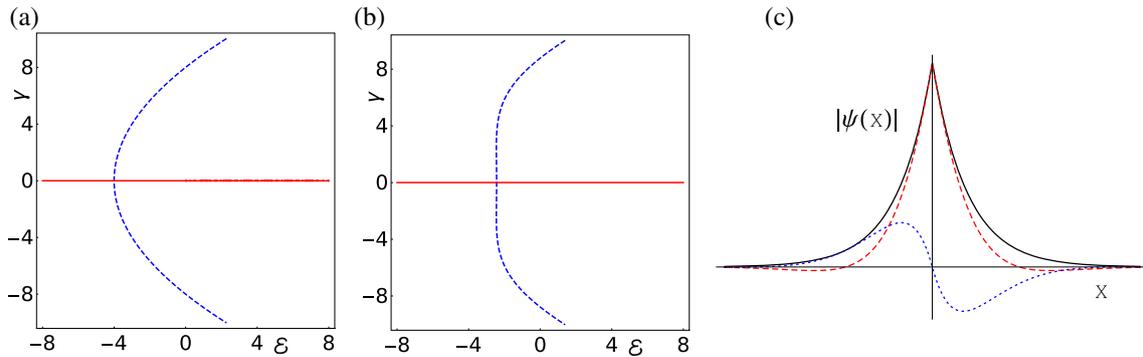
(a)

(b)

(c)

Figure 2. Contour plots in Argand plane $(\mathcal{E}, \gamma)$ for the stepped delta potential (3) (the blue and red lines cross only once). (**a**) $V_0 = -4$, $V_1 = 0$, $E_0 = -4$, (**b**) $V_0 = -4$, $V_1 = 5$, $E_0 = -2.4375$ and (**c**) $|\psi|$ (black solid line), $\mathbb{R}\psi$ (red-dashed line) and $\mathbb{I}\psi$ (blue-dotted line) for case (**b**).

$Ai(h)$ and $Bi(h)$ are called the Airy functions. Let us introduce

$$h_{1,2} = \frac{2m}{g\hbar^2}(E \pm i V_1).$$

By matching these solutions and their derivative at $x = \pm a$, we get linear equations in A, B, C and D. By eliminating them, we get an eliminant which gives the eigenvalue equation as

$$
\begin{aligned}
f(E) = {}& K_1 K_2 [Ai(h_2)Bi(h_1) - Ai(h_1)Bi(h_2)] \\
& - i K_1 \sqrt{g}[Ai'(h_2)Bi(h_1) - Ai(h_1)Bi'(h_2)] \\
& - i K_2 \sqrt{g}[Ai'(h_1)Bi(h_2) - Ai(h_2)Bi'(h_1)] \\
& - g[Ai'(h_1)Bi'(h_2) - Ai'(h_2)Bi'(h_1)] = 0.
\end{aligned}
\tag{8}
$$

The invariance of (8) under the interchange of (1 ↔ 2) complex conjugate $h_1 \leftrightarrow h_2$ ensures that $f(E)$ is real, which by the fundamental theorem of algebra will have real or complex conjugate roots. In figure 3, the contour plots for the case $V_1 = 5$ and $a = 2$, depict one CCPE and two real positive discrete spectra which are unbounded from above. $|\psi_n(x)|$ for the complex energy $E = E_{1-}$ is an asymmetric function peaking on the left of $x = 0$ (see figure 3b). Not shown in the figure is

the case for $E = E_{1+}$ when $\psi_{1+}(x)$ peaks on the right of $x = 0$. For real discrete energies, it is symmetric (see figure 1c), peaking at $x = 0$. At higher eigenvalues, oscillations increase but vanish asymptotically. We find that eigenstates corresponding to CCPE follow the invariance property

$$|\psi_-(x)| = N|\psi_+(+x)|, \quad N \in \mathbb{R}^+. \tag{9}$$

Presently, what is important here is that N is independent of x, and its dependence on other parameters is not known.

5. Winged rectangular potential

Earlier, the real discrete spectrum of antisymmetric and purely imaginary antisymmetric square well potential has been studied where the potential converges on both sides [19]. More interestingly, the Gamow–Sigert states with complex discrete spectrum $E_n - i\Gamma_n/2$, $\Gamma_n > 0$ discussed in [20] for both Hermitian and non-Hermitian short-range potentials by making the depth V_0 of the square well potential non-real as $V_0 = V_R + V_I$ are not similar to the results in [1–7]. In contrast to the
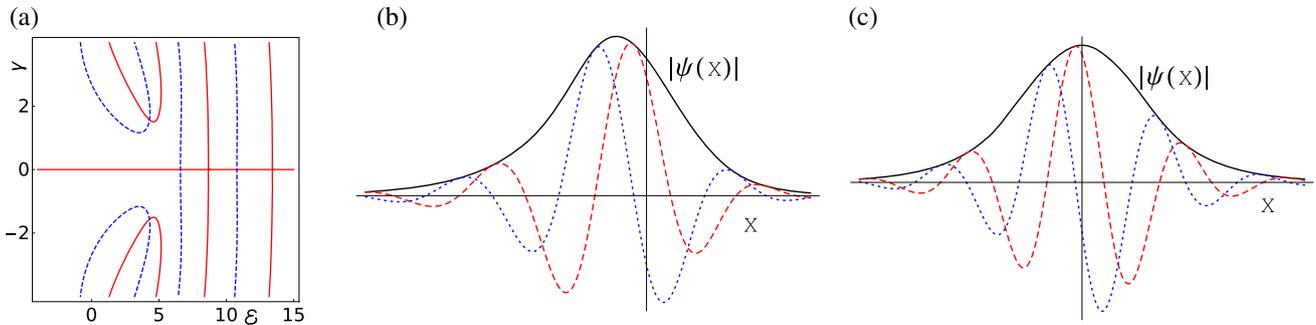
Figure 3. Linear three-piece step (6) potential having both real discrete spectrum and complex conjugate pairs of eigenvalues, when $V_1 = 5$ and $a = 2$. (**a**) CCPE ($E_{1\pm} = 4.2959 \pm 1.5653i$) along with unbounded real spectrum ($E_2 = 6.5952$, $E_3 = 10.7814$, ...) are plotted here. (**b**) $|\psi(x)|$ (black solid line), $\mathbb{R}\psi$ (blue-dotted line) and $\mathbb{I}(\psi(x))$ (red-dashed line) are plotted for $E = E_{1-}$, (**c**) same as in (**b**) for $E = E_2$. Notice for real energy, $|\psi|$ peaks at $x = 0$ and for complex energy E_{1-} the peak shifts to the left of $x = 0$.

aforementioned asymptotically vanishing square wells [19], here we discuss a winged rectangular potentials (well/barrier) which is an antisymmetric perturbation (wings for $|x| \geq a$) to the real Hermitian rectangular well/barrier constant imaginary saturation as the title of this paper suggests.

We can construct a winged PT-symmetric rectangular well/barrier potential with imaginary asymptotic saturation (wings for $|x| \geq a$) as

$$V(x < -a) = -iV_1,$$
$$V(-a \leq x \leq a) = -V_0,$$
$$V(x \geq a) = iV_1, \quad V_0, V_1, a \in \mathbb{R}. \tag{10}$$

Here $V_0 > 0$ means a well (see figure 1b). By defining

$$p = \sqrt{\frac{2m}{\hbar^2}(E + V_0)},$$

we can write the full solution of (1) for (10) as

$$\psi(x < -a) = Ae^{K_1 x},$$
$$\psi(-a \leq x \leq a) = C \sin px + D \cos px,$$
$$\psi(x \geq a) = Be^{-K_2 x}. \tag{11}$$

By matching these solutions and their derivatives at $x = \pm a$, we eliminate A, B, C and D to get the energy quantisation condition as

$$f(E) = p \cos pa[-(K_1 + K_2)p \cos pa$$
$$+ (-K_1 K_2 + p^2) \sin pa]$$
$$+ \sin pa[(-K_1 K_2 + p^2)p \cos pa$$
$$+ (K_2 p^2 + K_1 p^2) \sin pa)] = 0 \tag{12}$$

which simplifies to

$$f(E) = (K_1 + K_2)p \cos 2pa$$
$$+ (K_1 K_2 - p^2) \sin 2pa = 0. \tag{13}$$

This equation is invariant under the exchange of the subscripts 1 and 2. This means $f(E, K_1, K_2)$ is a real function commonly in both Hermitian and PT-symmetric quantum mechanics. So, by the fundamental theorem of algebra, the equation $f(E, K_1, K_2) = 0$ will have either real or complex conjugate roots. This in turn is the essence of complex PT-symmetric Hamiltonians [1], that they have either real or complex conjugate eigenvalues when the PT-symmetry is exact and broken, respectively, if this transition from real to CCPEs is always below or above a critical [1–7] value of a parameter of the potential. These critical values are also called exceptional points of the non-Hermitian potential. Further, for the Hermitian rectangular well potential $V_1 = 0$, $K_1 = K_2 = K = \sqrt{-2mE}/\hbar$ and

$$p = P = \sqrt{\frac{2m}{\hbar^2}(E + V_0)}.$$

In this special case, we recover the well-known eigenvalue formula (14) for even and odd states from (13) as

$$\tan 2Pa = \frac{2KP}{P^2 - K^2} \Leftrightarrow \tan Pa = \frac{K}{P}$$

and

$$\tan Pa = -\frac{P}{K}. \tag{14}$$

We fix $V_1 = 6$ and $a = 2$, and vary V_0 to find the intersection of blue and red curves which show real eigenvalues in the Argand plane $(\mathcal{E}, \gamma)$. In figure 4a, when $V_0 = 0$, one negative and three positive eigenvalues can be seen. In figure 4b, for $V_0 = 5$, three negative and two positive eigenvalues can be seen, because in this case the real part of the potential is a well in $(-a, a)$. In figure 4c, for $V_0 = -5$, three real positive eigenvalues can been seen when $E > 4$. In this case, it is a barrier of height 5 units in $(-a, a)$. We find that there are no
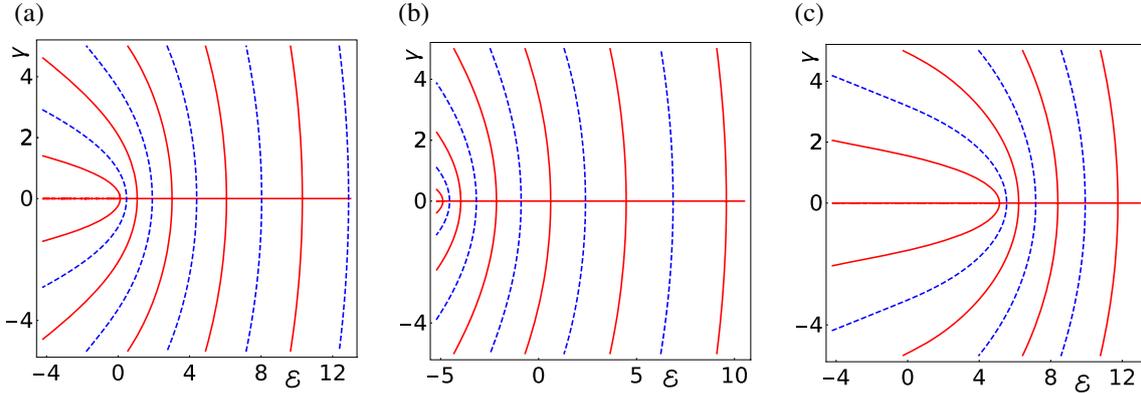
Figure 4. Winged rectangular potential (10) when $a = 2$, $V_1 = 6$ for (**a**) $V_0 = 0$, (**b**) $V_0 = 5$, (**c**) $V_0 = -5$. An eigenvalue occurs, if the blue curve cuts the red one. There are no CCPEs and there are many positive real eigenvalues on the higher side (not shown fully here).

CCPEs in the three cases of the winged potential and the spectrum contains many positive eigenvalues at higher energies. $|\psi_n(x)|$ are not plotted in these cases. They are symmetric, node-less and vanish asymptotically.

6. Stepped rectangular potential

We perturb rectangular well/barrier by the imaginary simple step potential (like we did for the Dirac delta potential in §3). We define the stepped potential as

$$V(x \leq -a) = -i V_1,$$
$$V(-a \leq x \leq 0) = -V_0 - i V_1,$$
$$V(0 \leq x \leq a) = -V_0 + i V_1,$$
$$V(x \geq a) = i V_1, \quad V_0, V_1, a \in \mathbb{R} \tag{15}$$

(see figure 1c). The solution of Schrödinger equation can be written as

$$\psi(x \leq -a) = A e^{K_1 x},$$
$$\psi(-a \leq x \leq 0) = B \sin px + C \cos px,$$
$$\psi(0 \leq x \leq a) = D \sin qx + C \cos qx,$$
$$\psi(x \geq a) = G e^{-K_2 x},$$
$$P_1 = \sqrt{2m(E + V_0 + i V_1)},$$
$$P_2 = \sqrt{2m(E + V_0 - i V_1)}. \tag{16}$$

These solutions are already continuous at $x = 0$ and for differentiability here we set $Bp = Dq$. Hence, there are only four unknowns in the linear simultaneous equations which come from the continuity and differentiability at $x = \pm a$. The consistency of these equations require a 4×4 determinant to vanish and we get the eliminant giving us the real eigenvalue equation for discrete energies as

$$f(E) = [(P_2^2 - K_1 K_2) P_1 \cos a P_1 \sin a P_2$$
$$- (K_1 + K_2) P_1 P_2 \cos a P_1 \cos a P_2]$$

$$+ [(P_1^2 - K_1 K_2) P_2 \sin a P_1 \cos a P_2$$
$$+ (K_1 P_2^2 + K_2 P_1^2) \sin a P_2 \sin a P_2] = 0. \tag{17}$$

Once again the eigenvalue equation (17) is invariant jointly under $1 \rightarrow 2$ (complex conjugation). Hence, it is real to possess real or CC roots. This transition from real to CCPEs is always below or above a critical [1–7] value of a parameter of the potential. These critical values are also called exceptional points of the non-Hermitian potential. When $K_1 = K_2 = K$ and $P_1 = P_2 = P$, eq. (17) degenerates to (14).

For stepped rectangular potential (15) we fix $a = 2$. In figure 5a when $V_0 = 5$, $V_1 = 0$, there are three real eigenvalues in $-5 < E < 0$. We find that, when $V_1 \lesssim 1.75$, there is at least one negative eigenvalue. Next to this, we get only CCPEs in the well as in figure 5b. For $V_1 = 2$, $V_0 = 5$ the initial real eigenvalues disappear and we get two CCPEs and two real positive eigenvalues, where lines of different colours are intersecting. So $V_{1c} = 1.75$ is a critical (exceptional point) value of V_1 above which negative eigenvalues disappear. In figure 5c, when $V_0 = -5$, $V_1 = 2$, we have a barrier in $(-a, a)$ and we get one CCPE and positive real eigenvalues.

7. The Rosen–Morse potential

Now we discuss the solved example of complex PT-symmetric Rosen–Morse potential [6,10]

$$V(x) = -s(s+1)\operatorname{sech}^2 x + 2ic \tanh x; \quad s \in \mathbb{R}^+, c \in \mathbb{R}. \tag{18}$$

Its real, discrete energy spectrum is,

$$E_n = -(n - s)^2 + \frac{c^2}{(n - s)^2}, \quad n = 0, 1, 2, 3, \dots < s, \tag{19}$$
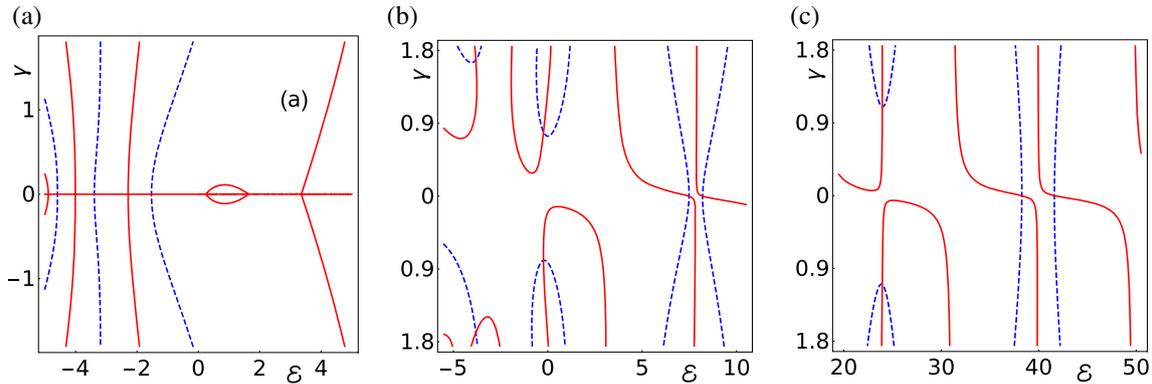
Figure 5. Stepped rectangular potential (15) for $a = 2$. (**a**) $V_0 = 5$, $V_1 = 0$ and there are three real eigenvalues in $-5 < E < 0$, (**b**) when $V_1 = 2$, the initial real eigenvalues disappear and we get two CCPEs and two real positive eigenvalues, where lines of different colours are intersecting and (**c**) $V_0 = -5$; $V_1 = 2$ and we get one CCPE and positive real eigenvalues at higher energies.
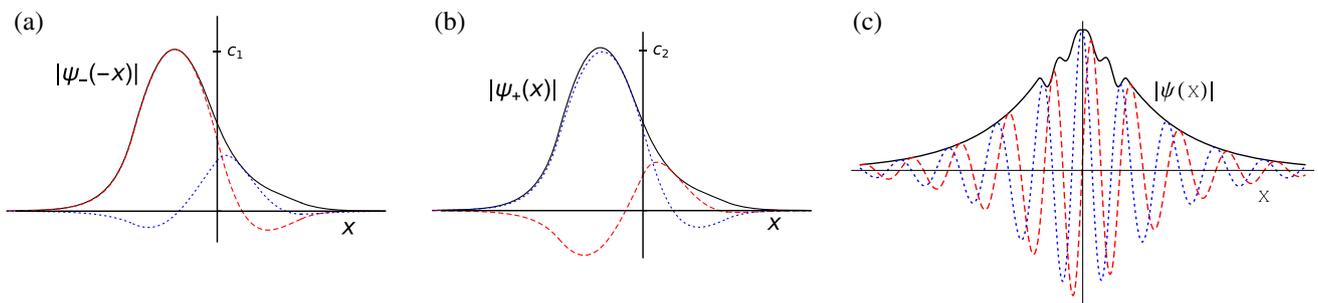

Figure 6. The wave functions for the case of figure 5b. Note that $|\psi_-|$ and $|\psi_+|$ correspond to the first CCPEs: $E_{\mp} = -3.80 \mp 1.68i$ in (**a**), (**b**). Here maximum numbers on the y-axis are $c_1 = 0.041$ and $c_2 = 0.271$ so that $N = c_1/c_2$. Eventually, we have $|\psi_-(-x)| = 0.151|\psi_+(+x)|$, the property (9), where $N = 0.151$. (**c**) is for real positive eigenvalue $E = 8.20$ where $|\psi(x)|$ is node-less and symmetric about $x = 0$.

with energy eigenstates as,

$$\psi(x) = A \, \text{sech}^{(s-n)} x e^{-icx/(s-n)}$$
$$\times P_n^{s-n+ic/(s-n),\,s-n-ic/(s-n)}(\tanh x), \quad (20)$$

where $P_n^{\alpha,\beta}(z)$ is the Jacobi's n degree polynomials in z. One interesting feature of (19) is that by making c larger, the negative energies can be pushed to positive energies. A large value of c makes the eigenfunction $\psi(x)$ more oscillatory. Even if c is small, the closeness of s to an integer brings in the same effect (see figure 7d). For instance, for $s = 3.2$, there will be four discrete energy states with eigenvalues as $E_0 = -10.14$, $E_1 = -4.63$, $E_2 = -0.74$, $E_3 = 24.96$, and in figure 7, we have plotted the energy eigenstates ($|\psi|$ and $\mathbb{R}\psi$) corresponding to these four discrete energies. Earlier, these have been welcomed as normal real discrete energies of a complex PT-symmetric potential. But surprisingly, more recently all such bound-state eigenvalues just being positive have been mistaken to be multiple spectral singularities. As mentioned in §2 at

a spectral singularity [12] the states become asymptotically out going on both sides. On the contrary, note that the part sech^{s-n}, $s < n$ (20), dampens the states asymptotically on both sides.

However, such a large and positive value for the last eigenvalue would attract one's attention. Similar to the case of a Hermitian potential, the higher eigenstates have large number of nodes and hence an oscillatory behaviour in the real and imaginary parts of $\psi_n(x)$. But as per the conjecture [12], $|\psi_n(x)|$ would be node-less in figures 6c and 7d.

The positive discrete energy states discussed here are: $\psi(x \sim -\infty) = Ae^{(\alpha+i\beta)x}$ and $\psi(x \sim \infty) = Be^{(-\alpha+i\beta)x}$. For example, the eigenstate $\psi(x) = Ae^{ix} \, \text{sech} \, x$ (20) of the PT-symmetric Rosen–Morse potential $V(x) = -2 \, \text{sech}^2 x - 2i \tanh x$ (18), has the position-dependent current density $J = (\hbar^2 |A|^2/\mu) \, \text{sech}^2 x$ which also vanishes asymptotically. At an energy slightly different from their discrete energy eigenvalue, these states may diverge at large distances or if they converge, they become non-differentiable at the joints of the potential. This is the general feature
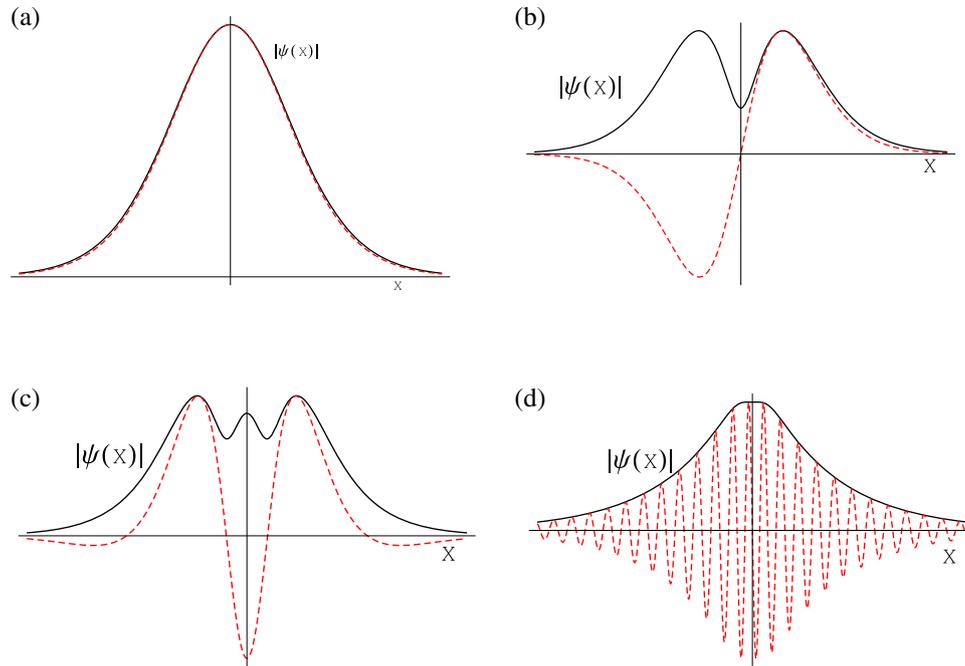
Figure 7. $|\psi|$ (black solid line) and $\mathbb{R}\psi$ (red-dashed line) of the first four eigenstates of the complex PT-symmetric potential (18) for $s = 3.2$ and $c = 1$. The energy eigenvalues are: $E_0 = -10.14$ (**a**), $E_1 = -4.63$ (**b**), $E_2 = -0.74$ (**c**), $E_3 = 24.96$ (**d**). Note the distinctive feature of fast oscillations in the last state with positive energy eigenvalue.

of the discrete energy eigenstates of the PT-symmetric eigenstates when the potential has imaginary asymptotic saturation.

8. Discussion and conclusion

It turns out that PT-symmetric potentials with imaginary asymptotic saturation are strictly devoid of scattering states. They not only vanish asymptotically but also their current density is position-dependent which vanishes asymptotically. This also rules out the existence of spectral singularity (zero-width resonance) in them. Earlier studies [10,11,16] have ignored these points and even found reflection and transmission amplitudes which, being inconsistent, do not degenerate to the correct eigenvalues [21] of the Hermitian Rosen–Morse potential (18) when c is set equal to zero or made imaginary. The potentials (6), (10) and (15) with imaginary asymptotic saturation can have real discrete spectrum with or without complex conjugate pairs (CCPEs) of eigenvalues. Earlier work [11] has overlooked the existence of CCPEs in them.

We find that $|\psi(x)|$ for real discrete eigenstates for this class of PT-symmetric potentials (4), (8), (11) and (16) are again integrable, symmetric and node-less as conjectured in ref. [12] for other kinds of PT symmetric potentials [1–7]. We further conjecture that, for CCPEs

($E_\mp$) the eigenstates $\psi_\mp$ satisfy property (9). Further, the states $|\psi_\mp(x)|$ corresponding to CCPEs ($E_\mp$), peak on the left (right) of $x = 0$ (see figures 3b, 6a and 6b). This interesting discriminatory feature of spontaneously broken PT-symmetry has been overlooked so far. We welcome a model-independent explanation of this phenomenon and property (9). If the PT-symmetry is exact, the eigenvalue is real, then $|\psi(x)|$ are symmetric functions of x, see e.g., figures 2c, 3c, 6c and 7.

It will not be surprising that the shift of the peak of $|\psi_\mp|$ on the left (right) and the interesting invariance property of $|\psi_\pm(x)|$ proposed here in eq. (9) for broken PT-symmetry, in general, could be harnessed experimentally in optics where complex PT-symmetric mediums have already thrown novel effects and surprises [22]. Moreover, new exactly solvable models presented here would attract attention in this regard.

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