Pramana – J. Phys. (2022) 96:141
https://doi.org/10.1007/s12043-022-02387-y

Conformal elliptic Universe

JOSÉ M CERVERÓ

Departamento de Física Fundamental, Universidad de Salamanca, 37008 Salamanca, Spain
E-mail: cervero@usal.es

MS received 31 August 2021; revised 27 January 2022; accepted 17 March 2022

Abstract. A model for cosmology that is conformal invariant from the beginning is presented. Breakdown of the conformal symmetry produces gravitation with parameters of mass. The system initially contains the Hilbert Lagrangian for general relativity plus quadratic terms in the Ricci tensor and in the scalar curvature. All these terms are supposed to create a space–time which cannot exist without gravity. So, masses appear when conformal invariance is broken. Massless Yang–Mills fields appear also as fundamental fields in the Lagrangian. An exact classical solution exists if a conformal scalar boson is introduced also in a conformal manner that couples gravity to the Yang–Mills sector. The paper claims that before the breakdown the model is renormalisable due to its conformal invariant nature.

Keywords. Space–time structure; cosmology.

PACS Nos **02.30.Ik; 02.90.+p; 03.65.Ge**

1. Introduction

Three universal constants associated with fundamental physics, the Planck constant ($\hbar$), the speed of light (c) and the size of the unit cell of space–time (L_p) are proposed. The well-known values of our simple units are

$$\hbar = 6.582119569 \times 10^{-16}\,\mathrm{eV}\cdot\mathrm{s}$$

$$c = 299792458\,\mathrm{m}\cdot\mathrm{s}^{-1}$$

$$L_p = 1.6162 \times 10^{-35}\,\mathrm{m}$$

$$\hbar c = 1.97326963 \times 10^{-7}\,\mathrm{eV}\cdot\mathrm{m}.$$

Space–time is then a physical object, not a mathematical construction existing only in our minds. It must always be considered as a net with a unit of cell equal to the Planck spacing. What physicists try always to do is to model this strange object in a way that may eventually confirm the observations in the first and final stages of the beginning of the Universe. We propose that the Universe began in a way that contains all the resources to account for gravity ($G = L_P^2 c^3/\hbar$) online as a consequence of the physical nature of space–time. So, gravity couples to fields even in a very early epoch when masses do not yet exist [1,2]. Only the fields come into being. For obvious reasons, we shall call this a conformally symmetric Universe in which masses are not yet present. The educated reader might also worry about the electric charge. Note however that the unit of charge is also a derived unit as can be very well described by

$$e^2 \approx \frac{\hbar c}{137},$$

where 137 is a dimensionless number.

However, one can still take advantage of the fundamental features of local field theory. In this framework, L_P is 20 orders of magnitude smaller than the size of, for example, any known hadrons. So, local field theory plays the role of an 'effective theory' and one can do in principle calculations and using Lagrangians that describe the physics from small distances to very large cosmological scale although one has to be careful in some instances. Non-Abelian Yang–Mills field can be introduced in the same manner as the scalar curvature and the Ricci tensor but always without a mass.

2. The model

The non-Abelian Yang–Mills fields appear also in the same manner as the effective local mathematical structures of this non-continuous space–time. With these hypotheses, one can write the most general conformal invariant Lagrangian as

Published online: 20 July 2022

$$\mathbf{S} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\lambda^2}{24}\varphi^4 - \frac{1}{12}\mathbf{R}\varphi^2 - \frac{f^2}{24}\mathbf{R^2} - \frac{h^2}{12}g^{\mu\rho}g^{\nu\sigma}\mathbf{R}_{\mu\nu}\mathbf{R}_{\rho\sigma} + \frac{1}{2}\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma}\}\right], \quad (1)$$

where the only undefined field is a massless scalar φ which is just a way to make the model consistent without breaking conformal invariance. The equations of motion are [3–5]:

$$-\frac{\varphi^2}{12}\left[\mathbf{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathbf{R}\right] + \frac{f^2}{12}\left[\mathbf{R}_{;\mu;\nu} - g_{\mu\nu}\mathbf{R}^{;\lambda}_{;\lambda} - \mathbf{R}\{\mathbf{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathbf{R}\}\right] + \frac{h^2}{12}\left[\mathbf{R}_{;\mu;\nu} - \frac{1}{2}g_{\mu\nu}\mathbf{R}^{;\lambda}_{;\lambda} - \mathbf{R}^{;\lambda}_{\mu\nu;\lambda} - 2\mathbf{R}^{\rho\sigma}\left[\mathbf{R}_{\rho\mu\sigma\nu} - \frac{1}{4}g_{\mu\nu}\mathbf{R}_{\rho\sigma}\right]\right] + \frac{1}{2}\theta^{\varphi}_{\mu\nu} + \frac{1}{2}\theta^{\mathbf{YM}}_{\mu\nu} = 0, \quad (2)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\right] - \frac{1}{6}\mathbf{R}\varphi + \frac{\lambda^2}{6}\varphi^3 = 0, \quad (3)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}\mathbf{F}_{\alpha\beta}\right] - g^{\mu\alpha}g^{\nu\beta}[\mathbf{F}_{\alpha\beta}, \mathbf{A}_\mu] = 0, \quad (4)$$

where the semicolon {; } in (2) indicates the covariant derivative. The conformally invariant Yang–Mills field $\mathbf{F}_{\mu\nu}$ [7] and the equally conformally invariant energy–momentum tensor for the scalar φ [8] are

$$\theta^{\mathbf{YM}}_{\mu\nu} = -2\mathbf{Tr}\left[g^{\rho\sigma}\mathbf{F}_{\mu\rho}\mathbf{F}_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}\mathbf{F}_{\rho\alpha}\mathbf{F}_{\sigma\beta}\right] \quad (5)$$

$$\theta^{\varphi}_{\mu\nu} = \left(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + \frac{\lambda^2}{24}\varphi^4\right) + \frac{1}{6}(g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta - \nabla_\mu\nabla_\nu)\varphi^2. \quad (6)$$

3. The geometry of the hypertorus $\mathbf{S^3} \otimes \mathbf{S^1}$

For the metric $g_{\mu\nu}$, we shall be working with the well-known cosmological Robertson–Walker metric of the Einstein equations, whose line element reads [6,9,10] as

$$\mathrm{d}^2\mathbf{s_{RW}} = \mathrm{d}^2t - \mathcal{A}^2(t) \times \left[\frac{\mathrm{d}^2r}{1-Kr^2} + r^2(\mathrm{d}^2\theta + \sin^2\theta\,\mathrm{d}^2\phi)\right] \quad \text{with} \quad K = +1. \quad (7)$$

To find conformal solutions of the entire systems (2)–(6) we use conformal coordinates

$$t_\pm = t \pm r \quad (8)$$

$$\omega = \arctan t_+ + \arctan t_-$$
$$\hat{\omega} = \arctan t_+ - \arctan t_- \quad (9)$$

in such a manner that the coordinates are homeomorphic to the $\mathbf{S^3} \otimes \mathbf{S^1}$ manifold:

$$\sin\omega = \frac{t_+ + t_-}{(1+t_+^2)^{\frac{1}{2}}(1+t_-^2)^{\frac{1}{2}}}; \quad \sin\hat{\omega} = \frac{t_+ - t_-}{(1+t_+^2)^{\frac{1}{2}}(1+t_-^2)^{\frac{1}{2}}} \quad (10)$$

$$\cos\omega = \frac{1 - t_+t_-}{(1+t_+^2)^{\frac{1}{2}}(1+t_-^2)^{\frac{1}{2}}}; \quad \cos\hat{\omega} = \frac{1 + t_+t_-}{(1+t_+^2)^{\frac{1}{2}}(1+t_-^2)^{\frac{1}{2}}} \quad (11)$$

and

$$r = \sin\hat{\omega}; \qquad \mathrm{d}r = \cos\hat{\omega}d\hat{\omega}; \qquad \mathrm{d}t = \mathcal{A}(\omega)\mathrm{d}\omega. \quad (12)$$

Then, the line element transforms in the following manner:

$$\mathrm{d}^2\mathbf{s_{RWC}} = \mathcal{A}^2(\omega)\left[\mathrm{d}^2\omega - \mathrm{d}^2\hat{\omega} - \sin^2\hat{\omega}(\mathrm{d}^2\theta + \sin^2\theta\,\mathrm{d}^2\phi)\right] \quad (13)$$

or

$$\{\text{diag}\}\mathbf{g}_{\mu\nu} = \mathcal{A}^2(\omega)\{1; -1; -\sin^2\hat{\omega}; -\sin^2\hat{\omega}\sin^2\theta\} \quad (14)$$

where

$\{\mu\nu\} = \{\omega, \hat{\omega}, \theta, \phi\}$.

For the scalar field we propose the general ansatz:

$$\varphi(\omega) = \frac{\sqrt{6}}{\lambda}\frac{f(\omega)}{\mathcal{A}(\omega)} \quad (15)$$

which in combination with field equations (2)–(6), leads to a unique condition for $f(\omega)$: a second-order differential equation of the form:

$$\frac{\mathrm{d}^2f(\omega)}{\mathrm{d}\omega^2} + f(\omega) - f^3(\omega) = 0$$

whose solutions are Jacobian elliptic functions. All equations of motion are then satisfied and the fields take the following form:

$$\varphi(\omega) = \frac{2\sqrt{3}}{\lambda\mathcal{A}(\omega)}\frac{1}{(1+m^2)^{\frac{1}{2}}}\left[\frac{\mathbf{dn}(\Omega|m^2)}{\mathbf{cn}(\Omega|m^2)}\right], \tag{16}$$

$$\mathbf{B}_i = \frac{1}{2}\frac{i\sigma_i}{\mathcal{A}^2(\omega)}\frac{1}{(1+m^2)} \times\left[\frac{\mathbf{dn^2}(\Omega|m^2)}{\mathbf{cn^2}(\Omega|m^2)} + m^2\frac{\mathbf{cn^2}(\Omega|m^2)}{\mathbf{dn^2}(\Omega|m^2)}\right], \tag{17}$$

$$\mathbf{E}_i = \frac{1}{2}\frac{\sigma_i}{\mathcal{A}^2(\omega)}\frac{1}{(1+m^2)} \times\left[\frac{\mathbf{dn^2}(\Omega|m^2)}{\mathbf{cn^2}(\Omega|m^2)} - m^2\frac{\mathbf{cn^2}(\Omega|m^2)}{\mathbf{dn^2}(\Omega|m^2)}\right], \tag{18}$$

where

$$\Omega = \frac{\omega-\omega_0}{(1+m^2)^{\frac{1}{2}}}. \tag{19}$$

The solution takes a simple form over the compact torus $\mathbf{S^3}\otimes\mathbf{S^1}$ and it has been constructed in a fully conformal manner. Now we should remember the conformal tensor and the Gauss–Bonet topological invariant. These quantities have the form:

$$\mathbf{C^2}_{\mu\nu\rho\sigma} = \mathbf{R^2}_{\mu\nu\rho\sigma} - 2\mathbf{R^2}_{\mu\nu} + \frac{1}{3}\mathbf{R^2} \tag{20}$$

$$\mathcal{G} = \mathbf{R^2}_{\mu\nu\rho\sigma} - 4\mathbf{R^2}_{\mu\nu} + \mathbf{R^2}. \tag{21}$$

As the solution is conformally invariant, the conformal tensor vanishes. We rewrite action (1) as

$$\mathbf{S} = \int \mathrm{d}^4x\sqrt{-g} \times\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\lambda^2}{24}\varphi^4 - \frac{1}{12}\mathbf{R}\varphi^2 - \frac{\gamma^2}{12}\mathbf{R^2} + \frac{1}{2}\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma}\}\right] - \frac{h^2}{24}\int \mathrm{d}^4x\sqrt{-g}\ \ \mathbf{C^2}_{\mu\nu\rho\sigma} + \frac{h^2}{24}\int \mathrm{d}^4x\sqrt{-g}\,[\mathcal{G}], \tag{22}$$

where the following relationships must hold among the couplings:

$$\frac{f^2}{2} + \frac{h^2}{3} = \gamma^2 \quad \text{and} \quad 1 + f^2 + h^2 = \frac{1}{\lambda^2} \tag{23}$$

which reduce the number of couplings of the theory [10]. Now, the term containing the quadratic conformal tensor vanishes for these solutions and the Gauss–Bonet topological invariant is just a well-defined number that has no effect on the equations of motion. Hence, the action is restricted to the first line of (22). The breaking of conformal invariance is obtained by giving a constant value to the scalar field φ. This can easily be achieved

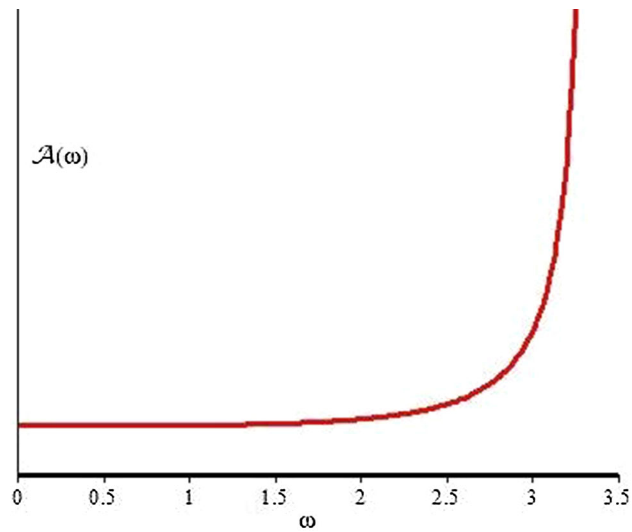

Figure 1. The conformal elliptic Universe. The plot shows the variation of $\mathcal{A}(\omega)$ with ω. The values in years on the horizontal axis should be multiplied by 10^{10} years. The present age of the Universe is estimated to be 1.37 in these units. From there on, the acceleration increases quite rapidly. Here $m^2 = 0.5$.

if we set

$$\varphi(\omega) = \frac{\sqrt{6}}{\lambda}\frac{f(\omega)}{\mathcal{A}(\omega)} = \frac{\sqrt{6}}{\lambda}. \tag{24}$$

The result is that we have set:

$$\mathcal{A}(\omega) = \frac{\sqrt{2}}{(1+m^2)^{\frac{1}{2}}}\frac{\mathbf{dn}(\Omega|m^2)}{\mathbf{cn}(\Omega|m^2)}. \tag{25}$$

This conformal factor is most welcome as it has several good properties. In the first place, it shows an accelerating Universe with no singularity at the beginning. Also the time scale factor $\mathcal{A}(\omega)$ remains almost constant from the initial value until quite large values of the cosmological time ω (see figure 1). This behaviour is compatible with practically no acceleration for a large number of years followed by rapid increase of the scale factor in the final evolution of the Universe. These features seem to be in accordance with the recent observations.

Also the conformal factor describes a Universe that blows up at the value $\mathbf{K(m)}$ where $\mathbf{K(m)}$ is the complete elliptic integral of the first kind. So the Universe expands very rapidly just before this value and blows up suddenly: the Universe has an age: see figure 1. Although it is open, it does not expand forever as in the usual Robertson–Walker case. Finally, with the chosen value for $\varphi(\omega)$, conformal invariance may be broken for negative scalar curvature to give rise a sort of Higgs potential and the masses appear.

4. Conclusions

Under the assumption of a conformal space–time that contains gravity as an intrinsic property and therefore is accordingly treated as a physical object, we have derived the evolution in the conformal time of such a Universe as following the behaviour of an elliptic function, instead of a harmonic (open or closed) function as in the usual Robertson–Walker case. The problem of acceleration, and hence the hypothesis of dark energy, can be solved due to the appearance of inertial forces in the $\mathbf{S^3}$-sphere as this sphere rotates around the cosmological time $\mathbf{S^1}$. This model not only expands in the physical Universe $\mathbf{S^3}$, but also has another source of motion: the centripetal motion caused by the rotation in $\mathbf{S^1}$. Therefore, when time goes on, it also influences the internal motion of the observable Universe. Furthermore, the breaking of conformal invariance is a natural consequence of the model, and is responsible for the appearance of masses in a description of a Universe without initial singularity.

Acknowledgements

This research has been supported in part by MICINN (Grant PID2019-106820RBC22) and Junta de Castilla y León (Grant SA256P18).

References

[1] S Fubini, *Nuovo Cimento A* **34**, 521 (1976); V de Alfaro, S Fubini and G Furlan, *Nuovo Cimento A* **34**, 569 (1976)
[2] V de Alfaro, S Fubini and G Furlan, *Phys. Lett. B* **83**, 163 (1976)
[3] J M Cerveró, *Phys. Lett. B* **108**, 108 (1982)
[4] J M Cerveró, *Lett. Math. Phys.* **8**, 233 (1984)
[5] J M Cerveró, *Lett. Nuovo Cimento* **39**, 125 (1984); *Eur. Phys. Lett.* **1**, 425 (1986)
[6] V de Alfaro, S Fubini and G Furlan, *Nuovo Cimento B* **57**, 227 (1980)
[7] C N Yang and Robert L Mills, *Phys. Rev.* **96**, 191 (1954)
[8] C G Callan, S Coleman and R Jackiw, *Ann. Phys.* **59**, 42 (1970)
[9] J M Cerveró and L Jacobs, *Phys. Lett. B* **78**, 427 (1978)
[10] M Ibison, *J. Math. Phys.* **48**, 122501 (2007)