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# Mass spectra and thermodynamic properties of some heavy and light mesons

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**Abstract.** An energy eigenvalue expression for a sextic anharmonic interaction potential is derived by solving the Schrödinger equation analytically within the framework of the Nikiforov–Uvarov (NU) method. The resultant expression is then used to calculate the mass spectra of some heavy and light mesons, viz., $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $c\bar{s}$, $b\bar{s}$ and $b\bar{q}$, and the results are found to be in good agreement with the other theoretical and experimental studies. The energy eigenvalue expression is further used to compute some important thermodynamic quantities like partition function, specific heat capacity, free energy, mean energy, entropy and magnetisation.

**Keywords.** Schrödinger equation; sextic potential; Nikiforov–Uvarov method; mesons; mass spectra; thermodynamic properties.

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## 1. Introduction

Exact expressions of eigenfunctions and eigenvalues, if derived as solutions to the Schrödinger equation (SE) for a physical system, are of prime importance. These expressions provide a window to look into the underlying dynamics of a system. Also, with the help of bound state energy eigenvalues, one can obtain the partition function which can further be utilised to calculate other thermodynamic properties, such as specific heat capacity, free energy, mean energy, entropy etc. In reality, however there are only a few potentials for which the radial SE can be solved exactly [1]. Numerical and analytical methods are thus complementary to find exact or approximate solutions to the SE with/without the centrifugal term, $1/r^2$, in the potential $V(r)$ for a particle. In the case of a quarkonium system, one can reproduce with a reasonable accuracy, the experimentally observed mass spectra within the realm of the non-relativistic quantum mechanics with an ansatz for interaction potential. To this effect, many studies were carried out earlier and a brief overview of such studies is given now.

Inyang *et al* [2] obtained the mass spectra of heavy mesons using a class of Yukawa potential by employing the series expansion method. Here potential was made to be temperature-dependent by replacing the screening parameter with Debye mass. Rahmani *et al* [3] obtained the masses, magnetic moments and relative decay widths of baryons with two heavy quarks invoking non-relativistic quark model. Within the framework of the Cornell potential, Gupta and Mehrotra [4] numerically solved the non-relativistic SE and calculated the mass spectra of mesons. By using the perturbation method, Moazami *et al* [5] studied mass spectra and decay properties of light and heavy mesons by considering a new potential, a combination of Cornell, Gaussian and inverse square interaction terms. Kumar and Chand [6–8] determined the mass spectra of heavy mesons and energy spectra of quantum dots by solving the $N$-dimensional radial SE using power series and the asymptotic iteration methods within the framework of Coulomb plus harmonic interaction potential.

Rani *et al* [9] obtained the mass spectra of heavy and light mesons by considering a general interaction potential (extended Cornell potential) by solving the non-relativistic SE via the asymptotic iteration method in three dimensions. Shady *et al* [10] investigated the mass spectra and thermodynamic properties of various mesons using extended Cornell potential by solving the $N$-dimensional radial SE using Nikiforov–Uvarov (NU)

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method. Exploiting the quasipotential approach, Ebert *et al* [11] studied mass spectra of various mesons employing relativistic quark model. Arda *et al* [12] analytically computed some thermodynamic quantities for the KG equation with a linear plus inverse-linear, scalar potential with Euler–MacLaurin formula. Recently, some experimental studies on the quarkonium systems are also reported in refs [13–18]. Intending to expand the catalogue of analytically solvable interaction potentials and to generate mass spectra of quark–antiquark systems, in the present work, we propose a polynomial potential of the form

$$V(r) = a_1 r^2 + b_1 r^4 + c_1 r^6, \quad (1)$$

where the coefficients $a_1$, $b_1$ and $c_1$ are arbitrary positive numbers. To the best of our knowledge, this form of potential was not used in the past to study the mass spectra of mesons. Although this potential does not include the Coulomb term which describes the short-range behaviour of mesons, interestingly it still provides good results for some initial states which are comparable with other studies.

In the past, this potential has been a part of many important studies. Almeida and Martin [19] obtained approximate coefficients for eigenvalues of ground and first excited states of a sextic anharmonic potential with multipoint quasirational approximation. Using higher-order JWKB approximation, Budaca [20] analytically derived energy spectra of the prolate $r$-rigid Bohr–Mottelson Hamiltonian, an oscillator potential with a sextic anharmonicity in $\beta$-shape variable. Other studies on the sextic potential can be found in refs [21–23]. Further, the study of thermodynamic properties helps us to understand the quark–antiquark interactions. In literature, several important studies are available to derive thermodynamic properties for various interaction potentials [24–27].

Over the years, several exact and approximate analytical methods to solve the SE for various interaction potentials, such as the asymptotic iteration method [28,29], supersymmetric quantum mechanics [30], the Laplace transform [31,32], the NU method [33–36], the series expansion method [37,38], the parametric NU method [39], the Nikiforov–Uvarov functional analysis method (NUFA) [40,41] and many more have been developed. Here, however we use the NU method to analytically solve the SE for potential (1) and further calculate the mass spectra and thermodynamic quantities of some heavy and light mesons. The reason for choosing the NU method for the present study is that it is more handy and can predict the results better.

The organisation of the paper is as follows: In §2, a brief description of the NU method is given. The energy eigenvalues and the eigenfunctions of the sextic anharmonic potential (1) are obtained in §3. The mass spectra of six heavy and light mesons are computed and compared with other recent theoretical and experimental studies in §4. Analytic expressions of various thermodynamic parameters are described in §5. Results and discussion of this work are presented in §6. Section 7 presents the conclusion.

## 2. The NU method

The NU method is a powerful mathematical tool to solve second-order differential equations [42]. This is based on the solutions to the Schrödinger-like general second-order linear differential equations with special orthogonal functions. Using appropriate transformation of variable $s = s(x)$, the following second-order differential equation can be solved via NU method:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \quad (2)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ must be at the most second-degree polynomials, $\tilde{\tau}(s)$ is at the most a first-degree polynomial and $\psi(s)$ is a hypergeometric-type function. In order to find an exact solution to eq. (2), one may start with the wave function

$$\psi(s) = \Phi(s)y(s). \quad (3)$$

Substitution of eq. (3) in eq. (2) reduces it into a hypergeometric equation of the form

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0. \quad (4)$$

The wave function $\Phi(s)$ in eq. (3) is a logarithmic derivative and at the most a first-order polynomial defined as

$$\frac{\Phi'(s)}{\Phi(s)} = \frac{\pi(s)}{\sigma(s)}. \quad (5)$$

Here, $\pi(s)$ will be at the most a first degree polynomial and is written as

$$\pi(s) = \frac{1}{2}[\tau(s) - \tilde{\tau}(s)]. \quad (6)$$

The wave function $y(s)$ in eq. (3) is a hypergeometric function and its polynomial solution is given by the Rodrigues relation

$$y_n(s) = \frac{N_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\rho^n(s)\sigma(s)], \quad (7)$$

where $N_n$ is the normalisation constant and $\rho(s)$ is a weight function which must satisfy the condition

$$\frac{\mathrm{d}}{\mathrm{d}s}[\sigma(s)\rho(s)] = \rho(s)\tau(s). \quad (8)$$

It should be noted that the derivative of $\tau(s)$ with respect to $s$ should be negative. The eigenfunctions and eigenvalues can be obtained by using functions $\pi(s)$ and $\lambda$ as

$$\pi(s) = \frac{\sigma'(s) - \tau(s)}{2} \pm\sqrt{\left(\frac{\sigma'(s) - \tau(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \quad (9)$$

and

$$\lambda = k + \pi'(s). \quad (10)$$

The value of $k$ can be obtained by setting the discriminant of eq. (9) equal to zero. As such, the new eigenvalue equation is written as

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\rho''(s), \quad n = 0, 1, 2, 3, \ldots. \quad (11)$$

To obtain an eigenvalue solution through the NU method, the relations for $\lambda$ and $\lambda_n$ must be set equal by means of eqs (10) and (11).

## 3. Bound-state solutions

The radial SE for two particles interacting via symmetric potential in $N$-dimensional space is given as

$$\left[\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr} - \frac{L(L+N-2)}{r^2} + 2\mu(E - V(r))\right]\psi(r) = 0, \quad (12)$$

where $L$, $N$ and $\mu$ are respectively the angular momentum quantum number, the dimensionality number and reduced mass of a two-particle system.

Using wave function $\psi(r) = r^{\frac{1-N}{2}}R(r)$, the above equation reduces to

$$\left[\frac{d^2}{dr^2} + 2\mu\left(E - V(r) - \frac{\left(L+\frac{N-2}{2}\right)^2 - \frac{1}{4}}{2\mu r^2}\right)\right] \times R(r) = 0. \quad (13)$$

By substituting eq. (1) into eq. (13), we obtain

$$\left[\frac{d^2}{dr^2} + 2\mu\left(E - a_1 r^2 - b_1 r^4 - c_1 r^6 - \frac{\left(L+\frac{N-2}{2}\right)^2 - \frac{1}{4}}{2\mu r^2}\right)\right]R(r) = 0. \quad (14)$$

Let us assume that $r = 1/x$ and $r_0$ is the characteristic radius of a particle. Next, using the power series expansion of $1/x$ around $r_0$, i.e. $\delta = 1/r_0$, eq. (14) converts to

$$\left[\frac{d^2}{dx^2} + \frac{2x}{x^2}\frac{d}{dx} + \frac{2}{x^4}(-Ax^2 + Bx - C)\right]R(x) = 0, \quad (15)$$

where

$$A = \left[\frac{3a_1}{\delta^4} + \frac{10b_1}{\delta^6} + \frac{21c_1}{\delta^8} + \frac{\left(L+\frac{N-2}{2}\right)^2 - \frac{1}{4}}{2\mu}\right]\mu, \quad (16)$$

$$B = \left[\frac{8a_1}{\delta^3} + \frac{24b_1}{\delta^5} + \frac{48c_1}{\delta^7}\right]\mu \quad (17)$$

and

$$C = \left[\frac{6a_1}{\delta^2} + \frac{15b_1}{\delta^4} + \frac{28c_1}{\delta^6} - E\right]\mu. \quad (18)$$

Comparison of eq. (15) with eq. (2) leads to

$$\tilde{\tau}(s) = 2x,$$
$$\sigma(s) = x^2,$$
$$\tilde{\sigma}(s) = 2(-Ax^2 + Bx - C).$$

Therefore, in view of eq. (6) and the above results, eq. (9) becomes

$$\pi(s) = \pm\sqrt{(k+2A)x^2 - 2Bx + 2C}. \quad (19)$$

Here, a quadratic equation in $x$ is available for the constant $k$ which can be determined by setting the discriminant of this equation equal to zero. After determining $k$, i.e. $k+2A = (B^2/2C)$, the polynomial $\pi(s)$ is obtained from eq. (9), and $\tau'(s)$ and $\lambda$ are then obtained from eqs (6) and (10), respectively as

$$\tau'(s) = 2 - \frac{2B}{\sqrt{2C}}, \quad (20)$$

$$\lambda = k + \pi'(s) = \frac{B^2}{2C} - 2A - \frac{2}{\sqrt{2C}} \quad (21)$$

and

$$\lambda_n = -n\left(2 - \frac{2B}{\sqrt{2C}}\right) - n(n-1). \quad (22)$$

To obtain an eigenvalue solution through the NU method, the relationship between $\lambda$ an $\lambda_n$ must be set equal by using eqs (21) and (22) along with eqs (16)–(18). The energy eigenvalues in $N$-dimensional space is thus written as

$$E=\frac{6a_1}{\delta^2}+\frac{15b_1}{\delta^4}+\frac{28c_1}{\delta^6}-\frac{2\mu\left(\frac{8a_1}{\delta^3}+\frac{24b_1}{\delta^5}+\frac{48c_1}{\delta^7}\right)^2}{\left[(1+2n)+\sqrt{1+\frac{24a_1\mu}{\delta^4}+\frac{80b_1\mu}{\delta^6}+\frac{168c_1\mu}{\delta^8}+4\left[\left(L+\frac{N-2}{2}\right)^2-\frac{1}{4}\right]}\right]^2}. \quad (23)$$

The above equation is reduced for three-dimensional systems as

$$E=\frac{6a_1}{\delta^2}+\frac{15b_1}{\delta^4}+\frac{28c_1}{\delta^6}-\frac{2\mu\left(\frac{8a_1}{\delta^3}+\frac{24b_1}{\delta^5}+\frac{48c_1}{\delta^7}\right)^2}{\left[(1+2n)+\sqrt{1+\frac{24a_1\mu}{\delta^4}+\frac{80b_1\mu}{\delta^6}+\frac{168c_1\mu}{\delta^8}+4L(L+1)}\right]^2}. \quad (24)$$

The corresponding radial wave function is obtained by substituting Eqs. (5) and (8) into Eq. (7) as

$$R_{nl}(r)=N_n r^{\frac{-B}{\sqrt{2C}}-1}\mathrm{e}^{\sqrt{2C}r}\left(-r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)^n \times\left(r^{-2n+\frac{2B}{\sqrt{2C}}}\mathrm{e}^{-2\sqrt{2C}r}\right), \quad (25)$$

where $N_n$ is the normalisation constant.

## 4. Mass spectra of mesons

As a possible application of potential (1) to real physical systems, here we obtain the mass spectra of heavy and light mesons utilising the following relation [6]:

$$M=m_q+m_{\bar{q}}+E_{nl}, \quad (26)$$

where $m_q$ and $m_{\bar{q}}$ are masses of quark and antiquark respectively. So, in view of eq. (24), the above equation becomes

$$M=m_q+m_{\bar{q}}+\frac{6a_1}{\delta^2}+\frac{15b_1}{\delta^4}+\frac{28c_1}{\delta^6}-\frac{2\mu\left(\frac{8a_1}{\delta^3}+\frac{24b_1}{\delta^5}+\frac{48c_1}{\delta^7}\right)^2}{\left[(1+2n)+\sqrt{1+\frac{24a_1\mu}{\delta^4}+\frac{80b_1\mu}{\delta^6}+\frac{168c_1\mu}{\delta^8}+4L(L+1)}\right]^2}. \quad (27)$$

In our work, we have chosen $m_c$ = 1.209 GeV, $m_b$ = 4.823 GeV, $m_s$ = 0.419 GeV and $m_d = m_u$ = 0.220 GeV and values of $a_1$, $b_1$ and $c_1$ are taken from ref. [9].

The complete mass spectra of heavy and light mesons are given in tables 1–6. We observe that the mass spectra of $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $c\bar{s}$, $b\bar{s}$ and $b\bar{q}$ are in good agreement with experimental data and other recent theoretical studies. The variation of interaction potential with inter-quark separation $r$ for charmonium is shown in figure 1. We have also observed similar behaviour of sextic potential for other mesonic systems.

## 5. Thermodynamic properties

With a view to extend the scope of the present study, in this section, we derive expressions for different thermodynamic quantities such as partition function $Z(\beta)$, mean energy $U(\beta)$, specific heat capacity $C(\beta)$, free energy $F(\beta)$, entropy $S(\beta)$ and magnetisation $M(\beta)$ using the energy eigenvalues (24).

### 5.1 *Partition function* ($Z(\beta)$)

It is pertinent to mention here that partition function is a mathematical function. As such, it has no lateral physical significance. However, it plays a key role in deriving expressions for various thermodynamic quantities. The partition function is an exponential type function that represents the sum of all possible energies for that system and also expresses the number of thermally accessible states that a system provides to carriers.

The exact formula of partition function can be written as [24,25]

$$Z(\beta)=\int_0^{\lambda}\mathrm{e}^{-\beta E_{nl}}\,\mathrm{d}n, \quad (28)$$

where $E_{nl}$ is the total energy of the system in the respective microstate, $\beta$ is a thermodynamic parameter, i.e. $\beta = 1/k_{\mathrm{B}}T$ and $n$ is the principal quantum number varying from 0 to λ and

$$\lambda=\frac{1}{2}\left[\sqrt{\frac{\wedge_2}{\wedge_1}}-\wedge_3\right]$$

**Table 1.** Mass spectra of charmonium (in GeV) ($a_1 = 0.0297$ GeV, $b_1 = 0.2209$ GeV, $\delta = 2.292$ GeV).

| States | $c_1$ | Present work | Experimental [14] | Relativistic [11] | Ref. [7] | Ref. [9] |
|---|---|---|---|---|---|---|
| 1S | 3.4665 | 3.095 | 3.096 | 3.096 | 3.078 | 3.096 |
| 1P | 3.9045 | 3.240 | – | 3.510 | 3.415 | 3.214 |
| 1D | 4.6411 | 3.405 | – | 3.798 | 3.752 | 3.412 |
| 2S | 4.6322 | 3.427 | 3.686 | 3.686 | 4.187 | 3.686 |
| 2P | 5.0702 | 3.521 | 3.773 | 3.929 | 4.143 | 3.773 |
| 3S | 5.7980 | 3.674 | 4.040 | 4.088 | 5.297 | 4.275 |
| 4S | 6.9637 | 3.901 | 4.263 | – | 6.407 | 4.865 |

**Table 2.** Mass spectra of bottomonium (in GeV) ($a_1 = 0.0289$ GeV, $b_1 = 0.3555$ GeV, $\delta = 2.323$ GeV).

| States | $c_1$ | Present work | Experimental [14] | Relativistic [11] | Ref. [7] | Ref. [9] |
|---|---|---|---|---|---|---|
| 1S | 5.1080 | 9.465 | 9.460 | 9.460 | 9.510 | 9.460 |
| 1P | 5.2998 | 9.831 | – | 9.892 | 9.862 | 9.492 |
| 1D | 5.6621 | 10.091 | – | 10.153 | 10.214 | 9.551 |
| 2S | 6.0602 | 10.493 | 10.023 | 10.023 | 10.627 | 10.023 |
| 2P | 6.2520 | 10.597 | – | 10.255 | 10.468 | 10.038 |
| 3S | 7.0124 | 10.886 | 10.355 | 10.355 | 11.726 | 10.585 |
| 4S | 7.9646 | 11.146 | 10.580 | – | 12.834 | 11.148 |

**Table 3.** Mass spectra of $b\bar{c}$ (in GeV) ($a_1 = 0.0204$ GeV, $b_1 = 0.2209$ GeV, $\delta = 2.463$ GeV).

| States | $c_1$ | Present work | Experimental [14] | Relativistic [11] | Ref. [10] | Ref. [9] |
|---|---|---|---|---|---|---|
| 1S | 4.0087 | 6.278 | 6.277 | 6.332 | 6.277 | 6.277 |
| 1P | 4.3500 | 6.466 | – | 6.734 | 7.042 | 6.340 |
| 1D | 4.9539 | 6.601 | – | 7.072 | – | 6.452 |
| 2S | 5.1208 | 6.683 | – | 6.881 | 7.383 | 6.814 |
| 2P | 5.4621 | 6.750 | – | 7.126 | 6.663 | 6.851 |
| 3S | 6.2329 | 6.877 | – | 7.235 | 7.206 | 7.351 |
| 4S | 7.3450 | 7.032 | – | – | – | 7.889 |

**Table 4.** Mass spectra of $c\bar{s}$ (in GeV) ($a_1 = 0.0122$ GeV, $b_1 = 0.0688$ GeV, $\delta = 2.875$ GeV).

| States | $c_1$ | Present work | Experimental [15] | Relativistic [18] | Ref. [10] | Ref. [9] |
|---|---|---|---|---|---|---|
| 1S | 18.2251 | 2.511 | – | 2.129 | 1.968 | 2.512 |
| 1P | 22.0717 | 2.721 | – | 2.549 | 2.565 | 2.649 |
| 1D | 27.9841 | 3.022 | 2.859 | 2.899 | 2.857 | 2.859 |
| 2S | 26.1194 | 2.936 | 2.709 | 2.732 | 2.709 | 2.709 |
| 2P | 29.9659 | 3.129 | – | 3.018 | – | 2.860 |
| 3S | 34.0136 | 3.332 | – | 3.193 | 2.932 | 2.906 |
| 4S | 41.9079 | 3.729 | – | 3.575 | – | 3.102 |

is an upper bound of $n$.

Here,

$$\wedge_1 = \left(\frac{6a_1}{\delta^2} + \frac{15b_1}{\delta^4} + \frac{28c_1}{\delta^6}\right),$$

$$\wedge_2 = \left(2\mu\left(\frac{8a_1}{\delta^3} + \frac{24b_1}{\delta^5} + \frac{48c_1}{\delta^7}\right)^2\right), \wedge_3 = \left(1 + \sqrt{1 + \frac{24a_1\mu}{\delta^4} + \frac{80b_1\mu}{\delta^6} + \frac{168c_1\mu}{\delta^8} + 4L(L+1)}\right)^2. \quad (29)$$

We can write the exact partition function at temperature $T$ after using eq. (24) into eq. (28) along with the definitions of eq. (29) as

$$Z(\beta)=\int_0^{\lambda} \mathrm{e}^{-\beta\left[\wedge_1-\frac{\wedge_2}{(2n+\wedge_3)^2}\right]}\,\mathrm{d}n. \tag{30}$$

The integration of this equation directly provides us exact expression for partition function for a quark–antiquark system as

$$\begin{aligned}Z(\beta)=&\frac{1}{2}\mathrm{e}^{\left(\frac{-\beta(\wedge_1\wedge_4^2-\wedge_2)}{\wedge_4^2}\right)}-\frac{1}{2}\mathrm{e}^{\left(\frac{-\beta(\wedge_1\wedge_3^2+\wedge_2)}{\wedge_3^2}\right)}\\&+\frac{1}{2}\mathrm{e}^{-\beta\wedge_1}\sqrt{\wedge_2}\sqrt{\pi}\sqrt{\beta}\quad \mathrm{erf}i\left(\frac{\sqrt{\wedge_2}\sqrt{\beta}}{\wedge_3}\right)\\&-\frac{1}{2}\mathrm{e}^{-\beta\wedge_1}\sqrt{\wedge_2}\sqrt{\pi}\sqrt{\beta}\quad \mathrm{erf}i\left(\frac{\wedge_2\sqrt{\beta}}{\wedge_4}\right),\end{aligned} \tag{31}$$

where

$$\wedge_4=\wedge_3+2\lambda$$

and

$$\mathrm{erf}i(x)=-I\,\mathrm{erf}(Ix)=\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t$$

is an error function. The plots $Z(\beta)$ vs. $\beta$ for constant $\lambda$ (figure 2) and $Z(\lambda)$ vs. $\lambda$ for constant $\beta$ (figure 3) are consistent with other similar studies [24,43].

### 5.2 *Mean energy* $(U(\beta))$

The mean energy can be calculated by simply taking derivative of $\ln Z(\beta)$ with respect to $\beta$ as

$$U(\beta)=-\frac{\partial}{\partial\beta}\ln Z(\beta). \tag{32}$$

A straightforward calculation leads to

$$\begin{aligned}U(\beta)=&\left[\frac{M_1(\wedge_1\wedge_4^2-\wedge_2)}{\wedge_4}-\frac{N_1(\wedge_1\wedge_3^2-\wedge_2)}{\wedge_3}\right.\\&+\sqrt{\beta}\wedge_1(M-N)\\&+\frac{(N-M)}{2\sqrt{\beta}}+\frac{1}{2}\mathrm{e}^{-\beta\wedge_1}\mathrm{e}^{\frac{\wedge_2\beta}{\wedge_3^2}}\\&\left.\left(\frac{-\wedge_2}{\wedge_3}+\frac{\wedge_2^{\frac{3}{2}}}{\wedge_4}\right)\right]\Big/\\&\left(M_1\wedge_4-N_1\wedge_3+M\sqrt{\beta}-N\sqrt{\beta}\right)\end{aligned} \tag{33}$$

where $M$, $N$, $M_1$ and $N_1$ are defined as

$$\begin{aligned}M&=\mathrm{e}^{-\beta\wedge_1}\sqrt{\wedge_2}\sqrt{\pi}\,\mathrm{erf}i\left(\frac{\sqrt{\wedge_2}\sqrt{\beta}}{\wedge_3}\right),\\N&=\mathrm{e}^{-\beta\wedge_1}\sqrt{\wedge_2}\sqrt{\pi}\,\mathrm{erf}i\left(\frac{\wedge_2\sqrt{\beta}}{\wedge_4}\right),\\M_1&=\mathrm{e}^{\frac{-\beta(\wedge_1\wedge_4^2-\wedge_2)}{\wedge_4^2}},\\N_1&=\mathrm{e}^{\frac{-\beta(\wedge_1\wedge_3^2+\wedge_2)}{\wedge_3^2}}.\end{aligned} \tag{34}$$

### 5.3 *Specific heat capacity* $(C(\beta))$

Specific heat capacity is the amount of heat to be supplied to or taken out of a unit mass of a system to increase or decrease its temperature by one degree. It can also be identified as the ratio of the heat capacities of a substance at a given temperature and of a reference substance at a reference temperature. Specific heat capacity in terms of partition function is written as

$$C(\beta)=k_\beta\beta^2\frac{\partial^2}{\partial\beta^2}\ln Z(\beta). \tag{35}$$

Using eq. (31) in eq. (35), we obtain a relation for $C(\beta)$ as

$$\begin{aligned}C(\beta)=&k_\beta\beta^2\left\{\frac{2}{M_1\wedge_4-N_1\wedge_3+M\sqrt{\beta}-N\sqrt{\beta}}\right.\\&\times\left[\frac{M_1(\wedge_1\wedge_4^2-\wedge_2)^2}{2\wedge_4^2}-\frac{N_1(\wedge_1\wedge_3^2+\wedge_2)^2}{2\wedge_3^3}\right.\\&+\wedge_1^2\beta(M/2-N/2)+\frac{\wedge_1}{\sqrt{\beta}}\left(\frac{N}{2}-\frac{N}{2}\right)\\&+\frac{\mathrm{e}^{-\beta\wedge_1}\wedge_2\,\mathrm{e}^{\frac{\wedge_2\beta}{\wedge_3^2}}}{\wedge_3}\left(\wedge_1+\frac{1}{4\beta}+\frac{\wedge_2}{2\wedge_3}\right)\\&+\frac{1}{b^{3/2}}\left(\frac{N}{8}-\frac{N}{8}\right)+\frac{\mathrm{e}^{-\beta\wedge_1}\wedge_2^{3/2}\,\mathrm{e}^{\frac{\wedge_2^2\beta}{\wedge_4^2}}}{\wedge_4}\\&\left.\times\left(\wedge_1-\frac{1}{4\beta}-\frac{\wedge_2^2}{2\wedge_3^2}\right)\right]-\left[\frac{1}{2}\frac{M_1(\wedge_1\wedge_4^2-\wedge_2)}{\wedge_4}\right.\\&+\frac{N_1(\wedge_1\wedge_3^2+\wedge_2)}{\wedge_3}+\wedge_1\sqrt{\beta}\left(\frac{N}{2}-\frac{M}{2}\right)\\&+\frac{1}{\sqrt{\beta}}\left(\frac{M}{4}-\frac{N}{4}\right)\\&\left.+\frac{\mathrm{e}^{-\beta\wedge_1\wedge_3}}{2}\left(\frac{\mathrm{e}^{\frac{\wedge_2\beta}{\wedge_3^2}}}{\wedge_3}-\frac{\wedge_2^{1/2}\mathrm{e}^{\frac{\wedge_2^2\beta}{\wedge_4^2}}}{\wedge_4}\right)\right]\Big/\end{aligned}$$

**Table 5.** Mass spectra of $b\bar{s}$ (in GeV) ($a_1 = 0.0372$ GeV, $b_1 = 0.2796$ GeV, $\delta = 2.688$ GeV).

| States | $c_1$ | Present work | Experimental [14] | Relativistic [17] | Ref. [9] |
|---|---|---|---|---|---|
| 1S | 1.7943 | 5.416 | 5.415 | 5.450 | 5.415 |
| 1P | 3.3520 | 5.564 | 5.830 | 5.857 | 5.830 |
| 1D | 4.9816 | 5.695 | – | 6.182 | 6.264 |
| 2S | 3.4444 | 5.591 | – | 6.012 | 6.819 |
| 2P | 5.0021 | 5.711 | – | 6.279 | 6.786 |
| 3S | 5.0946 | 5.723 | – | 6.429 | 8.225 |
| 4S | 6.7447 | 5.849 | – | 6.773 | 9.629 |

**Table 6.** Mass spectra of $b\bar{q}$ ($q = u, d$) (in GeV) ($a_1 = 0.0178$ GeV, $b_1 = 0.1589$ GeV, $\delta = 2.376$ GeV).

| States | $c_1$ | Present work | Experimental [14] | Relativistic [17] | Ref. [9] |
|---|---|---|---|---|---|
| 1S | 1.9283 | 5.416 | 5.325 | 5.371 | 5.325 |
| 1P | 3.7065 | 5.702 | 5.723 | 5.777 | 5.723 |
| 1D | 5.5315 | 5.898 | – | 6.110 | 6.131 |
| 2S | 3.7657 | 5.717 | – | 5.933 | 6.413 |
| 2P | 5.5440 | 5.996 | – | 6.197 | 6.486 |
| 3S | 5.6032 | 6.006 | – | 6.355 | 7.501 |
| 4S | 7.4407 | 6.293 | – | 6.703 | 8.589 |

$$\left(\frac{M_1\wedge_4}{2} - \frac{N_1\wedge_3}{2} + \frac{M\sqrt{\beta}}{2} - \frac{N\sqrt{\beta}}{2}\right)^2\Bigg\}. \quad (36)$$

### 5.4 *Free energy* ($F(\beta)$)

Free energy is the energy available to a system to do useful work and it is different from the total energy change of a chemical reaction. It is that portion of any energy which is available to perform thermodynamic work at constant temperature. Free energy is an extensive property, which means its magnitude depends on the amount of substance in a given thermodynamic state. The changes in free energy are useful to determine the direction of spontaneous change and evaluate the maximum work done. An expression for the free energy is derived from the relation

$$F(\beta) = -kT \ln Z(\beta). \quad (37)$$

The final expression of free energy is written as

$$F(\beta) = -\frac{1}{\beta}\ln\left(\frac{1}{2}e^{\frac{-\beta(\wedge_1\wedge_4^2-\wedge_2)}{\wedge_4^2}} - \frac{1}{2}e^{\frac{-\beta(\wedge_1\wedge_3^2+\wedge_2)}{\wedge_3^2}} + \frac{1}{2}e^{-\beta\wedge_1}\sqrt{\wedge_2}\sqrt{\pi}\sqrt{\beta}\mathrm{erf}i\left(\frac{\sqrt{\wedge_2}\sqrt{\beta}}{\wedge_3}\right) - \frac{1}{2}e^{-\beta\wedge_1}\sqrt{\wedge_2}\sqrt{\pi}\sqrt{\beta}\mathrm{erf}i\left(\frac{\wedge_2\sqrt{\beta}}{\wedge_4}\right)\right). \quad (38)$$

The plot between free energy and $\beta$ for various values of $\lambda$ is shown in figure 4 and the variation is similar to the previous study [44].

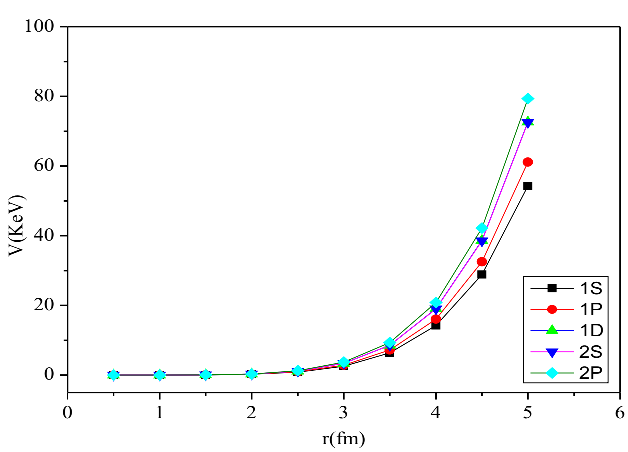


**Figure 1.** Variation of interaction potential $V(r)$ with $r$.

### 5.5 *Entropy* ($S(\beta)$)

Entropy is a state variable whose change is defined for a reversible process at a particular temperature. It is a measure of the amount of energy which is unavailable to do work and is a measure of the disorder and multiplicity of a system. Entropy is also a measure of how many possible configurations the atoms may have in a structure. Thus, when a system goes from a more orderly state to a less orderly state, then its entropy increases. By using the partition function, entropy of the system is

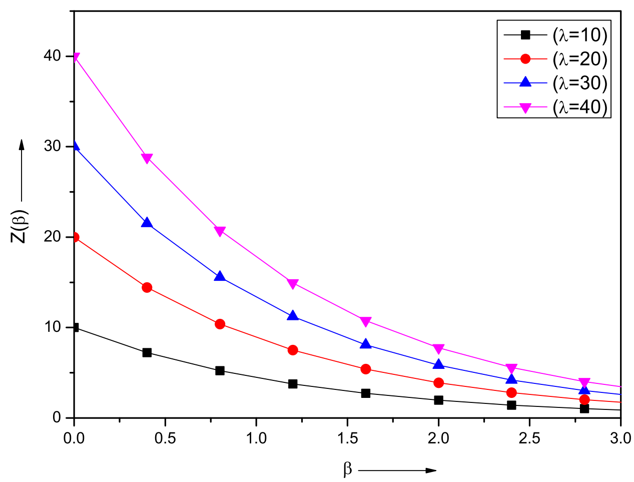


**Figure 2.** Variation of partition function ($Z$) with respect to $\beta$ for different values of $\lambda$.

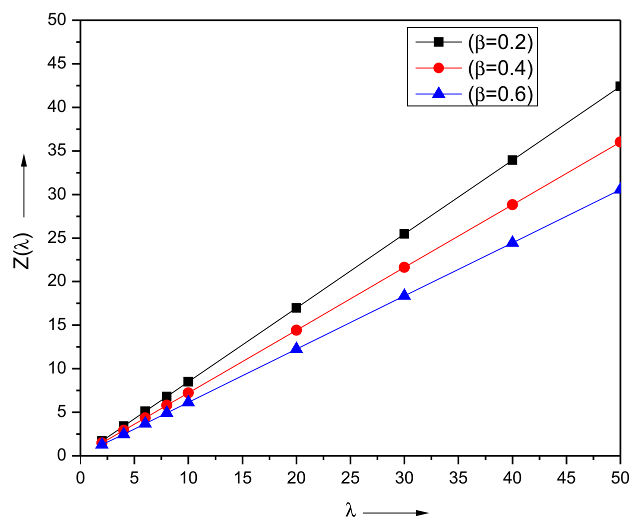


**Figure 3.** Variation of partition function ($Z$) with respect to $\lambda$ for different values of $\beta$.

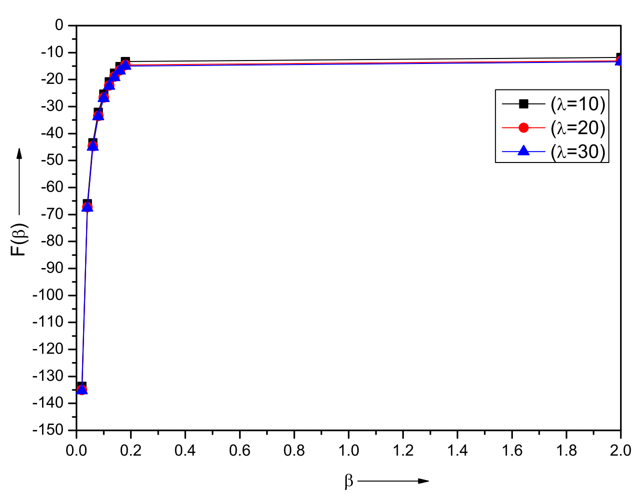


**Figure 4.** Variation of free energy ($F$) with respect to $\beta$.

written as

$$S(\beta) = k_\beta \ln Z(\beta) - k_\beta \beta \frac{\partial}{\partial \beta} Z(\beta). \tag{39}$$

Using eq. (31) and after solving the above equation, we obtain

$$\begin{aligned} S(\beta) = k_\beta \Bigg\{ &\ln\left(\frac{M_1 \wedge_4}{2} - \frac{N_1 \wedge_3}{2} + \frac{M\sqrt{\beta}}{2} - \frac{N\sqrt{\beta}}{2}\right) \\ &-\beta\Bigg(-\frac{1}{2}\frac{M_1(\wedge_1 \wedge_4^2 - \wedge_2)}{\wedge_4} + \frac{1}{2}\frac{N_1(\wedge_1 \wedge_3^2 + \wedge_2)}{\wedge_3} \\ &-\frac{1}{2} M \wedge_1 \sqrt{\beta} + \frac{M}{4\sqrt{\beta}} \\ &+\frac{1}{2} N \wedge_1 \sqrt{\beta} - \frac{N}{4\sqrt{\beta}} \\ &+\frac{1}{2}\frac{e^{-\beta\wedge_1} \wedge_2 e^{\frac{\wedge_2 \beta}{\wedge_3^2}}}{\wedge_3} - \frac{1}{2}\frac{e^{-\beta\wedge_1} \wedge_3^{\frac{3}{2}} e^{\frac{\wedge_2^2 \beta}{\wedge_4^2}}}{\wedge_4}\Bigg) \Bigg/ \\ &\left(\frac{M_1 \wedge_4}{2} - \frac{N_1 \wedge_3}{2} + \frac{M\sqrt{\beta}}{2} - \frac{N\sqrt{\beta}}{2}\right)\Bigg\}. \end{aligned} \tag{40}$$

A plot between entropy and $\beta$ is shown in figure 6. This graph is similar to an earlier work [24].

### 5.6 *Magnetisation* ($M(\beta)$)

It is a measure of the density of permanent or induced dipole moment in a given magnetic material or the amount of net magnetic moment in a unit volume of a substance. Origin of magnetic moment can be either electric current resulting from the motion of electrons in an atom or the spin of nucleus or electron. The magnetisation in terms of partition function can be written as

$$M(\beta) = \frac{1}{\beta}\frac{1}{Z(\beta)}\left(\frac{\partial}{\partial \beta} Z(\beta)\right). \tag{41}$$

Now, by using eq. (31) in the above equation, $M(\beta)$ is written as

$$\begin{aligned} M(\beta) = \Bigg( &\frac{-1}{2}\frac{M_1(\wedge_1 \wedge_4^2 - \wedge_2)}{\wedge_4} + \frac{1}{2}\frac{N_1(\wedge_1 \wedge_3^2 + \wedge_2)}{\wedge_3} \\ &- \frac{1}{2} M \wedge_1 \sqrt{\beta} + \frac{1}{4}\frac{M}{\sqrt{\beta}} + \frac{1}{2}\frac{e^{-\beta\wedge_1} \wedge_2 e^{\frac{\beta\wedge_2}{\wedge_3^2}}}{\wedge_3} \\ &+ \frac{1}{2} N \wedge_1 \sqrt{\beta} - \frac{1}{4}\frac{N}{\sqrt{\beta}} - \frac{1}{2}\frac{e^{-\beta\wedge_1} \wedge_2^{\frac{1}{2}} e^{\frac{\beta\wedge_2^2}{\wedge_4^2}}}{\wedge_4}\Bigg) \Bigg/ \\ &\beta\left(\frac{1}{2} M_1 \wedge_4 - \frac{1}{2} N_1 \wedge_3 + \frac{1}{2} M\sqrt{\beta} - \frac{1}{2} N\sqrt{\beta}\right). \end{aligned} \tag{42}$$

The variation of $M$ as a function of $\beta$ is shown in figure 7 and the trend is similar to the study reported in ref. [24].

## 6. Results and discussion

A close perusal of the mass spectra data of tables 1–6, generated within the framework of potential (1), for various heavy and light mesons clearly reveals a close agreement between the results of the present work and other experimental and theoretical studies. Figures 2–7 are plotted with Maple 15 software to get a better insight about the variation of some of the analytically derived thermodynamic quantities as a function of temperature, state energy and quantum number. Figure 2 shows that partition function, $Z$, increases exponentially with increase in temperature for a given state quantum number, λ. Its overall value increases with increase in λ. However, the variation with $\beta$ remains the same. Figure 3 shows that the partition function increases linearly with state quantum number λ at a given temperatur. It has higher value at higher temperature (smaller $\beta$). Same behaviour is observed in the previous case.

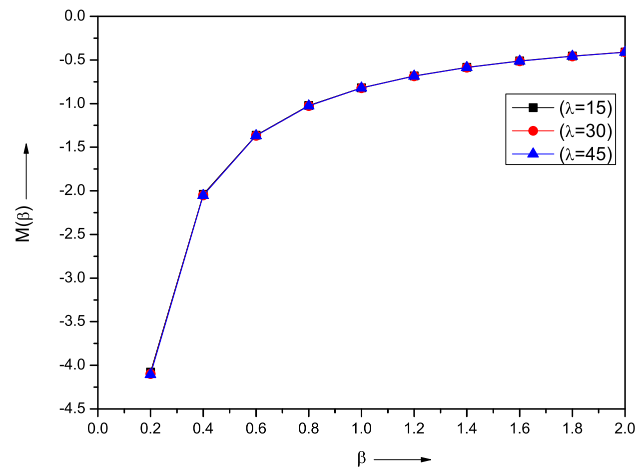


**Figure 7.** Variation of magnetisation ($M$) with respect to $\beta$ for different values of λ.

Figure 4 shows that the free energy, $F$, increases very sharply at high temperature. The graph starts to level off with further decrease in temperature and ultimately saturates at low temperature. The free energy is almost insensitive to λ. Figure 5 depicts the variation of partition function and free energy $F$ as a function of energy of various states, 1S to 4S, at constant temperature (300 K). This constancy of $Z$ and $F$ is not an unusual one because the values of energy eigenvalues of the reported states are not much different. Figure 6 establishes the characteristic behaviour of entropy, i.e. it increases with increase in temperature. However, its variation with λ is negligible. The magnetisation of the system, as shown in figure 7, decreases with increase in temperature and independent of λ. This trend is on the expected line.

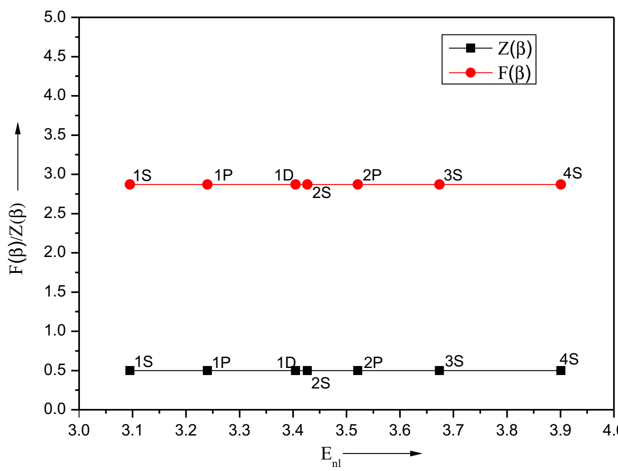


**Figure 5.** Variation of partition function and free energy with respect to $E_{nl}$.

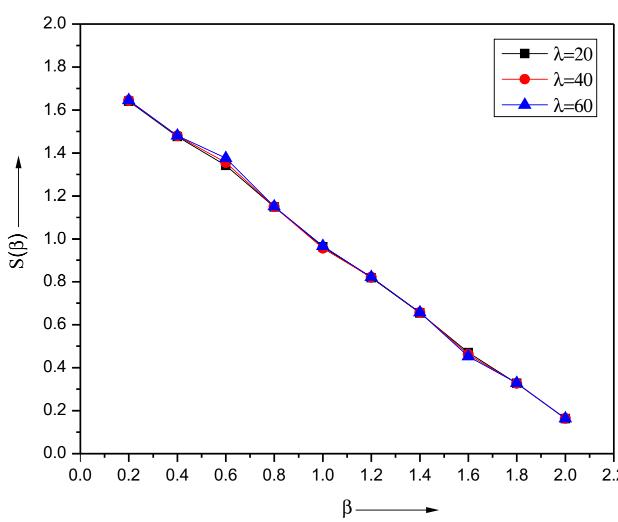


**Figure 6.** Variation of entropy ($S$) with respect to $\beta$ for different values of λ.

## 7. Conclusion

In the present work, we have successfully employed the NU method to solve the non-relativistic SE and calculated the bound-state energy eigenvalues and eigenfunctions of a system governed by a sextic anharmonic potential. To check the suitability of the choice of potential and veracity of the NU method, we utilised the energy eigenvalue expression of eq. (24) to compute mass spectra of heavy and light mesons like $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $c\bar{s}$, $b\bar{s}$ and $b\bar{q}$. For this purpose, taking insight from literature, suitable value of $a_1$, $b_1$ and $c_1$ were identified. We have also derived analytical expressions for various thermodynamic quantities using energy eigenvalue equations such as partition function, mean energy, specific heat capacity, free energy, entropy and magnetisation using Maple. The variation of sextic potential with interquark separation is plotted in figure 1.

The results of the present work (see tables 1–6) are in good agreement with experimental and theoretical finding of recent studies on mesonic systems. So through our calculated results, it can be claimed that the sextic anharmonic potential with suitable potential parameters can be used to generate the mass spectra and thermodynamic properties of mesons.

Further, to explore more applications of potential (1), we also tried eq. (24), to generate electronic states of some diatomic molecules but the results were not encouraging. Work is in progress to surmount issues encountered in these calculations. The behaviour of various thermodynamic properties (see figures 2–7) is in tune with other works [22,39]. Finally, it is worth to mention that the interaction potential (1) does not contain any Coulombic term but interestingly it still provides the mass spectra of heavy and lights mesons with reasonable accuracy for some initial states but same is not true for higher states. This observation needs further investigations.

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