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Conservation laws and solution of the geodesic system of Gödel's metric via Lie and Noether symmetries

F ALKINDI[1], A H KARA[2],* and M ZIAD[3]

[1]General Directorate of Education of A'ddakhiliyah Province, Ministry of Education, Sultanate of Oman
[2]School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa
[3]Department of Mathematics, College of Science, Sultan Qaboos University, Muscat, Sultanate of Oman
*Corresponding author. E-mail: Abdul.Kara@wits.ac.za

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Abstract. We consider the geodesic system for Gödel's metric as a toy model and solve it analytically using its Lie point symmetries. It is shown that the differential invariants of these symmetries reduce the second-order non-linear system to a single second-order ordinary differential equation (ODE). Invariance of the latter under a one-dimensional Lie point symmetry group reduces it to an integrable first-order ODE. A complete solution of the system is then achieved. The sub-algebras of Noether symmetries and isometries are then found with their corresponding first integrals.

Keywords. Geodesic equations; Lie symmetries; Noether symmetries; isometries.

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1. Introduction

Lie point symmetries provide operative tools to find solutions of differential equations [1–9]. Various techniques are applied using Lie point symmetries to integrate ordinary differential equations (ODEs), which under some considerations, can be applied for systems of ODEs [1,7,8]. If a system admits a transitive solvable Lie algebra of the Lie point symmetries of enough dimensions, then the solution can be found by line integrals. If there are several symmetries but they do not satisfy the condition of solvability, method of successive reduction of order using canonical forms of the symmetry generators may be purposive. Furthermore, invariants of Lie point symmetries can, in some cases, provide invariant solutions or first integrals obtained from differential invariants. However, if the Lagrangian of the system is known, then the variational symmetries (if exist) provide first integrals via Noether theorem. Similarly, if the ambient space–time of the system admits enough isometries, then according to Cartan theory [1] these readily provide the reduction of order of the system.

We show the efficiency of using Lie point symmetries to find a complete analytic solution of the geodesic system of Gödel's metric. Gödel's metric belongs to the family of the Einstein and the de Sitter space–times which are well known homogeneous Petrov type-D static metric [10–14]. It admits five space–time isometries and is stationary compared to the other two members of its family. It gained popularity and attracted much attention of many investigators [15–23] since its emergence because of its unusual features. The stress–energy tensor of this metric represents a rotating matter without having a singularity. Another notable characteristic of this space–time metric is that a portion of the trajectories of the free particles in it appears as time-like closed curve. Here we exhibit this feature of Gödel's metric explicitly.

We provide a complete solution of the geodesic system of Gödel's metric in a cylindrical coordinate system (t, r, ϕ, z), where $t < \infty$, $0 \le r \le \infty$, $0 \le \phi \le 2\pi$, $-\infty < z < \infty$, given by [15,18]

$$\mathrm{d}s^2 = a^2([\mathrm{d}t + \sqrt{2}\sinh^2 r\,\mathrm{d}\phi]^2 - \mathrm{d}r^2 - \mathrm{d}z^2 - \sinh^2 r\cosh^2 r\,\mathrm{d}\phi^2), \tag{1}$$

with the aid of Lie point symmetries. Here $a = 2/\omega$, where ω is a measure of the constant rotation of the matter flow. Assuming $\omega = 2$, the geodesic equations for this metric turn out to be

$$\ddot{t} + \frac{4\sinh r}{\cosh r}\dot{t}\dot{r} + \frac{2\sqrt{2}\sinh^3 r}{\cosh r}\dot{r}\dot{\phi} = 0, \tag{2a}$$

$$\ddot{r} + 2\sqrt{2}\sinh r\cosh r\,\dot{t}\dot{\phi}$$

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$$-\sinh r\cosh r(1-2\sinh^2 r)\dot{\phi}^2=0, \tag{2b}$$

$$\ddot{\phi}-\frac{2\sqrt{2}}{\sinh r\cosh r}\dot{t}\dot{r}+\frac{2}{\sinh r\cosh r}\dot{r}\dot{\phi}=0, \tag{2c}$$

$$\ddot{z}=0, \tag{2d}$$

where dot over the variables t, r, ϕ and z denote the derivatives with respect to the arc-length parameter s. In the next section, Lie point symmetry generators of geodesic system (2) are found which form a ten-dimensional Lie algebra and a brief analysis of the resulting Lie algebra is provided in terms of its various sub-algebras and their applicability to solve the system. In §3, the system is reduced, with the aid of the admitted Lie point symmetries, to an integrable autonomous first-order ODE which is then analytically solved in details. In the following section, Noether symmetries of the geodesic system besides the isometries of the space as well as their corresponding first integrals are found.

2. Lie point symmetries of the geodesic system

Finding Lie point symmetries (using Einstein summation convention)

$$X=\xi(s,x^a)\frac{\partial}{\partial s}+\eta^a(s,x^a)\frac{\partial}{\partial x^a} \tag{3}$$

of a system of k second-order ordinary differential equations

$$E_i(s,x^a,\dot{x}^a,\ddot{x}^a)=0,\quad i,a=1,\ldots,k, \tag{4}$$

means finding the general solution $\xi(s,x^a)$ and $\eta^a(s,x^a)$ of the determining equations obtained from the symmetry condition

$$\hat{X}(E_i)=0, \tag{5}$$

where $\hat{X}$ is the extension of the symmetry operator X written as

$$\hat{X}=\xi(s,x^a)\frac{\partial}{\partial s}+\eta^a(s,x^a)\frac{\partial}{\partial x^a}+(\eta^a)'(s,x^a,\dot{x}^a)\frac{\partial}{\partial \dot{x}^a}+(\eta^a)''(s,x^a,\dot{x}^a)\frac{\partial}{\partial \ddot{x}^a}$$

and

$$(\eta^a)^{(n)}=\frac{\mathrm{d}(\eta^a)^{(n-1)}}{\mathrm{d}s}-(x^a)^{(n)}\frac{\mathrm{d}\xi}{\mathrm{d}s}.$$

The solution of symmetry condition (5) applied onto system (2) gives a ten-dimensional Lie algebra (L_{10}) whose symmetry generators are

$$\begin{aligned}
&X_1=s\frac{\partial}{\partial s},\ X_2=z\frac{\partial}{\partial s},\ X_3=\frac{\partial}{\partial s},\\
&X_4=-\sqrt{2}\tanh r\sin\phi\frac{\partial}{\partial t}+\cos\phi\frac{\partial}{\partial r}\\
&\qquad-\frac{2\cosh^2 r-1}{\sinh r\cosh r}\sin\phi\frac{\partial}{\partial\phi},\\
&X_5=\sqrt{2}\tanh r\cos\phi\frac{\partial}{\partial t}\\
&\qquad+\sin\phi\frac{\partial}{\partial r}+\frac{2\cosh^2 r-1}{\sinh r\cosh r}\cos\phi\frac{\partial}{\partial\phi},\\
&X_6=\frac{\partial}{\partial\phi},\ X_7=\frac{\partial}{\partial t},\ X_8=s\frac{\partial}{\partial z},\\
&X_9=z\frac{\partial}{\partial z},\ X_{10}=\frac{\partial}{\partial z},
\end{aligned} \tag{6}$$

with non-vanishing commutators given by

$$\begin{aligned}
&[X_1,X_2]=-X_2,\ \ [X_1,X_3]=-X_3,\\
&[X_1,X_8]=X_8,\ \ [X_2,X_8]=X_9-X_1,\\
&[X_2,X_9]=-X_2,\ \ [X_2,X_{10}]=-X_3,\\
&[X_3,X_8]=X_{10},\ \ [X_4,X_5]=2\sqrt{2}X_7+4X_6,\\
&[X_4,X_6]=X_5,\ \ [X_5,X_6]=X_4,\\
&[X_8,X_9]=X_8,\ \ [X_9,X_{10}]=-X_{10}.
\end{aligned}$$

The derived algebra for L_{10} is L_{10} itself showing that it is neither solvable nor nilpotent. The sub-algebra

$$H_7=\langle X_1,X_2,X_3,X_6,X_7,X_9,X_{10}\rangle \tag{7}$$

is the maximal solvable sub-algebra of L_{10}. With a change of basis, L_{10} can be written as

$$L_{10}=\left\langle X_1+X_9,\ X_1-X_9,\ X_2,\ X_3,\ X_4,\ X_5,\ X_6+\frac{\sqrt{2}}{2}X_7,\ X_7,\ X_8,\ X_{10}\right\rangle.$$

The symmetries involving the arc length s would necessarily be gauge-dependent and those that correspond to zero gauge form a sub-algebra as, viz., [20]

$$L_7=\left\langle X_3,\ X_4,\ X_5,\ X_6+\frac{\sqrt{2}}{2}X_7,\ X_7,X_8,\ X_{10}\right\rangle. \tag{8}$$

Equation (2d) is exclusive from the rest of the ODEs of the system and can be integrated readily to give z as a linear function of s. Thus, we consider eqs (2a)–(2c) which form a coupled system admitting

$$H_6=\langle X_1,X_3,\ X_4,\ X_5,\ X_6,X_7\rangle. \tag{9}$$

In the next section, we show the employment of H_6 on solving geodesic systems (2a)–(2c).

3. Solving the geodesic system using H_6

The generators X_6 and X_7 show that the variables ϕ and t do not appear explicitly in the system. Thus, these generators will be no more useful for applying further reduction of order. So is the case with the generators X_4 and X_5 as their coefficients are dependent on the variable ϕ. Thus, we have only two generators X_1 and X_3 to be taken in the reduction process. The commutator

$$[X_1, X_3] = -X_3,$$

sets the order X_3, then X_1 is to be taken into consideration. Using X_3, which is in the normal form with respect to arc-length parameter s, the system can be reduced to first order with respect to all dependent variables by change of variables

$$\tau = r, \quad v_1 = t, \quad v_2 = s, \quad v_3 = \phi. \tag{10}$$

Then, transforming the system into the space of the canonical variables (τ, u_1, u_2, u_3) with respect to generator X_1 using the transformation:

$$\tau = \tau, \quad u_1 = \ln \dot{v}_2, \quad u_2 = \dot{v}_1, \quad u_3 = \dot{v}_3, \tag{11}$$

where $\dot{v}_i = (\mathrm{d}v_i/\mathrm{d}\tau)$ reduces the system into the form

$$\dot{u}_1 = 2\sqrt{2}\sinh\tau\cosh\tau\; u_2u_3 - \sinh\tau\cosh\tau(1-2\sinh^2\tau)u_3^2, \tag{12a}$$

$$\dot{u}_2 = -\frac{4\sinh\tau\; u_2 + 2\sqrt{2}\sinh^3\tau\; u_3}{\cosh\tau} + 2\sqrt{2}\sinh\tau\cosh\tau\; u_2^2u_3 - \sinh\tau\cosh\tau(1-2\sinh^2\tau)u_2u_3^2, \tag{12b}$$

$$\dot{u}_3 = \frac{2\sqrt{2}\;u_2 - 2\;u_3}{\cosh\tau} + 2\sqrt{2}\sinh\tau\cosh\tau\; u_2u_3^2 - \sinh\tau\cosh\tau(1-2\sinh^2\tau)u_3^3, \tag{12c}$$

where $\dot{u_i} = \mathrm{d}u_i/\mathrm{d}\tau$. The last two equations of system (12) form a system of two first-order ODEs in the two variables u_2 and u_3. In principle, solving this latter system for u_2 and u_3, substituting into eq. (12a) and transforming back into the original variables give the complete solution of geodesic system (2). Practically, this is a non-autonomous system of two nonlinear first-order ODEs which are in a non-integrable form. Moreover, it does not admit a Lie point symmetry. Thus, we turn to use the differential invariants which appear to be practically operative [7,8].

The differential invariants of the transitive Lie subgroup H_6, which form the solution $\Lambda_i(\mathbf{x})$ of the system

$$\hat{X}_N \Lambda_i(\mathbf{x}) = 0, \quad N = 2, \ldots, 6,$$

where $\mathbf{x} = (s, t, r, \phi, \dot{t}, \dot{r}, \dot{\phi})$ and $\hat{X_N}$ is the prolonged form of X_N up to the first derivatives of the dependent variables, are

$$\Lambda_1 = \dot{t} + \sqrt{2}\sinh^2 r\dot{\phi}, \tag{13a}$$

$$\Lambda_2 = \dot{r}^2 + \sinh^2 r\cosh^2 r\dot{\phi}^2. \tag{13b}$$

Invariants (13) satisfy geodesic system (2). Thus, they are first integrals, which are analogous to the first integrals found in ref. [15] according to his form of Gödel's metric. To reduce the order of the system, one requires a full set of first integrals which needs to be three in this case. However, one can use them to reduce the number of equations in the system. Substituting invariant solutions (13) into the geodesic system reduces the system of three equations to a single ordinary differential equation given as

$$\ddot{r} + 2\sqrt{2}\Lambda_1\sqrt{\Lambda_2 - \dot{r}^2} - \frac{2(\Lambda_2 - \dot{r}^2)}{\tanh 2r} = 0, \tag{14}$$

where Λ_i are constants. The above equation admits the only symmetry generator $\partial/\partial s$ by which it is reduced, via the transformation $v = \dot{r}, \; x = r$, to the first-order ODE

$$\frac{v\dot{v} + 2\sqrt{2}\Lambda_1\sqrt{\Lambda_2 - v^2}}{2\left(\Lambda_2 - v^2\right)} = \frac{2}{\tanh 2x}, \tag{15}$$

where $\dot{v} = (\partial v/\partial x)$. The solution of eq. (15) is given implicitly as

$$\sqrt{2}\Lambda_1\cosh 2x - \sqrt{\Lambda_2 - v^2}\sinh 2x + \Lambda_3 = 0, \tag{16}$$

where Λ_3 is a constant. Substituting back into the space of the original variables $s, r, \dot{r}$ gives

$$\dot{r}^2 = \Lambda_2 - \frac{\left(\sqrt{2}\Lambda_1\cosh 2r + \Lambda_3\right)^2}{\sinh^2 2r}. \tag{17}$$

Equation (17) offers the third functionally independent first integral of system (2) which, with eqs (13), provide a complete solution of system (2). This is given in detail as follows.

Equation (17) can be written as

$$\dot{r}^2 = \frac{4\Lambda_2\sinh^2 r\cosh^2 r - (\sqrt{2}\Lambda_1(\sinh^2 r + \cosh^2 r) + \Lambda_3)^2}{4\sinh^2 r\cosh^2 r} \tag{18}$$

or

$$(2\dot{r}\sinh r\cosh r)^2 = 4\Lambda_2\sinh^2 r\cosh^2 r - [\sqrt{2}\Lambda_1(\sinh^2 r + \cosh^2 r) + \Lambda_3]^2.$$

The left-hand side of this equation is nothing but $\left(\frac{\mathrm{d}}{\mathrm{d}s}\sinh^2 r\right)^2$. Thus, we can write it as

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}y\right)^2 = 4\Lambda_2 y(1+y) \\ -[\sqrt{2}\Lambda_1(y+y+1)+\Lambda_3]^2,$$

where $y=\sinh^2 r$. Hence

$$\dot{y}^2 = 4\Lambda_2(y^2+y) - [\sqrt{2}\Lambda_1(2y+1)+\Lambda_3]^2. \quad (19)$$

Expanding and regrouping the terms, this can be written as

$$\dot{y}^2 = -4(2\Lambda_1^2-\Lambda_2) \\ \times\left[y^2 - \frac{\Lambda_2 - 2\Lambda_1^2 - \sqrt{2}\Lambda_1\Lambda_3}{2\Lambda_1^2-\Lambda_2}y \\ + \frac{(\sqrt{2}\Lambda_1+\Lambda_3)^2}{4(2\Lambda_1^2-\Lambda_2)}\right]. \quad (20)$$

From eqs (13), it is clear that $\Lambda_2 \geq 0$, and

$$2\Lambda_1^2 - \Lambda_2 = 2(\dot{t}+\sqrt{2}\dot{\phi}^2\sinh^2 r)^2 - \dot{r}^2 \\ -\dot{\phi}^2\sinh^2 r\cosh^2 r. \quad (21)$$

Dividing eq. (1) by $\mathrm{d}s^2$

$$1 = (\dot{t}+\sqrt{2}\dot{\phi}^2\sinh^2 r)^2 - \dot{r}^2 \\ -\dot{\phi}^2\sinh^2 r\cosh^2 r - \dot{z}^2 \quad (22)$$

and subtracting from eq. (21) gives

$$2\Lambda_1^2 - \Lambda_2 = (\dot{t}+\sqrt{2}\dot{\phi}^2\sinh^2 r)^2 + 1 + \dot{z}^2 > 1. \quad (23)$$

This proves that $2\Lambda_1^2-\Lambda_2$ is a positive quantity. Moreover, from eq. (20)

$$\Lambda_2 - 2\Lambda_1^2 - \sqrt{2}\Lambda_1\Lambda_3 \\ = \frac{\dot{y}^2 + 4(2\Lambda_1^2-\Lambda_2)y^2 + (\sqrt{2}\Lambda_1+\Lambda_3)^2}{4y} > 0. \quad (24)$$

Thus, one can write eq. (20) as

$$\dot{y}^2 = -4A^2\left[y^2 - B^2 y + C^2\right], \quad (25)$$

where

$$A^2 = 2\Lambda_1^2-\Lambda_2, \quad B^2 = \frac{\Lambda_2 - 2\Lambda_1^2 - \sqrt{2}\Lambda_1\Lambda_3}{2\Lambda_1^2-\Lambda_2},$$

$$C^2 = \frac{(\sqrt{2}\Lambda_1+\Lambda_3)^2}{4(2\Lambda_1^2-\Lambda_2)}. \quad (26)$$

An alternative form of eq. (25) is

$$\dot{y}^2 = 4A^2\left[\frac{B^4-4C^2}{4} - \left(y-\frac{B^2}{2}\right)^2\right], \quad (27)$$

from which we can deduce that

$$B^4 - 4C^2 > 0. \quad (28)$$

Hence

$$\dot{y} = 2A\sqrt{\frac{B^4-4C^2}{4} - \left(y-\frac{B^2}{2}\right)^2}, \quad (29)$$

whose integral

$$y = \frac{B^2}{2} + \frac{\sqrt{B^4-4C^2}}{2}\cos(2As+s_\circ) = \sinh^2 r. \quad (30)$$

In terms of Λs,

$$y(s) = \frac{\Lambda_2 - 2\Lambda_1^2 - \sqrt{2}\Lambda_1\Lambda_3}{2\left(2\Lambda_1^2-\Lambda_2\right)} \\ + \frac{\sqrt{\Lambda_2\left(\Lambda 2 - 2\Lambda_1^2 + \Lambda_3^2\right)}}{2\left(2\Lambda_1^2-\Lambda_2\right)} \\ \times\cos\left(2\sqrt{2\Lambda_1^2-\Lambda_2}\,s + s_0\right). \quad (31)$$

To find $\dot{t}$ and $\dot{\phi}$ we use eqs (17) and (13b) from which we can find that

$$\dot{\phi}^2 = \frac{\left[\sqrt{2}\Lambda_1(\sinh^2 r+\cosh^2 r)+\Lambda_3\right]^2}{4\sinh^4 r\cosh^4 r}. \quad (32)$$

Taking the square root of both sides and regrouping the terms we can write it as

$$\dot{\phi} = \frac{\sqrt{2}\Lambda_1-\Lambda_3}{2\cosh^2 r} + \frac{\sqrt{2}\Lambda_1+\Lambda_3}{2\sinh^2 r}. \quad (33)$$

Substituting eq. (30) gives

$$\phi(s) = \frac{\sqrt{2}\Lambda_1-\Lambda_3}{2}I + \frac{\sqrt{2}\Lambda_1-\Lambda_3}{2}J, \quad (34)$$

where

$$I = \frac{\mathrm{d}s}{1+\frac{B^2}{2}+\frac{\sqrt{B^4-4C^2}}{2}\cos(2As+s_0)}$$

and

$$J = \frac{\mathrm{d}s}{\frac{B^2}{2}+\frac{\sqrt{B^4-4C^2}}{2}\cos(2As+s_0)}.$$

This gives

$$\phi(s) = \tan^{-1}\left(\sqrt{\frac{2+B^2-\sqrt{B^4-4C^2}}{2+B^2+\sqrt{B^4-4C^2}}}\tan\left(As+\frac{s_0}{2}\right)\right) \\ + \tan^{-1}\left(\sqrt{\frac{B^2-\sqrt{B^4-4C^2}}{B^2+\sqrt{B^4-4C^2}}}\tan\left(As+\frac{s_0}{2}\right)\right) + \phi_0. \quad (35)$$

Now substituting eq. (33) into eq. (13a) we get

$$t(s) = -\frac{\sqrt{2}\left(\Lambda_3 - \sqrt{2}\Lambda_1\right)}{\sqrt{B^4 - 4C^2}} I - \Lambda_1 s, \tag{36}$$

yielding

$$t(s) = \sqrt{2}\tan^{-1}\left(\sqrt{\frac{2+B^2-\sqrt{B^4-4C^2}}{2+B^2+\sqrt{B^4-4C^2}}} \times \tan\left(As+\frac{s_0}{2}\right)\right) - \Lambda_1 s + t_0. \tag{37}$$

In terms of the constants Λs, the time-like trajectories in the Gödel Universe in (t, r, ϕ)-hypersurface are given as

$$\begin{aligned}(t, r, \phi) = \Bigg(&\sqrt{2}\tan^{-1}\left(\sqrt{\frac{2+B^2-\sqrt{B^4-4C^2}}{2+B^2+\sqrt{B^4-4C^2}}} \times\tan\left(As+\frac{s_0}{2}\right)\right) - \Lambda_1 s + t_0,\\
&\sinh^{-1}\left(\sqrt{\frac{B^2}{2} + \frac{\sqrt{B^4-4C^2}}{2}\cos(2As+s_0)}\right),\\
&\tan^{-1}\left(\sqrt{\frac{2+B^2-\sqrt{B^4-4C^2}}{2+B^2+\sqrt{B^4-4C^2}}}\tan\left(As+\frac{s_0}{2}\right)\right)\\
&+\tan^{-1}\left(\sqrt{\frac{B^2-\sqrt{B^4-4C^2}}{B^2+\sqrt{B^4-4C^2}}}\tan\left(As+\frac{s_0}{2}\right)\right)+\phi_0\Bigg),\end{aligned} \tag{38}$$

where A, B and C are as defined in eq. (26).

4. Noether/gauge symmetries, conservation laws and isometries

It is well known that for a variational system, the Lie pint symmetry that leaves the action integral invariant leads to a conservation law via Noether's theorem. In this case, they are called variational or Noether symmetries [1,3,4,12]. Since the geodesic equations are derived from a metric, we have a 'natural' Lagrangian, with $a = 1$, arising from (1), viz.,

$$L = \dot{t}^2 - \dot{r}^2 - \dot{z}^2 + (2\sinh^4 r - \sinh^2 r\cosh^2 r)\dot{\phi}^2 + (2\sqrt{2}\sinh^2 r)\dot{t}\dot{\phi}. \tag{39}$$

L is substituted into a Killing-type equation given by

$$XL + LD_s\xi = D_s f, \tag{40}$$

where D_s is the total derivative and $f(s, t, r, z, \phi)$ is a gauge term. The conserved quantity or first integral is then given by

$$T = L\xi + (\eta^\alpha - \dot{x}^\alpha\xi)\frac{\partial L}{\partial \dot{x}^\alpha} - f, \tag{41}$$

where, by definition,

$$D_s T = 0 \tag{42}$$

along the solutions of geodesic equations (2).

Alternative to determining X by eq. (40), one may utilise the 'multiplier approach' [4] which determines the multipliers that render a system of equations to be closed/conserved. Here, each multiplier is of the form $(\eta^\alpha - \dot{x}^\alpha\xi)$ and is a 'symmetry' since the system is variational. The multiplier approach leads to table 1. The method does not explicitly generate the gauge functions.

The 'energy', from table 1 is $T = -(\dot{t} + \sqrt{2}\dot{\phi}\sinh^2 r)$. The Ts in table 1 are constants along the solutions of the geodesic equations.

Moreover, according to Cartan's theory, there exists a first integral, $X_a\dot{x}^a$ for each Lie point symmetry generator $X = \xi_a\partial_a$ satisfying the Killing equations [1].

$$X_{a;b} + X_{b;a} = 0, \tag{43}$$

where solution X of eq. (43) is an isometry of the space–time metric and is called a Killing vector. It was shown that for a natural Lagrangian, the Killing vectors of the metric form a sub-algebra of the Noether symmetries [23]. If the metric admits enough isometries, they are used to instantly reduce the order of the geodesic system by one, leading to a first-order system. Gödel's metric possesses five isometries which are the symmetry generators X_i where $i = 4, \ldots, 7, 10$ in eq. (6), providing five corresponding first integrals. This reduces the geodesic system to

$$\dot{t} = c_2\left[1 - \frac{2\sinh^2 r}{\cosh^2 r}\right] + \frac{\sqrt{2}c_1}{\cosh^2 r}, \tag{44a}$$

$$\dot{\phi} = \frac{\sqrt{2}c_2}{\cosh^2 r} - \frac{c_1}{\sinh^2 r\cosh^2 r}, \tag{44b}$$

$$\dot{r}^2 = c_2^2 - c_3 - \left(\frac{\sqrt{2}c_2\sinh r}{\cosh r} - \frac{c_1}{\sinh r\cosh r}\right)^2, \tag{44c}$$

$$\dot{r} = -(a\cos\phi + b\sin\phi), \tag{44d}$$

where c_1, c_2, c_3, a and b are constants. It is notable that

Table 1. First integrals corresponding to Noether/gauge symmetries of system (2).

X	T
X_3	$\dot{t}^2 - \dot{r}^2 - \dot{z}^2 + \dot{\phi}^2 \sinh^2 r(\sinh^2 -1) + 2\sqrt{2}\dot{t}\dot{\phi}\sinh^2 r$
X_4	$\dot{r}\cos\phi + 2\sqrt{2}\dot{t}\sin\phi\sinh r\cosh r + \dot{\phi}\sin\phi\sinh r\cosh r(2\sinh^2 r - 1)$
X_5	$\dot{r}\sin\phi - 2\sqrt{2}\dot{t}\cos\phi\sinh r\cosh r - \dot{\phi}\cos\phi\sinh r\cosh r(2\sinh^2 r - 1)$
X_6	$-\sinh^2 r((\sinh^2 r - 1)\dot{\phi} + \sqrt{2}\dot{t})$
X_7	$-(\dot{t} + \sqrt{2}\dot{\phi}\sinh^2 r)$
X_8	$s\dot{z}$
X_{10}	$s\dot{z} - z$

eq. (44c) is identical to eq. (17) with

$$\Lambda_1 = c_2, \quad \Lambda_2 = c_2^2 - c_3, \quad \Lambda_3 = -(2c_1 + \sqrt{2}c_2).$$

Thus, solving this equation with eqs (44a) and (44b) gives the complete solution for the geodesic system.

5. Conclusion

The geodesic system for Gödel's metric is shown to admit a 10-dimensional Lie algebra which contains a solvable sub-algebra of dimension seven. We focussed our attention on solving this system in the hypersurface $z =$ constant by making use of the admitted Lie point symmetries. Applying Lie's integration theorem requires the admitted transitive solvable Lie algebra to be of at least six dimensions, which is not the case here and thus the solution cannot be found through line integrals. Method of successive reduction reduces the problem to a system of two nonlinear first-order ODEs. Differential invariants provide two first integrals which reduce the problem to an autonomous second-order ODE. The invariance of this latter ODE under translation along the independent variable reduces it to an integrable first-rder ODE, which form an additional functionally-independent first integral for the geodesic system. Accordingly, these first integrals provide a complete analytic solution for the time-like geodesics. Besides, variational symmetries via Noether theorem are found which constitute a sub-algebra of Lie point symmetries. These provide conserved quantities (first integrals) of the geodesic system. Furthermore, isometries of the space–time, which form a sub-algebra of Noether symmetries, are singled out of the Lie point symmetries and used to find the first integrals, thus reducing the order of the geodesic system by one.

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