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Influence of nonlinearity on the Berry phase and thermal entanglement in deformed Jaynes–Cummings model

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Abstract. Deformed Jaynes–Cummings model (JCM) has a physical importance in quantum optics. Hence, we investigated the nonlinear JCM including the intensity-dependent coupling constant and the additional Kerr term. The cavity was assumed to be in thermal equilibrium with a heat reservoir at temperature T. Using the generators of a closed algebra which reduces to the $su(1, 1)$ and Heisenberg–Weyl algebras at limiting cases, and considering the total excitation number as a constant of motion, the total Hilbert space decomposes into two subspaces. So the eigenvalues and the corresponding eigenvectors were obtained. We derived the thermal density matrix and analysed the fidelity and thermal entanglement using negativity measure. Furthermore, we studied the Berry phase of the nonlinear atom–field system and explored the influence of nonlinearity on the quantum phase transition (QPT) point and entanglement. It is found that the deformation parameter can strongly affect the fidelity, negativity and QPT point.

Keywords. Berry phase; deformed Jaynes–Cummings model; fidelity; negativity; nonlinearity; thermal entanglement.

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1. Introduction

Entanglement which was introduced by Einstein *et al* in 1935 [1] is one of the most striking phenomenon in quantum physics. It is a key resource for quantum information theory such as quantum teleportation [2], communication and quantum computing [3,4]. Therefore, the generation of entangled state is an important subject. One of the most intriguing models in quantum optics for generating entangled states is the Jaynes–Cummings model (JCM) which was introduced by Jaynes and Cummings in 1963 [5]. The JCM describes the interaction of a two-level atom with a single-mode quantised radiation field which leads to a strong correlation between the atomic and field properties [6]. Due to the significance of JCM in quantum optics, this model has been extended to multiphoton JCM [7–9], multimode fields [10], multilevel atoms [11,12] and two-atom interactions [13]. As an important theoretical model, the JCM has been generalised to q-deformed quantum harmonic oscillator [14], f-deformed [15], parity deformed JCM [16] and nonlinear JCM [10,17,18]. Sivakumar [19] considered the nonlinear JCM where the interaction between the atom and electromagnetic field includes the Kerr nonlinearity, and then investigated quantum dynamical features such as atomic inversion, quantum statistics and squeezing. These nonlinear effects are one of the ways for producing entangled states, which are extensively applied in quantum information, especially in quantum communication. Dynamics of the entanglement in a nonlinear JCM with intensity-dependent coupling constant has been studied in [19–21].

Recently, the evolution of entanglement has been studied in cavities surrounded by the thermal environments which was named thermal entanglement. The thermal entanglement has been studied for the JCM by other researchers. For example, Abbasi and Golshan have studied the thermal entanglement between a two-level atom and photons [22]. Abbasi [23] studied the entanglement between two-coupled two-level atoms in a single-mode cavity, filled with a Kerr medium.

In 1984, Berry introduced a geometric phase and showed that after cyclic and adiabatic evolutions, the quantum state may acquire a geometric phase called the

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Berry phase in addition to the dynamical phase [24]. It has very interesting applications such as in quantum computation to construct a universal quantum logic gate that may be robust to certain kinds of errors [25,26]. One of the potential candidates for the verification of Berry phase is the two-level atom interacting with quantised field [27,28]. The Berry phase and entropy of the ground state for arbitrary photon number of a two-atom JCM with Kerr-like medium was studied by Bu *et al* [29]. It can be controlled by the atom–cavity coupling strength and the detuning parameter, which can be important in applications in geometric quantum computing [30].

In the present work, we intend to turn our attention to the nonlinear JCM, whose interaction term is intensity-dependent [20] when the cavity is held at temperature T. In this case, the Hamiltonian can be expressed through the generators of a closed algebra which reduces to $su(1,1)$ and Heisenberg–Weyl algebras at limiting cases. In order to examine the thermal atom–photon entanglement, we consider the total excitation number which decomposes the total Hilbert space into two subspaces. So the matrix representation of the Hamiltonian becomes block diagonal, a 1×1 block and N, 2×2 blocks. By evaluating the eigenvalues and corresponding eigenvectors, we derived the thermal density matrix and then obtained the negativity as a suitable measure of entanglement. The effect of nonlinearity on thermal entanglement was examined. To explore the overlap of the thermal atom–field system and the system at zero temperature, fidelity is calculated. Furthermore, we obtained the Berry phase of the atom–field system and analysed the influence of deformation parameter on the quantum phase transition (QPT) and the entanglement.

The paper is organised as follows: In §2, the nonlinear JCM with an intensity-dependent coupling constant is described in terms of the generators of a closed algebra. In §3, the thermal density matrix is obtained and the fidelity is explored as a numerical measure of the closeness of two quantum states. Thermal entanglement of the atom–field system is discussed in §4. In §5, the Berry phase of the system is evaluated after an adiabatic and cyclic evolution by introducing the phase shifter operator. Finally, some conclusions are given in §6.

2. Description of the nonlinear Jaynes–Cummings model

The interaction between a single two-level atom coupled to a single-mode quantised electromagnetic field is governed with the JCM. In the dipole and rotating wave approximations, the JCM Hamiltonian is given by [5]

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \tag{1}$$

with

$$\hat{H}_0 = \hbar\left(\omega\hat{a}^\dagger\hat{a} + \frac{\omega_0}{2}\hat{\sigma}_z\right)$$

and

$$\hat{H}_{\text{int}} = \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\sigma_-),$$

where $\hat{a}(\hat{a}^\dagger)$ stands for the annihilation(creation) operators for the field quanta, ω and ω_0 are the field and atomic transition frequencies, respectively. g is the atom–field coupling constant. The operators $\hat{\sigma}_\pm$ and $\hat{\sigma}_z$ are atomic operators for the two-level atom with ground state $|g\rangle$ and excited state $|e\rangle$ satisfying the commutation relations $[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$ and $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$. Here, we consider the $su(1,1)$ algebra with deformed single-photon operators where the generators are multiplied by an operator-valued function of the number operator [31,32]. Let us consider the Holstein–Primakoff realisation of the $su(1,1)$ Lie algebra:

$$\hat{K}_0 = \hat{a}^\dagger\hat{a} + j,$$
$$\hat{K}_+ = \hat{a}^\dagger\sqrt{\hat{a}^\dagger\hat{a} + 2j},\ \hat{K}_- = \hat{K}_+^\dagger, \tag{2}$$

where j is a constant and we set it equal to $\frac{1}{2}$ in the following discussion. For the Holstein–Primakoff realisation (with $j = 0$), the relevant interaction is

$$\hat{H} = \hat{H}_0 + \hbar g(\hat{\sigma}_+\sqrt{\hat{a}^\dagger\hat{a}}\hat{a} + \hat{\sigma}_- a^\dagger\sqrt{\hat{a}^\dagger\hat{a}}). \tag{3}$$

This interaction describes the single-photon interaction with nonlinear coupling between the atom and the field. We introduce an additional parameter k in the set of operators defined in eq. (2) [20]:

$$\hat{K}_+ := \hat{a}^\dagger\sqrt{1 + k\hat{a}^\dagger\hat{a}}, \quad \hat{K}_- := \sqrt{1 + k\hat{a}^\dagger\hat{a}}\,\hat{a}, \tag{4}$$

where the operators $\hat{K}_+$ and $\hat{K}_-$ satisfy the following commutation relations:

$$[\hat{K}_-, \hat{K}_+] = 2\hat{K}_0, \quad [\hat{K}_0, \hat{K}_\pm] = \pm k\hat{K}_\pm\,, \tag{5}$$

with $\hat{K}_0 = k\hat{a}^\dagger\hat{a} + \frac{1}{2}$. Clearly, the operators $\hat{K}_0$, $\hat{K}_+$ and $\hat{K}_-$ form a closed algebra. For $k = 1$, the commutation relations (5) form the $su(1,1)$ Lie algebra and the well-known Heisenberg–Weyl algebra for $\hat{a}$ and $\hat{a}^\dagger$ is held for $k = 0$. Therefore, a more general Hamiltonian including the nonlinear JCM with an intensity-dependent coupling constant and Kerr term is obtained as follows:

$$\begin{aligned}\hat{H} = \hat{H}_0 + \hbar\chi\hat{a}^{\dagger 2}\hat{a}^2 + g(\hat{\sigma}_+\sqrt{1 + k\hat{a}^\dagger\hat{a}}\hat{a} \\ +\hat{\sigma}_-\hat{a}^\dagger\sqrt{1 + k\hat{a}^\dagger\hat{a}}),\end{aligned} \tag{6}$$

in which the Kerr coupling is $\chi = k\omega$ and k is non-negative and less than or equal to unity ($0 < k \leq 1$). When $k = 0$, Hamiltonian (6) becomes the usual JC Hamiltonian and when $k = 1$, the usual

Holstein–Primakoff realisation is obtained. Using eq. (4), Hamiltonian (6) can be rewritten as

$$\hat{H} = \hbar\left(\omega\hat{K}_+\hat{K}_- + \frac{\omega_0}{2}\hat{\sigma}_z + g(\hat{K}_+\hat{\sigma}_- + \hat{K}_-\hat{\sigma}_+\right). \quad (7)$$

The total excitation number is $\hat{N} = \hat{K}_+\hat{K}_- + 2\hat{K}_0\hat{\sigma}_+\hat{\sigma}_-$, which commutes with Hamiltonian (7), $[\hat{H}, \hat{N}] = 0$. Recognising the total excitation number, the Hilbert space of the atom–field system with the basis $\{|i, n\rangle|_{n=0}^{\infty}, (i = e, g)\}$ is decomposed into two subspaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. The subspace $\mathcal{H}_1$ corresponds to the no-excitation subspace spanned by $|g, 0\rangle$, and the subspace $\mathcal{H}_2$ is spanned by $\{|e, n\rangle, |g, n+1\rangle\}$. Thus, the matrix form of the Hamiltonian becomes block diagonal with 1×1 and 2×2 blocks. In this manner, the Hamiltonian (7) may be written as $\hat{H} = \hat{H}_1 \oplus \hat{H}_2$ where $\hat{H}_1$ and $\hat{H}_2$ are given by

$$\begin{aligned}\hat{H}_1 &= -\frac{\hbar\omega_0}{2}|g, 0\rangle\langle g, 0|,\\ \hat{H}_2 &= \hbar\{\Omega_{11}\,|e, n\rangle\,\langle e, n| + \Omega_{22}\,|g, n+1\rangle\,\langle g, n+1|\\ &\quad + (\Omega_{12}\,|e, n\rangle\,\langle g, n+1| + \text{H.C.})\},\end{aligned} \quad (8)$$

with

$$\begin{aligned}\Omega_{11} &= \frac{\omega_0}{2} + \omega(kn^2 - kn + n),\\ \Omega_{22} &= -\frac{\omega_0}{2} + \omega(kn^2 + kn + n + 1),\\ \Omega_{12} &= g\sqrt{(1+n)(1+kn)}.\end{aligned} \quad (9)$$

For the sake of simplicity, we use Ω_1 and Ω_2 instead of Ω_{11} and Ω_{22}. The no-excitation block 1×1 corresponds to the separable ground state with the energy $-\hbar\omega_0/2$. For the 2×2 block, the eigenvalues can be readily evaluated as

$$E_{n,k}^{\pm} = \frac{\hbar}{2}(\Omega_1 + \Omega_2 \pm \Omega_{n,k}), \quad (10)$$

with $\Omega_{n,k} = \sqrt{\Delta_n^2 + 4g^2(1+n)(1+kn)}$ and $\Delta_n = \Delta - 2kn\omega$, where $\Delta = \omega_0 - \omega$ is the detuning parameter. The corresponding eigenvectors are given by

$$\begin{aligned}|\psi_n^+\rangle &= \cos\theta_n|e, n\rangle + \sin\theta_n|g, n+1\rangle,\\ |\psi_n^-\rangle &= \sin\theta_n|e, n\rangle - \cos\theta_n|g, n+1\rangle,\end{aligned} \quad (11)$$

in which

$$\cos\theta_n = \frac{2g\sqrt{(1+n)(1+kn)}}{\sqrt{(\Omega_{n,k} - \Delta_n)^2 + 4g^2(1+n)(1+kn)}}. \quad (12)$$

3. Thermal density matrix

The density operator in thermal equilibrium at temperature T, is given by

$$\rho(T) = \frac{1}{Z}\mathrm{e}^{-\frac{\hat{H}}{K_\mathrm{B}T}}, \quad (13)$$

where $Z = \mathrm{Tr}(\mathrm{e}^{-\frac{\hat{H}}{K_\mathrm{B}T}})$ is the partition function and K_B is the Boltzmann's constant. The thermal density matrix in terms of the eigenvectors and the corresponding eigenvalues can be expressed as

$$\begin{aligned}\rho_{\mathrm{af}}(T) &= \frac{1}{Z}\sum_{n=0}^{\infty}\{\alpha_n\,|e, n\rangle\,\langle e, n| + \beta_n\,|g, n\rangle\,\langle g, n|\\ &\quad + (\gamma_n\,|e, n\rangle\,\langle g, n+1| + \text{H.C.})\},\end{aligned} \quad (14)$$

with $\beta_0 = \mathrm{e}^{\frac{\hbar\omega_0}{2K_\mathrm{B}T}}$ and

$$\begin{aligned}\alpha_n &= \mathrm{e}^{-\frac{\hbar(\Omega_1+\Omega_2)}{2K_\mathrm{B}T}}\\ &\quad\times\left(\cosh\left(\frac{\hbar\Omega_{n,k}}{2K_\mathrm{B}T}\right) + \frac{\Omega_2 - \Omega_1}{\Omega_{n,k}}\sinh\left(\frac{\hbar\Omega_{n,k}}{2K_BT}\right)\right),\\ \beta_{n+1} &= \mathrm{e}^{-\frac{\hbar(\Omega_1+\Omega_2)}{2K_\mathrm{B}T}}\\ &\quad\times\left(\cosh\left(\frac{\hbar\Omega_{n,k}}{2K_\mathrm{B}T}\right) - \frac{\Omega_2 - \Omega_1}{\Omega_{n,k}}\sinh\left(\frac{\hbar\Omega_{n,k}}{2K_BT}\right)\right),\\ \gamma_n &= \mathrm{e}^{-\frac{\hbar(\Omega_1+\Omega_2)}{2K_\mathrm{B}T}}\\ &\quad\times\left(\frac{-2g\sqrt{(1+n)(1+kn)}}{\Omega_{n,k}}\sinh\left(\frac{\hbar\Omega_{n,k}}{2K_\mathrm{B}T}\right)\right),\\ Z &= \mathrm{e}^{\frac{\hbar\omega_0}{2K_\mathrm{B}T}} + 2\sum_{n=0}^{\infty}\mathrm{e}^{-\frac{\hbar(\Omega_1+\Omega_2)}{2K_\mathrm{B}T}}\cosh\left(\frac{\hbar\Omega_{n,k}}{2K_\mathrm{B}T}\right).\end{aligned} \quad (15)$$

At absolute zero temperature ($T = 0$), the atom–field thermal state collapses into its ground state, which has the following form:

$$\rho_{\mathrm{af}}^{(1)}(T) = \frac{\mathrm{e}^{-\frac{\hat{H}_1}{K_\mathrm{B}T}}}{\mathrm{Tr}(\mathrm{e}^{-\frac{\hat{H}_1}{K_\mathrm{B}T}})} = |g, 0\rangle\,\langle g, 0|\,, \quad (16)$$

where it corresponds to the no-excitation subspace $\mathcal{H}_1$. To quantify the distance between two quantum states, we use the fidelity. It is a numerical measure of how close one state of a quantum system ρ is to another state σ [33] which is defined as

$$F(\rho, \sigma) = \mathrm{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2. \quad (17)$$

If one of the states becomes pure, $\rho = |\psi\rangle\langle\psi|$, the fidelity is given by

$$F(\rho, \sigma) = \langle\psi|\sigma|\psi\rangle, \quad (18)$$

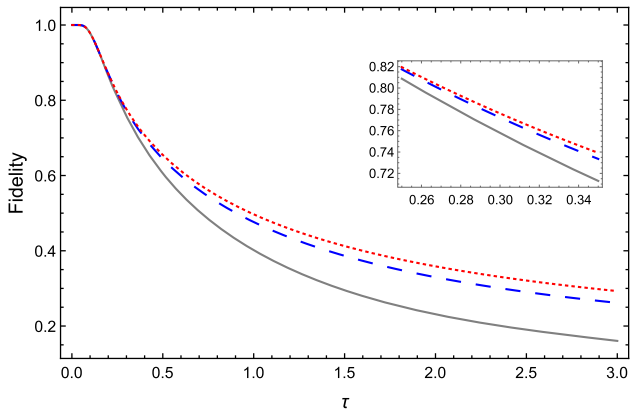

Figure 1. Fidelity vs. scaled dimensionless temperature $\tau = (K_B T/\hbar g)$, for $\omega = g$, $\omega_0 = 2g$ when $k = 0$ (full line), $k = 0.5$ (dashed line) and $k = 1$ (dotted line).

where $0 \leq F(\rho, \sigma) \leq 1$ and $F(\rho, \sigma) = 1$ if and only if $\rho = \sigma$. Using eq. (18), the fidelity between $\rho(T = 0)$ and $\rho(T)$ is obtained as

$$F(\rho(0), \rho_{\text{af}}(T)) = \frac{e^{\frac{\hbar\omega_0}{2K_B T}}}{Z}. \tag{19}$$

Figure 1 shows the plot of the fidelity vs. scaled dimensionless temperature, $\tau = (K_B T/\hbar g)$.

From figure 1, when the temperature is absolute zero ($T = 0$), the fidelity is maximum, i.e. $F_{\max} = 1$. This confirms that when $T = 0$, the atom–field thermal system collapses into its ground state. The fidelity between two states decreases at higher temperature. Moreover, by increasing the nonlinear parameter k, the fidelity decreases slowly.

4. Thermal atom–photon entanglement

Here, we evaluate the thermal entanglement in terms of the negativity as a suitable measure for any mixed states which is defined as [34]

$$\mathcal{N} = \sum_n \max(0, -\lambda_n), \tag{20}$$

where λ_n's are the eigenvalues of the partially transposed density matrix. Note that the partial transpose for a bipartite density matrix is given by: $\langle i, j| (\rho_{12})^{T_1} |k, l\rangle = \langle k, j| \rho_{12} |i, l\rangle$. Moreover, if the partially transposed density matrix does not have negative eigenvalues, the state is separable. The partially transposed density matrix, $(\rho_{\text{af}}(T))^{T_a}$ (partially transposed with respect to the atomic state) is obtained as

$$(\rho_{\text{af}}(T))^{T_a} = \frac{1}{Z}\left\{\alpha_0 |e, 0\rangle \langle e, 0| + \sum_{n=0}^{\infty} (\rho_n)^{T_a}\right\} \tag{21}$$

and

$$(\rho_n)^{T_a} = \alpha_{n+1} |e, n+1\rangle \langle e, n+1| + \beta_n |g, n\rangle \langle g, n| + (\gamma_n |e, n+1\rangle \langle g, n| + \text{H.C.}), \tag{22}$$

in which $(\rho_n)^{T_a}$ is a 2×2 matrix which has the following eigenvalues:

$$\lambda_n^{\pm} = \frac{1}{2Z}\{\alpha_{n+1} + \beta_n \pm ((\alpha_{n+1} - \beta_n)^2 + 4\gamma_n^2)^{1/2}\}. \tag{23}$$

From eq. (15), we deduce that $\alpha_0 > 0$. So, the only negative eigenvalue is λ_n^- under the condition $\alpha_{n+1}\beta_n < {\gamma_n}^2$. By introducing the scaled dimensionless parameters as

$$\tau := \frac{K_B T}{\hbar g}, \quad d := \frac{\omega}{g}, \quad \delta := \frac{\Delta}{g}, \tag{24}$$

where the parameters d and δ are scaled field frequency and detuning parameter, respectively. For simplicity, we restrict ourselves to the case of linear JCM, i.e. $k = 0$. From eqs (15) and (24), we get

$$\begin{aligned}\alpha_{n+1}\beta_n - \gamma_n^2 = \exp\left[-\frac{(2n+1)d}{\tau}\right] \\ \times \left(\cosh\left[\frac{1}{2\tau}\sqrt{\delta^2 + 4(n+2)}\right]\cosh\left[\frac{1}{2\tau}\sqrt{\delta^2 + 4n}\right]\right. \\ + \frac{\delta \cosh[\frac{1}{2\tau}\sqrt{\delta^2 + 4(n+2)}]\sinh[\frac{1}{2\tau}\sqrt{\delta^2 + 4n}])}{\sqrt{\delta^2 + 4n}} \\ - \frac{\delta \cosh[\frac{1}{2\tau}\sqrt{\delta^2 + 4n}]\sinh[\frac{1}{2\tau}\sqrt{\delta^2 + 4(n+2)}]}{\sqrt{\delta^2 + 4(n+2)}} \\ - \frac{\delta^2 \sinh[\frac{1}{2\tau}\sqrt{\delta^2 + 4n}]\sinh[\frac{1}{2\tau}\sqrt{\delta^2 + 4(n+2)}]}{\sqrt{(\delta^2 + 4(n+2))(\delta^2 + 4n)}} \\ \left. - \frac{4(n+1)\sinh^2[\frac{1}{2\tau}\sqrt{\delta^2 + 4(n+1)}]}{\delta^2 + 4(n+1)}\right).\end{aligned} \tag{25}$$

Figure 2 indicates $\alpha_{n+1}\beta_n - \gamma_n^2$ as a function of τ for $\omega = g$, $\omega_0 = 2g$ and $n = 0$. From figure 2, it is found that for $0 < \tau < \tau_0 \approx 0.45$, the term $\alpha_{n+1}\beta_n - \gamma_n^2$ is negative and the thermal atom–field system is entangled, whereas for $\tau > \tau_0 \approx 0.45$, the term $\alpha_{n+1}\beta_n - \gamma_n^2$ is positive and thus, there are no negative eigenvalues for the partially transposed density matrix. Then the thermal atom–field system is separable. Moreover, the behaviour of negativity as a function of the scaled dimensionless temperature, $\tau = (K_B T/\hbar g)$ for $\omega = g$, $\omega_0 = 2g$ and given k is plotted in figure 3.

This figure clearly illustrates that at zero temperature and at higher temperatures the atom–photon entanglement disappears. From eq. (15), it can be deduced that at zero or higher temperatures, the coefficient $\gamma_n \simeq 0$, and there are no negative eigenvalues for partially transposed density matrix and the thermal atom–field system is separable. The separability of the atom–field system at zero temperature is due to the fact that our system reduces to the ground state which is initially separable. Moveover, by increasing the deformation parameter k, the maximum value of negativity increases and shifts to higher temperatures. Also, it can be seen that larger values of k lead to robust atom–photon entangled states.

5. Berry phase

In this section, we evaluate the Berry phase of the system after an adiabatic and cyclic evolution for arbitrary photon number. In the subspace $N = (1+n)(1+kn)$, the basis are $\{|e,n\rangle, |g,n+1\rangle\}$ then Hamiltonian (7) in

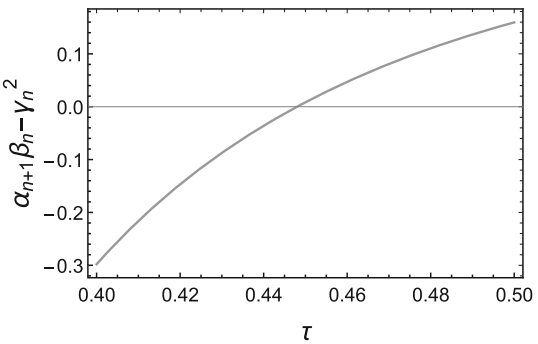

Figure 2. $\alpha_{n+1}\beta_n - \gamma_n^2$ as a function of τ for $\omega = g, \omega_0 = 2g$ and $n = 0$.

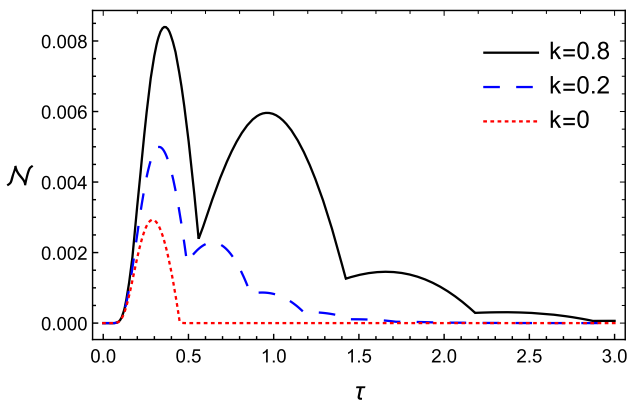

Figure 3. Negativity vs. scaled dimensionless temperature $\tau = (K_B T/\hbar g)$ for $\omega = g, \omega_0 = 2g$ and given k.

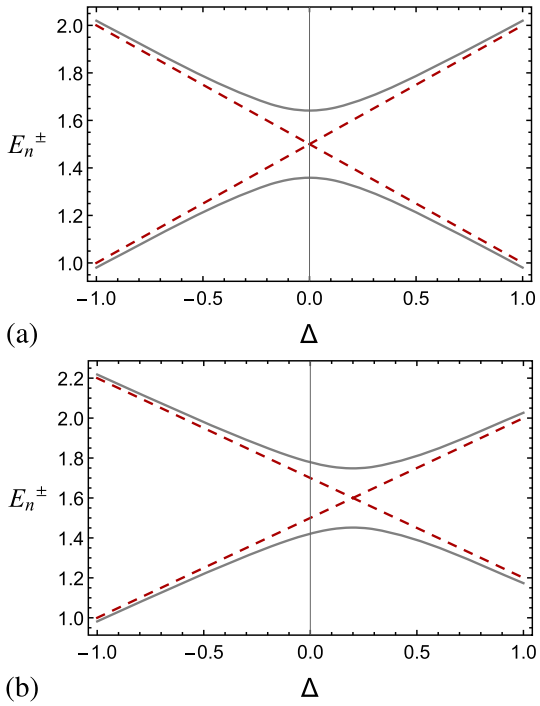

Figure 4. Energy eigenvalues vs. detuning parameter Δ for $\omega = 1$ and $n = 1$. (**a**) $k = 0$, (**b**) $k = 0.1$. The continuous and dashed curves correspond to $g = 0.1$ and $g = 0$, respectively. Lower part of the figure is for E_n^- and upper part for E_n^+.

that basis is written as follows ($\hbar = 1$):

$$\hat{H} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix}. \tag{26}$$

Note that for simplicity, we used Ω_1 and Ω_2 instead of Ω_{11} and Ω_{22}. The eigenvalues, $E_{n,k}^{\pm}$, and the corresponding eigenvectors $|\psi_n^{\pm}\rangle$ are given in eqs (10) and (11). The energy eigenvalues $E_{n,k}^{\pm}$ as a function of detuning parameter are plotted in figure 4. From figure 4, it can be seen that when there is no coupling between the atom and the field, i.e., when $g = 0$, the energy eigenvalues become degenerate by varying detuning parameter from negative to positive values. We say that the two energy levels cross each other. For non-zero coupling between the atom and the field, the degeneracy is split and the energy levels $E_{n,k}^+$ and $E_{n,k}^-$ no longer cross, the crossing is avoided. Furthermore, the comparison of figures 4a and 4b shows that the magnitudes of the energies shift to the upper values when the deformation parameter k gets bigger. On the other hand, the gap between the energies $E_{n,k}^+$ and $E_{n,k}^-$ decreases by increasing the deformation parameter. As can be seen from the figures, for linear JCM ($k = 0$, thus $\chi = k\omega = 0$) the level crossing occurs for exact resonance, i.e. $\Delta = 0$, while for the nonlinear JCM ($k \neq 0$) the level crossing shifts to positive detuning, $\Delta > 0$.

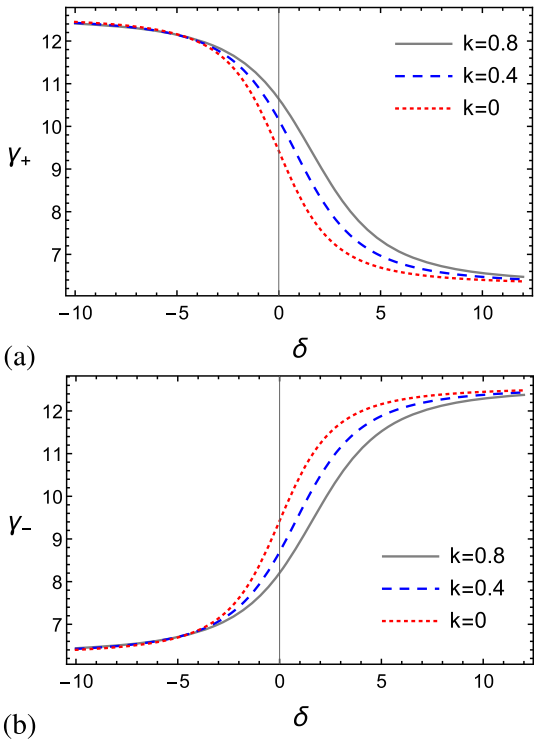

Figure 5. The Berry phases vs. rescaled detuning parameter $\delta = (\Delta/g)$, for $\omega = g$ and $n = 1$: (**a**) γ_+ and (**b**) γ_-.

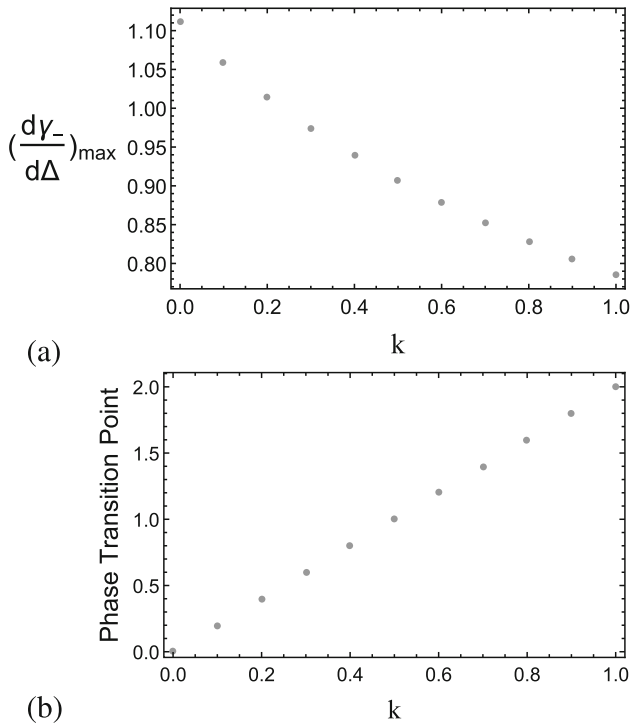

Figure 6. (**a**) $(\mathrm{d}\gamma_-/\mathrm{d}\Delta)_{\max}$ and (**b**) the phase transition point vs. k for $\omega = g$ and $n = 1$.

To evaluate the Berry phase for the atom–field system, we introduce the phase shifter operator as

$$\hat{U}(\varphi) = \mathrm{e}^{-i\varphi\hat{a}^\dagger\hat{a}}, \tag{27}$$

where φ changes slowly from zero to 2π to satisfy the cyclic and adiabatic conditions. The Berry phase of the atom–field system after the adiabatic and cyclic evolutions may be calculated as follows:

$$\gamma_\pm = i\oint \mathrm{d}\varphi\langle\psi_n^\pm(\varphi)|\dot{\psi}_n^\pm(\varphi)\rangle. \tag{28}$$

Then

$$\gamma_\pm = i\int_0^{2\pi}\mathrm{d}\varphi\left\langle\psi_n^\pm\left|U^\dagger(\varphi)\frac{\mathrm{d}}{\mathrm{d}\varphi}U(\varphi)\right|\psi_n^\pm\right\rangle, \tag{29}$$

where $\left|\psi_n^\pm\right\rangle$ are the energy eigenvectors of the Hamiltonian. By substituting eigenvectors (11) into (29), the Berry phases can be obtained a

$$\begin{aligned}\gamma_+ &= 2\pi(n+\sin^2\theta_n),\\ \gamma_- &= 2\pi(n+\cos^2\theta_n).\end{aligned} \tag{30}$$

Clearly, the Berry phase is dependent on n and for $n = 0$, the phases $\gamma_\pm$ is non-zero which means that the vacuum field introduces a correction in Berry phase and there is no counterpart in the classical physics [28]. To analyse the effect of deformation parameter, we plot the Berry phases $\gamma_\pm$ vs. rescaled dimensionless detuning parameter $\delta = (\Delta/g)$ for different values of k in figure 5.

From figure 5, it can be seen that the Berry phases illustrate different behaviours for $\Delta > 0$ and $\Delta < 0$. The ground-state Berry phase γ_- jumps up for positive and jumps down for negative detuning parameter. In other words, for $k = 0$ the QPT occurs at $\Delta = 0$. By increasing the deformation parameter k, the phase transition point shifts to $\Delta > 0$, i.e., it can strongly affect the QPT point. We analyse in detail the influence of the deformation parameter on the quantum phase transition point in figure 6.

It can be seen that when the parameter k increases, the Berry phase in the transition point decreases (see figure 6a) and the phase transition point shifts from $\Delta = 0$ to $\Delta > 0$ (see figure 6b). To compare the Berry phase and the entanglement, we calculate the von Neumann entropy as a measure for the pure entangled state. It is defined as

$$S_\mathrm{a} = -\mathrm{Tr}_\mathrm{a}(\rho_\mathrm{a}\log\rho_\mathrm{a}), \quad S_\mathrm{f} = -\mathrm{Tr}_\mathrm{f}(\rho_\mathrm{f}\log\rho_\mathrm{f}), \tag{31}$$

where $\rho_\mathrm{a} = \mathrm{Tr}_f(\rho_\mathrm{af})$ and $\rho_\mathrm{f} = Tr_\mathrm{a}(\rho_\mathrm{af})$ are the reduced density matrices of the atom and field, respectively, and $\rho_\mathrm{af} = |\psi\rangle\langle\psi|$ is the atom–field density matrix. Then, for the reduced density matrices of the atom and field, the degree of entanglement can be obtained as

$$S_\mathrm{a} = S_\mathrm{f} = -\lambda_1\log\lambda_1 - \lambda_2\log\lambda_2, \tag{32}$$

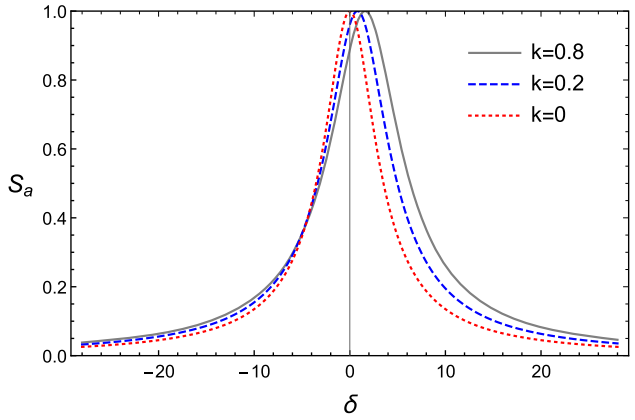

Figure 7. The von Neumann entropy vs. δ for $\omega = g, n = 1$ for different values of k.

in which λ_1 and λ_2 are the eigenvalues of the reduced density matrices ρ_a and ρ_f. The von Neumann entropy ranges from 0 for a separable state to 1 for a maximally entangled state. From eqs (11) and (32), the von Neumann entropy is obtained for the energy eigenvectors. In figure 7, the entropy vs. δ is plotted for $\omega = g, n = 1$ and different values of k. From figure 7, we see that for $k = 0$ the atom–field entanglement is maximal for the resonance case, $\Delta = 0$. By increasing the deformation parameter k, the maximum of the entropy shifts to $\Delta > 0$. A comparison of figures 5a, 5b and 7 shows that the Berry phase and the entropy show similar behaviour at point $\delta = 0$. For $k = 0$, the QPT occurs at resonance case and $\gamma_\pm = 3\pi$, where the entanglement is maximal. By increasing k, the QPT point and the maximum of entropy shift to $\Delta > 0$.

6. Conclusion

In summary, we introduced the nonlinear JCM with an intensity-dependent coupling constant and Kerr term at temperature T. Using the generators of a closed algebra and considering the total excitation number as a constant of motion, the total Hilbert space decomposed into two subspaces. So the eigenvalues and the corresponding eigenvectors of the Hamiltonian were calculated. We found that for $g = 0$, the energy eigenvalues become degenerate and for non-zero coupling between the atom and the field, the cross level is disappeared. Moreover, the deformation parameter shifted the energies to upper values. Then the thermal density matrix was obtained. The fidelity between the thermal atom–field system and the system at absolute zero temperature ($T = 0$) is maximum ($F_{\max} = 1$), confirming that the thermal atom–field system collapses into its ground state at $T = 0$. By increasing the nonlinear parameter k, the fidelity wanes slowly. The effects of the temperature and deformation parameter on the atom–field entanglement were studied using negativity measure. It was found that at $T = 0$ and at higher temperature, the atom–field system was separable. The presence of the deformation parameter enhances the maximum value of the entanglement and shifts it to higher temperatures. Furthermore, larger values of k generate robust atom–photon entangled states. Finally, we evaluated the Berry phase of the energy eigenvectors and found that for $n = 0$, the Berry phases $\gamma_\pm$ were non-zero which have no counterpart in classical physics. The diagrams of the Berry phase and the energy eigenvectors entropy vs. rescaled dimensionless detuning parameter δ demonstrated that for $k = 0$, QPT occurs at $\Delta = 0$ where the entanglement is maximal and by increasing the deformation parameter, the QPT point and the maximum of entanglement shift to $\Delta > 0$. Hence, we deduced that the deformation parameter can strongly affect the fidelity, entanglement and QPT point.

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