Pramana – J. Phys. (2022) 96:72
https://doi.org/10.1007/s12043-022-02319-w

Intrinsic decoherence for the displaced harmonic oscillator

ALEJANDRO R URZÚA* and HÉCTOR M MOYA-CESSA

Coordinación de Óptica, Instituto Nacional de Astrofísica, Óptica y Electrónica, Tonantzintla, Puebla 72840, Mexico
*Corresponding author. E-mail: arurz@inaoep.mx

MS received 25 August 2021; revised 18 November 2021; accepted 30 November 2021

Abstract. By using the complete solution of the Milburn equation (beyond the Lindblad form that it is generally used) that describes intrinsic decoherence, we studied the decaying dynamics of a displaced harmonic oscillator. We calculated the expectation values of position quadrature and the number operator in the initial coherent and squeezed states.

Keywords. Decoherence; harmonic oscillator; squeezed states.

PACS Nos 03.65.-w; 03.65.Yz; 03.70.+k; 03.65.Ud

1. Introduction

Decoherence is the main enemy of the non-classical properties that may be generated in quantum mechanical systems, and it is interesting to study it to learn how the properties that arise in such systems may be maintained. Some years ago, Milburn [1] proposed a modification of the Schrödinger equation that accounts for (intrinsic) decoherence. By assuming that the system evolves by a random sequence of unitary phase changes on sufficiently small time scales, Milburn produced a Lindblad equation with the Hamiltonian as the relevant operator. Since then, several researchers have studied the decay of coherences for different systems. For instance, it has been shown that in the atom–field interaction such loss of coherences prevents the revivals to occur for the atomic population inversion [2]. Yang *et al* [3] determined the performance of quantum Fisher information of the two-qutrit isotropic Heisenberg XY chain subject to decoherence. Mohamed *et al* [4] analysed the robustness of quantum correlations of the nearest-neighbour and the next-to-neighbour qubits in an intrinsic noise model describing the dynamics of decoherence for a system formed by three-qubit Heisenberg XY chain. Zheng and Zhang [5] applied Milburn's scheme to study the entanglement in the Jaynes–Cummings model, where a pair of atoms undergo Heisenberg-type interactions; He *et al* [6] studied the coherence dynamics of two atoms in a Kerr-like medium. Muthuganesan and Chandrasekar [7] applied intrinsic decoherence when studying an exactly solvable model of two interacting spin-$\frac{1}{2}$ qubits described by the Heisenberg anisotropic interaction. Chlih *et al* [8] used intrinsic decoherence to study a variety of initial states, where they obtain the temporal evolution of quantum correlations in a two-qubit XXZ Heisenberg spin chain model subject to a Dzyaloshinskii–Moriya (DM) interaction and to an external uniform magnetic field. Intrinsic decoherence scheme offers and alternative to the study of coherence phenomena under symmetry breaking, just as Gong *et al* [9] showed that in a ring arrangement of coupled harmonic oscillators. Guo-Hui and Bing Bing [10] estimated the quantum discord of two qubits that loose coherence through intrinsic mechanisms. Furthermore, Mohamed *et al* [11] have used the intrinsic decoherence effect for two qubits interacting with a coherent field, with the purpose to protect the entropy and entanglement from the dipole–dipole interaction. León-Montiel *et al* [12] have shown that noise may be helpful to assist energy transfer in coupled oscillators. Bayen and Mandal [13] reported the analytical solution of quartic anharmonic oscillator with driven force that show squeezing effects on the coherent states. Some time ago, Mandal [14] showed a way to calculate the quasidistribution of photons in the squeezed states of light when the parameters are taken as purely real. Germain *et al* [15] explicitly showed the effects of a non-commutative bath of oscillators on a magneto-oscillator. They gave the solution for decoherence without dissipation using the master's equation formulation. Recently,

Published online: 07 April 2022

Lu [16] reported the evolution of orthogonal coherent states given the solution of the master's equation, showing how the decoherence affects the negativity of the associate Wigner's function. Mohamed [17] and Abhignan and Muthuganesan [18] have studied the role of intrinsic decoherence in the subsystem correlations by using the quantum discord, finding how the decoherence parameter leads to sudden death and revivals of some geometrical and dynamical quantities, like entanglement.

The present paper is organised as follows: In §2 we give a review of the Milburn's intrinsic decoherence method. Section 3 introduces the displaced harmonic oscillator in terms of annihilation and creation operators. There, we calculate the expectation values for the position quadrature and the number operators for a general initial condition. Section 3.1 details the expectation values when we have an initial coherent state. Section 3.2 details the expectation values when we have an initial squeezed state for complex squeezing parameter; here, we hold the real part constant and vary the phase to look for changes in the dynamics. Section 4 discusses the results and conclusions.

2. Intrinsic decoherence equation and solution

Decoherence is the central research topic in quantum mechanics, as it shows how a quantum system starts to lose it quantumness because the system under study is usually surrounded by an environment that affects it. However, there may be other causes of decoherence that are intrinsic to the system [1]. We may say that there exist two main causes of decoherence [19]: First, those that are driven by environmental interactions, oscillators or spins bath, for example, where the Linblad master equations play the role of describing mathematical framework; and the other is intrinsic decoherence that is driven by the mere existence of the system dynamics in the equations that described their nature. We shall study here the second, by using the equation that Milburn [1] introduced to modifiy the Schrödinger equation and that describes (intrinsic) decoherence via the equation

$$\dot{\rho} = \gamma (\mathrm{e}^{-\mathrm{i}\frac{\hat{H}}{\gamma}} \rho \mathrm{e}^{\mathrm{i}\frac{\hat{H}}{\gamma}} - \rho), \tag{1}$$

where $\hat{H}$ is the system's Hamiltonian and γ is the intrinsic decoherence parameter that states the rate of decaying in the dynamics. By developing the exponential functions above in Taylor series, and keeping terms up to the second order, we obtain

$$\dot{\rho} \approx \gamma \left(\left[1 - \mathrm{i}\frac{\hat{H}}{\gamma} - \frac{\hat{H}^2}{\gamma^2} \right] \rho \left[1 + \mathrm{i}\frac{\hat{H}}{\gamma} - \frac{\hat{H}^2}{\gamma^2} \right] - \rho \right) \tag{2}$$

that can be rewritten in the Lindblad form as

$$\dot{\rho} = -\mathrm{i}[\hat{H}, \rho] - \frac{1}{\gamma}[\hat{H}, [\hat{H}, \rho]],$$

where the Schrödinger equation is recovered when $\gamma \to \infty$. This last equation shows us that a Linbladian may be obtained from the complete intrinsic decoherence evolution, describing the system with different levels of accuracy, since the parameter expansion leads to drop higher order of decoherence.

However, a complete solution of (1) may be obtained as

$$\rho(t) = \mathrm{e}^{-\gamma t} \mathrm{e}^{\hat{S}t} \rho(0), \tag{3}$$

where the superoperator

$$\hat{S}\rho = \gamma \mathrm{e}^{-\mathrm{i}\frac{\hat{H}}{\gamma}} \rho \mathrm{e}^{\mathrm{i}\frac{\hat{H}}{\gamma}},$$

has been used, such that

$$\mathrm{e}^{\hat{S}t}\rho(0) = \sum_{k=0}^{\infty} \frac{(\gamma t)^k}{k!} \rho_k,$$

with the kth element of the density matrix ρ defined by

$$\rho_k = |\psi_k\rangle\langle\psi_k|, \quad |\psi_k\rangle = \mathrm{e}^{-\mathrm{i}k\frac{\hat{H}}{\gamma}} |\psi(0)\rangle, \tag{4}$$

with $|\psi(0)\rangle$ as the initial wave function.

3. Displaced harmonic oscillator

We start with the Hamiltonian for a displaced harmonic oscillator (we set $\hbar = 1$)

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \lambda \left(\hat{a} + a^\dagger \right), \tag{5}$$

where ω is the natural frequency of the standard harmonic oscillator and λ is the amplitude of the displacement, meaning that the potential function is moved and scaled.

It is not difficult to show that Hamiltonian (5) may be rewritten as

$$\hat{H} = \omega \hat{D}^\dagger \left(\frac{\lambda}{\omega} \right) \hat{a}^\dagger \hat{a} \hat{D}^\dagger \left(\frac{\lambda}{\omega} \right) - \frac{\lambda^2}{\omega}, \tag{6}$$

where $\hat{D}\,(\lambda/\omega)$ is the usual displacement operator. With this last equation, we obtain the kth element of (4) as

$$|\psi_k\rangle = \mathrm{e}^{\mathrm{i}\frac{k\lambda^2}{\gamma\omega}} \hat{D}^\dagger \left(\frac{\lambda}{\omega} \right) \mathrm{e}^{-\mathrm{i}\hat{a}^\dagger \hat{a}\frac{k\omega}{\gamma}} \hat{D} \left(\frac{\lambda}{\omega} \right) |\psi(0)\rangle. \tag{7}$$

We now study two properties of quantum mechanical system, namely the position quadrature and the average number of photons. It is interesting first, to denote how the decoherence parameter affects the position quadrature, since it represents the real part of the complex amplitude of the annihilation operator $\hat{a}$; next, the effect on the photon number is explicit since we can observe how the population tends to decay in time depending on the strength of the decaying parameter γ.

Average of the position quadrature operator

We start by calculating the matrix elements of (twice) the position quadrature operator over the kth components of the wave function, and we have then

$$
\begin{aligned}
&\langle\psi_k|\hat{a}^\dagger+\hat{a}|\psi_k\rangle\\
&=\left[\langle\psi(0)|\hat{D}^\dagger\left(\frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}\hat{a}^\dagger\hat{a}}\hat{D}\left(\frac{\lambda}{\omega}\right)\right](\hat{a}^\dagger+\hat{a})\\
&\times\left[\hat{D}^\dagger\left(\frac{\lambda}{\omega}\right)\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}\hat{a}^\dagger\hat{a}}\hat{D}\left(\frac{\lambda}{\omega}\right)|\psi(0)\rangle\right]\\
&=\langle\psi(0)|\hat{D}^\dagger\left(\frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}\hat{a}^\dagger\hat{a}}\left[\hat{a}^\dagger+\hat{a}-2\frac{\lambda}{\omega}\right]\\
&\times\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}\hat{a}^\dagger\hat{a}}\hat{D}\left(\frac{\lambda}{\omega}\right)|\psi(0)\rangle\\
&=\langle\psi(0)|\hat{D}^\dagger\left(\frac{\lambda}{\omega}\right)\left[\hat{a}^\dagger\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}}+\hat{a}\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}}-2\frac{\lambda}{\omega}\right]\\
&\times\hat{D}\left(\frac{\lambda}{\omega}\right)|\psi(0)\rangle\\
&=\langle\psi(0)|\left[\left(\hat{a}^\dagger+\frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}}\right.\\
&\left.+\left(\hat{a}+\frac{\lambda}{\omega}\right)\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}}-2\frac{\lambda}{\omega}\right]|\psi(0)\rangle,
\end{aligned}
\tag{8}
$$

which finally gives

$$
\begin{aligned}
&\langle(\hat{a}^\dagger+\hat{a})\rangle=\mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\psi_k|\hat{a}^\dagger+\hat{a}|\psi_k\rangle\\
&=\mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\psi(0)|\left[\left(\hat{a}^\dagger+\frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}}\right.\\
&\left.+\left(\hat{a}+\frac{\lambda}{\omega}\right)\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}}-2\frac{\lambda}{\omega}\right]|\psi(0)\rangle.
\end{aligned}
\tag{9}
$$

Given a suitable initial wave function $|\psi(0)\rangle$, we can sum up the terms and find analytical expressions for the average dynamics.

Average of the number of photons operator

For the average of the number of photons, we follow the procedure depicted in eqs (8) and (9), first calculating the matrix elements of the operator $\hat{a}^\dagger\hat{a}$

$$
\begin{aligned}
&\langle\psi_k|\hat{a}^\dagger\hat{a}|\psi_k\rangle\\
&=\left[\langle\psi(0)|\hat{D}^\dagger\left(\frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}\hat{a}^\dagger\hat{a}}\hat{D}\left(\frac{\lambda}{\omega}\right)\right]\hat{a}^\dagger\hat{a}\\
&\times\left[\hat{D}^\dagger\left(\frac{\lambda}{\omega}\right)\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}\hat{a}^\dagger\hat{a}}\hat{D}\left(\frac{\lambda}{\omega}\right)|\psi(0)\rangle\right]\\
&=\langle\psi(0)|\hat{D}^\dagger\left(\frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}\hat{a}^\dagger\hat{a}}\\
&\times\left[\hat{a}^\dagger\hat{a}-\frac{\lambda}{\omega}(\hat{a}^\dagger+\hat{a})+\frac{\lambda^2}{\omega^2}\right]\\
&\times\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}\hat{a}^\dagger\hat{a}}\hat{D}\left(\frac{\lambda}{\omega}\right)|\psi(0)\rangle\\
&=\langle\psi(0)|\hat{D}^\dagger\left(\frac{\lambda}{\omega}\right)\\
&\times\left[\hat{a}^\dagger\hat{a}-\frac{\lambda}{\omega}\left(\hat{a}^\dagger\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}}+\hat{a}\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}}\right)+\frac{\lambda^2}{\omega^2}\right]\\
&\times\hat{D}\left(\frac{\lambda}{\omega}\right)|\psi(0)\rangle\\
&=\langle\psi(0)|\left[\left(\hat{a}^\dagger\hat{a}+\frac{\lambda}{\omega}\left\{\hat{a}^\dagger+\hat{a}\right\}+2\frac{\lambda^2}{\omega^2}\right)\right.\\
&\left.-\frac{\lambda}{\omega}\left(\left\{\hat{a}^\dagger+\frac{\lambda}{\omega}\right\}\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}}+\left\{\hat{a}+\frac{\lambda}{\omega}\right\}\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}}\right)\right]\\
&\times|\psi(0)\rangle,
\end{aligned}
\tag{10}
$$

that in the same fashion of eq. (9), we only need to give a suitable initial condition $|\psi(0)\rangle$ to obtain explicit expressions.

3.1 *Initial coherent state*

Let $|\psi(0)\rangle=|\alpha\rangle$ be a coherent state. We can use this initial condition to evaluate the position quadrature and number operators averages in the way that eq. (9) dictates.

First, we start by calculating the dynamics of (twice) the position quadrature operator, as $\langle\alpha|\hat{a}^\dagger|\alpha\rangle=\alpha^*$ and $\langle\alpha|\hat{a}|\alpha\rangle=\alpha$, and we obtain

$$
\begin{aligned}
&\langle\hat{a}^\dagger+\hat{a}\rangle=\mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\psi_k|\hat{a}^\dagger+\hat{a}|\psi_k\rangle\\
&=\mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\alpha|\\
&\times\left[\left(\hat{a}^\dagger+\frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}}+\left(\hat{a}+\frac{\lambda}{\omega}\right)\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}}-2\frac{\lambda}{\omega}\right]|\alpha\rangle\\
&=\mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\left[\left(\alpha^*+\frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}}\right.\\
&\left.+\left(\alpha+\frac{\lambda}{\omega}\right)\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}}-2\frac{\lambda}{\omega}\right]\\
&=\mathrm{e}^{-\gamma t}\left[\left(\alpha^*+\frac{\lambda}{\omega}\right)\mathrm{e}^{\gamma t\mathrm{e}^{\frac{\mathrm{i}\omega}{\gamma}}}\right.
\end{aligned}
$$

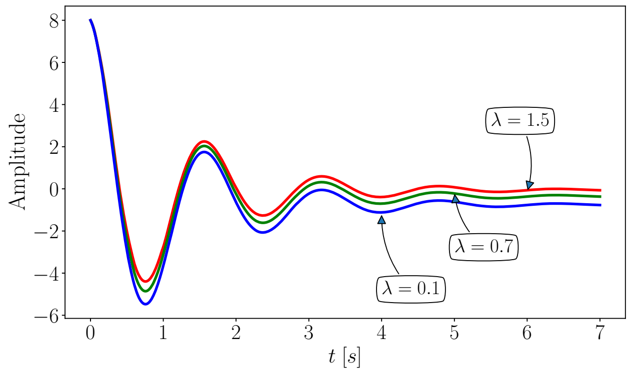

Figure 1. Expectation value of $\langle(\hat{a}^\dagger + \hat{a})\rangle$ given by (11). The parameters are $\alpha = 4, \omega = 4, \gamma = 10$. We vary the displacement strength $\lambda \in \{0.1, 0.7, 1.5\}$.

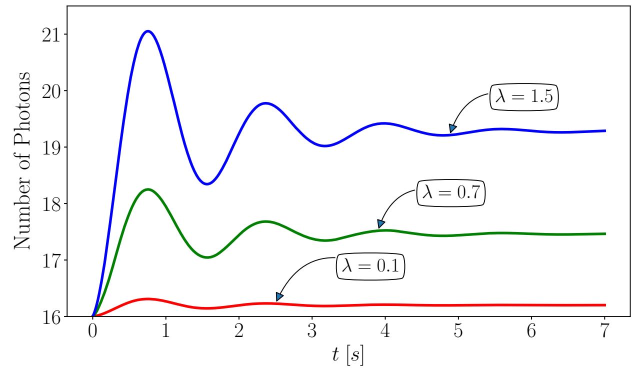

Figure 2. Expectation value of $\langle\hat{a}^\dagger\hat{a}\rangle$ given by (12). The parameters are $\alpha = 4, \omega = 4, \gamma = 10$. We vary the displacement strength $\lambda \in \{0.1, 0.7, 1.5\}$.

$$+\left(\alpha + \frac{\lambda}{\omega}\right) \mathrm{e}^{\gamma t \mathrm{e}^{-\frac{\mathrm{i}\omega}{\gamma}}} - 2\frac{\lambda}{\omega}\mathrm{e}^{\gamma t}\Bigg]$$
$$= \left(\alpha^* + \frac{\lambda}{\omega}\right) \mathrm{e}^{-\gamma t\left(1-\mathrm{e}^{\frac{\mathrm{i}\omega}{\gamma}}\right)}$$
$$+\left(\alpha + \frac{\lambda}{\omega}\right) \mathrm{e}^{-\gamma t\left(1-\mathrm{e}^{-\frac{\mathrm{i}\omega}{\gamma}}\right)} - 2\frac{\lambda}{\omega}, \tag{11}$$

that checks $\langle(\hat{a}^\dagger + \hat{a})\rangle = \alpha^* + \alpha \equiv 2\mathrm{Re}\{\alpha\}$, when $t = 0$. We plot the expectation value for $\langle(\hat{a}^\dagger + \hat{a})\rangle$ for several displacement strengths in figure 1, showing the same behaviour, i.e., a damping in the oscillations. Such damping is not amplified for larger displacement amplitudes.

For the average dynamics of the number of photons, we have

$$\langle\hat{a}^\dagger\hat{a}\rangle = \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\psi_k|\hat{a}^\dagger\hat{a}|\psi_k\rangle$$
$$= \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\alpha|$$
$$\times\Bigg[\left(\hat{a}^\dagger\hat{a} + \frac{\lambda}{\omega}\left\{\hat{a}^\dagger + \hat{a}\right\} + 2\frac{\lambda^2}{\omega^2}\right)$$
$$-\frac{\lambda}{\omega}\left(\left\{\hat{a}^\dagger + \frac{\lambda}{\omega}\right\}\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}} + \left\{\hat{a} + \frac{\lambda}{\omega}\right\}\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}}\right)\Bigg]|\alpha\rangle$$
$$= \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\Bigg[\left(|\alpha|^2 + \frac{\lambda}{\omega}\left\{\alpha + \alpha^*\right\} + 2\frac{\lambda^2}{\omega^2}\right)$$
$$-\frac{\lambda}{\omega}\left(\left\{\alpha + \frac{\lambda}{\omega}\right\}\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}} + \left\{\alpha^* + \frac{\lambda}{\omega}\right\}\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}}\right)\Bigg]$$
$$= \mathrm{e}^{-\gamma t}\Bigg[\left(|\alpha|^2 + \frac{\lambda}{\omega}\left\{\alpha + \alpha^*\right\} + 2\frac{\lambda^2}{\omega^2}\right)\mathrm{e}^{\gamma t}$$
$$-\frac{\lambda}{\omega}\left(\left\{\alpha + \frac{\lambda}{\omega}\right\}\mathrm{e}^{\gamma t\mathrm{e}^{\frac{\mathrm{i}\omega}{\gamma}}} + \left\{\alpha^* + \frac{\lambda}{\omega}\right\}\mathrm{e}^{\gamma t\mathrm{e}^{-\frac{\mathrm{i}\omega}{\gamma}}}\right)\Bigg]$$
$$= \left[|\alpha|^2 + \frac{\lambda}{\omega}\left\{\alpha + \alpha^*\right\} + 2\frac{\lambda^2}{\omega^2}\right]$$
$$-\frac{\lambda}{\omega}\Bigg[\left\{\alpha + \frac{\lambda}{\omega}\right\}\mathrm{e}^{-\gamma t\left(1-\mathrm{e}^{\frac{\mathrm{i}\omega}{\gamma}}\right)}$$
$$+\left\{\alpha^* + \frac{\lambda}{\omega}\right\}\mathrm{e}^{-\gamma t\left(1-\mathrm{e}^{-\frac{\mathrm{i}\omega}{\gamma}}\right)}\Bigg], \tag{12}$$

that also checks $\langle\hat{a}^\dagger\hat{a}\rangle = |\alpha|^2$, for $t = 0$. In figure 2 we plot the average number of photons which shows a similar behaviour as the quadrature, this is, a damping of the oscillations.

3.2 *Initial squeezed state*

For an initial squeezed state $|\alpha, z\rangle = \hat{S}(z)|\alpha\rangle$ we use $\hat{S}^\dagger(z)\hat{a}\hat{S}(z) = \mu\hat{a} - \nu\hat{a}^\dagger$ and $\hat{S}^\dagger(z)\hat{a}^\dagger\hat{S}(z) = \mu\hat{a}^\dagger - \nu^*\hat{a}$, and because $z = r\mathrm{e}^{\mathrm{i}\theta}$, we have $\mu = \cosh r$ and $\nu = \mathrm{e}^{\mathrm{i}\theta}\sinh r$.

For the average of (twice) the position quadrature operator, it is easy to show that the evolution gives

$$\langle\hat{a}^\dagger + \hat{a}\rangle = \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\psi_k|\hat{a}^\dagger + \hat{a}|\psi_k\rangle$$
$$= \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\alpha, r|\Bigg[\left(\hat{a}^\dagger + \frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}}$$
$$+\left(\hat{a} + \frac{\lambda}{\omega}\right)\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}} - 2\frac{\lambda}{\omega}\Bigg]|\alpha, r\rangle$$
$$= \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\alpha|\hat{S}^\dagger(r)\Bigg[\left(\hat{a}^\dagger + \frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{\mathrm{i}k\omega}{\gamma}}$$
$$+\left(\hat{a} + \frac{\lambda}{\omega}\right)\mathrm{e}^{-\frac{\mathrm{i}k\omega}{\gamma}} - 2\frac{\lambda}{\omega}\Bigg]\hat{S}(r)|\alpha\rangle$$

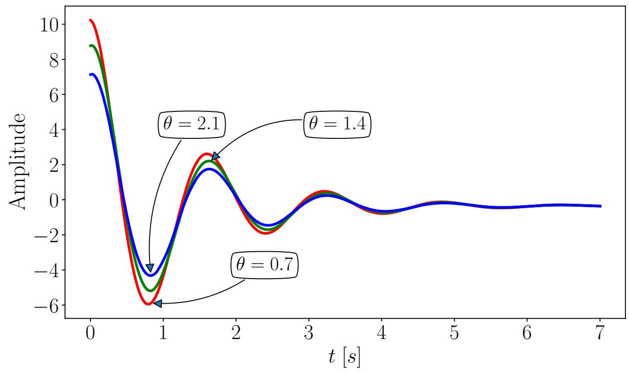

Figure 3. Expectation value of $\langle(\hat{a}^\dagger + \hat{a})\rangle$ given by (13). The parameters are $\alpha = 4, \omega = 4, \gamma = 10, \lambda = 0.7, r = 0.3$, where we choose to maintain the real part r of the squeeze coefficient and vary their complex phase θ.

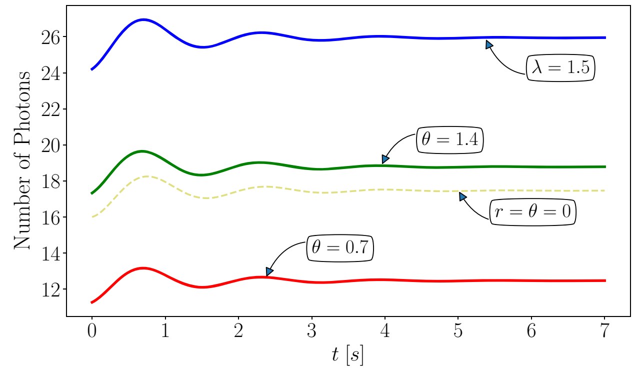

Figure 4. Expectation value of $\langle\hat{a}^\dagger\hat{a}\rangle$ given by (15). The parameters are $\alpha = 4, \omega = 4, \gamma = 10, \lambda = 0.7, r = 0.3$, where we choose to maintain the real part r of the squeeze coefficient and vary their complex phase θ

$$
\begin{aligned}
&= \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\left[\left(\mu\alpha^* - \nu^*\alpha + \frac{\lambda}{\omega}\right)\mathrm{e}^{\frac{ik\omega}{\gamma}}\right.\\
&\quad\left. + \left(\mu\alpha - \nu\alpha^* + \frac{\lambda}{\omega}\right)\mathrm{e}^{-\frac{ik\omega}{\gamma}} - 2\frac{\lambda}{\omega}\right]\\
&= \left(\mu\alpha^* - \nu^*\alpha + \frac{\lambda}{\omega}\right)\mathrm{e}^{-\gamma t\left(1-\mathrm{e}^{\frac{i\omega}{\gamma}}\right)}\\
&\quad + \left(\mu\alpha - \nu\alpha^* + \frac{\lambda}{\omega}\right)\mathrm{e}^{-\gamma t\left(1-\mathrm{e}^{-\frac{i\omega}{\gamma}}\right)} - 2\frac{\lambda}{\omega}.
\end{aligned}
\tag{13}
$$

Figure 3 shows that, as in the case of the coherent states, the oscillations decay fast, independently of the phase of the squeezed states.

For the average of the number operator, because the action of the squeezing operator results in

$$
\begin{aligned}
&\hat{S}^\dagger(z)\hat{a}^\dagger\hat{a}S(z) = \left(\mu\hat{a}^\dagger - \nu^*\hat{a}\right)\left(\mu\hat{a} - \nu\hat{a}^\dagger\right)\\
&= \mu^2\hat{a}^\dagger\hat{a} + |\nu|^2\hat{a}\hat{a}^\dagger - \mu\left(\nu\hat{a}^{\dagger 2} + \nu^*\hat{a}^2\right)\\
&= \left(\mu^2 + |\nu|^2\right)\hat{a}^\dagger\hat{a} - \mu\left(\nu\hat{a}^{\dagger 2} + \nu^*\hat{a}^2\right) + |\nu|^2,
\end{aligned}
\tag{14}
$$

we can then evaluate the average as

$$
\begin{aligned}
\langle\hat{a}^\dagger\hat{a}\rangle &= \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\psi_k|\hat{a}^\dagger\hat{a}|\psi_k\rangle\\
&= \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\alpha, r|\\
&\quad\times\left[\left(\hat{a}^\dagger\hat{a} + \frac{\lambda}{\omega}\left\{\hat{a}^\dagger + \hat{a}\right\} + 2\frac{\lambda^2}{\omega^2}\right)\right.\\
&\quad\left. - \frac{\lambda}{\omega}\left(\left\{\hat{a}^\dagger + \frac{\lambda}{\omega}\right\}\mathrm{e}^{\frac{ik\omega}{\gamma}} + \left\{\hat{a} + \frac{\lambda}{\omega}\right\}\mathrm{e}^{-\frac{ik\omega}{\gamma}}\right)\right]|\alpha, r\rangle\\
&= \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\langle\alpha|\hat{S}^\dagger(r)\\
&\quad\times\left[\left(\hat{a}^\dagger\hat{a} + \frac{\lambda}{\omega}\left\{\hat{a}^\dagger + \hat{a}\right\} + 2\frac{\lambda^2}{\omega^2}\right)\right.\\
&\quad - \frac{\lambda}{\omega}\left(\left\{\hat{a}^\dagger + \frac{\lambda}{\omega}\right\}\mathrm{e}^{\frac{ik\omega}{\gamma}}\right.\\
&\quad\left.\left. + \left\{\hat{a} + \frac{\lambda}{\omega}\right\}\mathrm{e}^{-\frac{ik\omega}{\gamma}}\right)\right]\hat{S}(r)|\alpha\rangle\\
&= \mathrm{e}^{-\gamma t}\sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\left[\left\{\mu^2 + |\nu|^2\right\}|\alpha|^2\right.\\
&\quad - \mu\left\{\nu\alpha^{*2} + \nu^*\alpha^2\right\}\\
&\quad + \frac{\lambda}{\omega}\left\{\left\{\mu - \nu^*\right\}\alpha + \left\{\mu - \nu\right\}\alpha^*\right\} + 2\frac{\lambda^2}{\omega^2} + |\nu|^2\\
&\quad - \frac{\lambda}{\omega}\left(\left\{\mu\alpha^* - \nu^*\alpha + \frac{\lambda}{\omega}\right\}\mathrm{e}^{\frac{ik\omega}{\gamma}}\right.\\
&\quad\left.\left. + \left\{\mu\alpha - \nu\alpha^* + \frac{\lambda}{\omega}\right\}\mathrm{e}^{-\frac{ik\omega}{\gamma}}\right)\right]\\
&= \left[\left\{\mu^2 + |\nu|^2\right\}|\alpha|^2 - \mu\left\{\nu\alpha^{*2} + \nu^*\alpha^2\right\}\right.\\
&\quad\left. + \frac{\lambda}{\omega}\left\{\left\{\mu - \nu^*\right\}\alpha + \left\{\mu - \nu\right\}\alpha^*\right\} + 2\frac{\lambda^2}{\omega^2} + |\nu|^2\right]\\
&\quad - \frac{\lambda}{\omega}\left[\left\{\mu\alpha^* - \nu^*\alpha + \frac{\lambda}{\omega}\right\}\mathrm{e}^{-\gamma t\left(1-\mathrm{e}^{\frac{i\omega}{\gamma}}\right)}\right.\\
&\quad\left. + \left\{\mu\alpha - \nu\alpha^* + \frac{\lambda}{\omega}\right\}\mathrm{e}^{-\gamma t\left(1-\mathrm{e}^{-\frac{i\omega}{\gamma}}\right)}\right].
\end{aligned}
\tag{15}
$$

In figure 4 we plot the average number of photons which also shows, as former examples, the rapid decay of oscillations due to intrinsic decoherence.

4. Conclusions

We have given a solution of the Milburn equation (beyond the Lindblad form that is generally used). This equation describes a modification of the Scrhödinger equation that accounts for (intrinsic) decoherence. We have studied the decaying dynamics of a displaced harmonic oscillator for initially coherent and squeezed states.

Acknowledgements

A R Urzúa acknowledges the financial support through Becas de Colaboración INAOE 2021.

References

[1] G J Milburn, *Phys. Rev. A* **44**, 5401 (1991)
[2] H Moya-Cessa, V Bužek, M S Kim and P L Knight, *Phys. Rev. A* **48**, 3900 (1993)
[3] Y Hong-Yang, Q Zheng and J Qi-Zhi, *Chin. Phys. B* **26**, 010601 (2017)
[4] A-B A Mohamed, A-H Abdel-Aty and H Eleuch, *Physica E* **128**, 114529 (2021)
[5] L Zheng and G-F Zhang, *Eur. Phys. J. D At. Mol. Opt. Phys.* **71**, 288 (2017).
[6] Q-L He, M Ding, Y-J Xiao and X-S Song, *Int. J. Theor. Phys.* **60**, 304 (2021)
[7] R Muthuganesan and V K Chandrasekar, *Quantum Inf. Process.* **20**, 46 (2021)
[8] A A Chlih, N Habiballah and M Nassik, *Quantum Inf. Process.* **20**, 1 (2021)
[9] Z Gong, Z Zhang, D Xu, N Zhao and C Sun, *Sci. China: Phys. Mech. Astron.* **61**, 1 (2018)
[10] Y Guo-Hui and Z Bing-Bing, *Int. J. Theor. Phys.* **55**, 2588 (2015)
[11] A-B A Mohamed, H A Hessian and H Eleuch, *Phys. Scr.* **95**, 075104 (2020)
[12] R de León-Montiel, M A Quiroz-Juárez, R Quintero-Torres, J L Domínguez-Juárez, H M Moya-Cessa, J P Torres and J L Aragón, *Sci. Rep.* **5**, 17339 (2015)
[13] D K Bayen and S Mandal, *Eur. Phys. J. Plus* **135**, 408 (2020)
[14] S Mandal, *Phys. Rev. A* **58**, 752 (1998)
[15] Y D Germain, A K Armel, A G Tene, N Isofa and M Tchoffo, *Phys. Scr.* **96**, 085705 (2021)
[16] D-M Lu, *Pramana – J. Phys*. **95**, 135 (2021)
[17] A-B A Mohamed, *Rep. Math. Phys.* **72**, 121 (2013)
[18] V Abhignan and R Muthuganesan, *Phys. Scr.* **96**, 125114 (2021)
[19] P C E Stamp, *Philos. Trans. R. Soc. A* **370**, 4429 (2012)