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Deformed special relativity with an invariant minimum speed as an explanation of the cosmological constant

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Abstract. This paper shows the need for the emergence of a universal minimum speed in the space–time by thoroughly investigating Dirac's large number hypothesis (LNH). We realise that there should be a minimum speed V with the same status of the invariance of the speed of light c. However, V has gravitational origin. Hence, such a minimum speed forms a new kinematic basis in the space–time, leading to a new deformed special relativity (DSR) for the quantum world so-called symmetrical special relativity (SSR). Furthermore, we show that such a new structure of space–time (SSR) reveals a connection between V and a preferred reference frame S_V of the background field, leading to the cosmological constant Λ, which can be associated with a cosmological antigravity. We also investigate the effect of the number of baryons N (Eddington number) of the observable Universe on the hydrogen atom. Finally, we show that SSR-metric plays the role of a de-Sitter (dS) metric with a positive cosmological constant, which could assume a tiny value.

Keywords. Cosmological constant; vacuum energy; invariant minimum speed; dS-metric.

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1. Introduction

We search for a new structure of space–time under a minimum speed V that behaves like a kinematic invariant for particles with low energies as is the speed of light c for high energies. To do that, we start with a deeper investigation of Dirac's large number hypothesis (LNH), where the relationship of the electric and gravitational interactions becomes more evident in the sense that such investigation of LNH will allow us to perceive that there should be a kinematic aspect of gravity represented by the invariant minimum speed $V(\propto G^{1/2}/\hbar)$. So we are led to think that such symmetry due to c and V forms the kinematic basis in a space–time that behaves like a de-Sitter (dS) space–time represented by a positive cosmological constant Λ. Because of such a connection between deformed special relativity (DSR) with a minimum speed and a dS space–time, we could get a tiny value of Λ, which is associated with a cosmological antigravity.

The study on the origin of the vacuum energy density ρ related to $\Lambda > 0$ in the scenario of an accelerated expanding Universe has been the centre of investigations [1,2], where the vacuum energy density $\rho = \Lambda c^2/8\pi G$.

The fine-structure constant α is associated with the cosmological constant Λ [3–5]. Thus, a possible variation of the fine-structure constant [6–10] would also point to a fundamental change in the subatomic structure, since α has a property of connecting the micro and macro-world, whose age is measured by the speed of light c, i.e., $R_{\rm H} = cT_{\rm H}$, where $T_{\rm H}$ ($\cong$ 13.7 Gyr) is the Hubble time and $R_{\rm H}$ ($\sim 10^{26}$ m) is the Hubble radius, i.e., the radius of the visible Universe.

The relationship between α and Λ, which is linked to the dark sector of the Universe is associated with models that aim to explain the antigravitational effects of the dark energy based on scalar fields [11–17].

The emergence of a minimum speed V in the space–time is associated with a preferred reference frame, thus leading to the birth of a new relativity with Lorentz symmetry violation at lower energies, i.e., the so-called symmetrical special relativity (SSR) [18–27].

It has also been shown that SSR has a relationship with the principle of Mach [25,28–30] within a quantum scenario due to the presence of the vacuum energy [23].

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The relationship between the fine-structure constant and the cosmological constant is $\Lambda \propto \alpha^{-6}$ [3–5]. Actually, Λ is also associated with other constants such as the mass of the electron m_e, Planck constant ($\hbar$) and the universal constant of gravity G. Thus, we can realise that Λ is connected to the constants of the standard model of elementary particles, i.e., $\Lambda \sim (G^2/\hbar^4)(m_e/\alpha)^6$ [3–5].

We shall investigate the global effect of the number of baryons N of the Universe (Eddington number) on the hydrogen atom within a Machian scenario [28–30].

The third section will be dedicated to the introduction of the space–time and velocity transformations for $(1+1)$D in SSR theory [18–25].

Our main goal is to show that the SSR metric plays the role of a dS metric, so that Λ emerges naturally from the SSR theory. Thus, Λ is related to the cosmological antigravity. The tiny value of Λ could be estimated.

2. Minimum speed as a universal constant

First of all, we intend to show the existence of universal minimum speed V as a new fundamental constant of nature. Such a speed has the same status of the invariance of the speed of light c [18]. However, V is given for lower energies. We shall show the emergence of V within the Dirac's LNH scenario.

The relationship between the minimum speed V and the cosmological constant Λ should be understood by means of the so-called ultrareferential S_V, i.e., a preferred reference frame shown in figure 1 [18–24], which is associated with V. So V is related to the background field of gravitational origin as the foundation of the cosmological constant Λ [25,28–30].

We conclude that there should be an equivalence between the SSR metric and the de-Sitter (dS) metric in the presence of Λ which is to be investigated later.

To obtain the universal minimum speed V, we start from Dirac's LNH that introduces the ratio of the electric and gravitational forces between the electron and proton in the hydrogen atom [3–5], i.e.,

$$\frac{F_e}{F_g} = \frac{e^2}{Gm_pm_e} = \frac{q_e^2}{4\pi\epsilon_0 Gm_pm_e}, \tag{1}$$

where $F_e/F_g \sim 10^{40}$. The masses m_e and m_p are the masses of electron and proton respectively. We write $q_e^2/4\pi\epsilon_0 = e^2$.

The large number of the order of 10^{40} is the well-known Dirac's large number. This number coincides exactly with $\sqrt{N}$ ($\sim 10^{40}$), where N ($\sim 10^{80}$) is the well-known Eddginton number, i.e., the number of protons (baryons) in the Universe.

Is is interesting to note that such a large number ($\sqrt{N}$) can be obtained by other means, for instance, the ratio $F_e/F_g \sim R_{\mathrm{H}}/r_p \sim 10^{40}$, where r_p is the proton radius and R_{H} is the Hubble radius. This indicates that this large number connects the length scales of the quantum world (the proton radius) with the cosmological scale, i.e., the Hubble radius.

By using the theorem of work (energy) to provide the works of the electric and gravitational forces to ionise the hydrogen atom and a hypothetical hydrogen atom with only gravitational interaction between the proton and electron, which in turn would be taken from its Bohr radius a_0 to infinite, we obtain the ratios of the works of both applied forces and their kinetic energies within the Dirac's LNH scenario to get the minimum speed V of gravitational origin ($V \propto G^{1/2}/\hbar$), i.e.,

$$\frac{\mathcal{W}_{F_e(\infty\to a_0)}}{\mathcal{W}_{F_g(\infty\to a_0)}} = \frac{\frac{q_e^2}{4\pi\epsilon_0}\int_\infty^{a_0}\frac{1}{r^2}\mathrm{d}r}{Gm_pm_e\int_\infty^{a_0}\frac{1}{r^2}\mathrm{d}r} = \frac{F_e}{F_g} = \frac{\frac{1}{2}m_e v_{\mathrm{B}}^2}{\frac{1}{2}m_e V^2}, \tag{2}$$

where v_{B} ($= e^2/\hbar \sim 10^5$ m/s) is the speed of the electron in the fundamental state of the hydrogen atom, which is a universal constant, the so-called Bohr velocity. It can be alternatively written as $v_{\mathrm{B}} = \alpha c$, where $\alpha = e^2/\hbar c$ ($\approx 1/137$) is the well-known fine-structure constant.

On the other hand, the speed V shown in eq. (2) should be understood as the most fundamental speed related to a small classical kinetic energy, since it has its origin from the work of the weakest force of nature (the gravitational force) as being the negative of the same applied force to ionise a hypothetical gravitational hydrogen atom, where we just consider the gravitational interaction between the proton and electron that form the most stable and simple bind structure in the Universe, i.e., the hydrogen atom.

Therefore, it does not seem by chance that Dirac's LNH with its strong 'coincidences' of ratios between scales of space–time with the same order of magnitude of 10^{40} starts just from the hydrogen atom as the fundamental (stable) structure that justifies the emergence of the universal minimum speed V with the same status of the speed of light c.

Furthermore, the extended Dirac's LNH given in eq. (2) shows clearly a new fundamental symmetry with respect to the existence of a higher energy scale related to the Bohr velocity ($v_{\mathrm{B}} = \alpha c$) with electric origin, i.e., a kinetic energy of Colombian origin to ionise the hydrogen atom, namely $(1/2)m_e v_{\mathrm{B}}^2$, and a lower (zero-point) energy related to the minimum speed V, which is associated with a minimum classical kinetic energy of ionisation of a hypothetical hydrogen atom by just considering the gravitational force between the masses of the proton (m_p) and electron (m_e). This leads to a very low kinetic energy $(1/2)m_eV^2$ of gravitational origin.

With respect to the symmetry provided by the extended Dirac's LNH, we have a good reason for considering the speed V as the kinematic invariant given for any particle that never reaches V at lower energies, V being related to a preferred reference frame (figure 1) associated with the vacuum energy that leads to a cosmological constant, as we shall see later. This motivates us to build the so-called symmetrical special relativity (SSR) [22] for describing the motion of the particles in the quantum world in the presence of the vacuum energy associated with the ultra-referential S_V (figure 1).

However, if we admit the classical idea of rest in the quantum world by making $V \to 0$, this would violate the Dirac's LNH, as gravity and vacuum energy would vanish and thus the large number would diverge ($F_e/F_g \to \infty$), so that the Universe also would be infinite ($N \to \infty$) with an infinite Hubble radius ($R_{\mathrm{H}} \to \infty$). Therefore, we realise that SR is not consistent with the Dirac's LNH scenario, as rest ($v = 0$) is naturally conceived by the classical theory, where there is no minimum speed, i.e., we just make $V = 0$ to recover SR as a particular case of SSR. Thus, only SSR is consistent with Dirac's LNH scenario.

Actually, we should realise that the extended Dirac's LNH shows that the Universe would be inconceivable if there was no minimum speed or if rest were possible to conceive in the quantum world, since the existence of a non-null minimum speed V (a zero-point energy) in the quantum world is essentially due to gravity, thus leading to the kinematic basis of quantum gravity at lower energies for describing the vacuum and its cosmological implications (the cosmological constant).

By substituting the Bohr velocity $v_{\mathrm{B}}(= e^2/\hbar)$ in eq. (2) and, by performing the calculations we finally obtain V as follows:

$$V = \sqrt{\frac{Gm_em_p}{4\pi\epsilon_0}}\frac{q_e}{\hbar}, \tag{3}$$

from where we can write $q_e/\sqrt{4\pi\epsilon_0} = e$.

From eq. (3), we get $V \cong 4.58 \times 10^{-14}$ m/s.

It is important to know that the minimum speed V is directly related to the Planck minimum length of quantum gravity (L_{P}), i.e.,

$$V = \sqrt{Gm_pm_e}e/\hbar = (e\sqrt{m_pm_ec^3/\hbar^3})L_{\mathrm{P}} = L_{\mathrm{P}}/\tau,$$

where

$$L_{\mathrm{P}} = \sqrt{G\hbar}/c^3 \sim 10^{-35}\,\mathrm{m}$$

and

$$\tau = (e\sqrt{m_pm_ec^3/\hbar^3})^{-1} \sim 10^{-21}\,\mathrm{s},$$

such that, $L_{\mathrm{P}} \to 0$ leads to $V \to 0$ and thus we recover the classical space–time of SR without quantum gravity effect. This leads us to perceive that there is a connection between the minimum speed V and the Planck length (L_{P}) by means of the constant of gravity (G) and the Planck constant ($\hbar$), as we find $V \propto L_{\mathrm{P}} \propto G^{1/2}$. Thus, both V and L_{P} are respectively the kinematic and space ingredients of quantum gravity, as L_{P} is well-known as the fundamental length of quantum gravity.

In eq. (3), if $G \to 0$, we find $V \to 0$. But, as gravity does not vanish anywhere, rest is in fact forbidden due to a zero-point energy related to the fundamental vacuum energy. Hence, we are led to postulate V as a kinematic invariant connected to the fundamental vacuum at very low energies.

By combining eq. (2) given for the Dirac's LNH with eq. (3), we obtain

$$\frac{F_e}{F_g} = \frac{v_{\mathrm{B}}^2}{V^2} = \frac{4\pi\epsilon_0 v_{\mathrm{B}}^2\hbar^2}{Gm_em_pq_e^2} = \frac{q_e^2}{4\pi\epsilon_0 Gm_em_p}, \tag{4}$$

which is in fact consistent with eq. (2), as it is already known that $v_{\mathrm{B}} = q_e^2/4\pi\epsilon_0\hbar = e^2/\hbar$.

2.1 *Effect of Eddington number on hydrogen atom*

From eq. (4), we can see the most fundamental speed V ($\cong 1.5322\times10^{-22}$ c) of gravitational origin is a function of Bohr velocity v_{B} ($\cong 137^{-1}$ c) of Coulombian origin, as follows:

$$V = \frac{\sqrt{4\pi\epsilon_0 Gm_em_p}}{q_e}v_{\mathrm{B}}, \tag{5}$$

where

$$v_{\mathrm{B}} = q_e^2/4\pi\epsilon_0\hbar$$

and

$$\begin{aligned}\xi &= V/c = \sqrt{4\pi\epsilon_0 Gm_em_p}v_{\mathrm{B}}/q_e \\ &= \sqrt{Gm_em_p/4\pi\epsilon_0}q_e/\hbar c \cong 1.5322 \times 10^{-22}.\end{aligned}$$

The dimensionless constant ξ, the so-called fine-tuning constant [18], is a fine-structure constant of gravito-electromagnetic origin.

Now it is important to further clarify why the minimum speed V must depend specially on the proton mass instead of other lighter particles, so that there should not be a minimum speed with a value smaller than V. Thus, for instance, if we think about a positronium atom, the mass of proton in eq. (5) would be replaced by the mass of electron or positron, so that we get the minimum speed V' smaller than V. However, it must be emphasised that positronium is highly unstable as it decays rapidly. If so, this would lead to a rapid collapse of the space–time. Due to such a collapse which would prevent the existence of the Universe for a much longer time, it is easy to conclude that $V'(< V)$ cannot be considered for our observable Universe with the age of about 13.7 Gyr.

In summary, the only bound state of two particles that is lighter and also completely stable is the hydrogen atom. That is why we postulate V as the universal minimum speed.

Other examples of bound states can also be provided for verifying that only the hydrogen atom has complete stability in its ground state, and is simultaneously the lightest stable element with highly stable particles such as proton electron.

From eqs (3)–(5), we can write

$$v_B = \frac{q_e^2}{4\pi\epsilon_0\hbar} = \sqrt{\frac{F_e}{F_g}}V = \frac{q_e}{\sqrt{4\pi\epsilon_0 G m_e m_p}}V, \quad (6)$$

where $F_e/F_g \sim \sqrt{N} \sim 10^{40}$ is the Dirac's large number, so that we have $\sqrt{F_e/F_g} \sim \sqrt[4]{N} \sim 10^{20}$ and $N \sim 10^{80}$ as being the well-known Eddington number that represents the order of magnitude of the number of baryons in the Universe. Thus, we can write eq. (6) as $v_B \sim \sqrt[4]{N}V$, which connects the quantum world (hydrogen atom) to the cosmological quantity represented by the number N of baryons in the Universe and also the vacuum energy associated with the universal mininum speed V related to the cosmological constant Λ.

The numerical coefficient $\sqrt[4]{N} \sim 10^{20}$ is known as the Weyl number. This is the great evidence of the connection of local quantities (quantum quantities) with the cosmological quantities. Thus, we can also say that this is an achievement of a kind of Machian principle [28–30], where the global distribution of mass (number N of baryons) determines the local properties of atoms, as is the case of the hydrogen atom.

We can also obtain the Bohr radius (a_0) in terms of cosmological quantities such as the Weyl number. Thus, knowing that $v_B = q_e^2/4\pi\epsilon_0\hbar \sim \sqrt[4]{N}V$ [eq. (7)] and also $a_0 = \hbar/m_e v_B$, we find the Bohr radius (a_0) in terms of V and N, as

$$a_0(N) \sim \frac{\hbar}{\sqrt[4]{N} m_e V} = \frac{\hbar^2}{\sqrt[4]{N}\sqrt{G m_e^3 m_p}e}, \quad (7)$$

where $V = \sqrt{G m_e m_p}e/\hbar$, with $e = q_e/\sqrt{4\pi\epsilon_0}$ [eq. (3)].

In eq. (7), when $N = 1$, we would find a giant hydrogen atom with a radius in the order of magnitude of about ten (10) stars like the Sun, i.e, $a_0(N = 1) \sim 10^{10}$ m.

If $N \to 0$, we would not have any hydrogen atom, i.e., there would not be baryons in the Universe, so that we would just have an enormous quark condensate for representing the Universe. This would be similar to a giant collapsed star made only of quarks.

If $N \to \infty$, we would not have any hydrogen atom, because it would be completely crushed, so that $a_0 \to 0$. This shows clearly how the entire mass of the Universe represented by the number N is able to locally compress the hydrogen atom due to a kind of 'pressure' of Machian (non-local) origin that would go to infinity if N diverges by reducing the Bohr radius to zero. Here we might also think about the emergence of a singularity due to the formation of a completely collapsed structure.

In view of the dependence of the Bohr velocity v_B with the Weyl number ($\sqrt[4]{N} \sim 10^{20}$) or even the Eddginton number N, we also should have a dependence of the fine-structure constant α with N, which can be obtained from eq. (7), as follows:

$$\alpha(N) = \frac{v_B}{c} \sim \sqrt[4]{N}\frac{V}{c} = \sqrt[4]{N}\xi = \sqrt[4]{N}\sqrt{G m_e m_p}\frac{e}{\hbar c}, \quad (8)$$

where we have $\xi = V/c = \sqrt{G m_e m_p}e/\hbar c$ [18] and $V = \sqrt{G m_e m_p}e/\hbar$, with $e = q_e/\sqrt{4\pi\epsilon_0}$.

3. Space–time and velocity transformations in SSR

In this section, first of all we carefully investigate the concepts of reference frame in SSR and their implications such as the new transformations of space–time and velocity.

We also show the equivalence between the SSR metric and a dS metric, which is associated with a certain positive cosmological constant Λ (> 0), thus leading to a cosmological antigravity. So, we conclude that SSR generates a background metric that plays the role of a dS metric in the presence of Λ. To realise such fundamental equivalence, we use a toy model which will be presented in the next section.

Violation of Lorentz symmetry for very low energies [18] generated by the presence of a background field related to S_V (figure 1) creates a new causal structure of space–time with an invariant minimum speed $V = \sqrt{G m_p m_e}e/\hbar$ [eq. (3)], which is an unattainable limit of speed for all particles at lower energies.

Since the minimum speed V is an invariant quantity as is the speed of light c, V does not alter the speed v of any particle, as we shall show later. Therefore, we denominate ultra-referential S_V as being the preferred reference

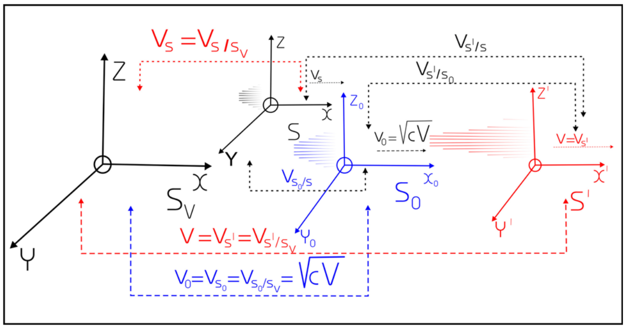

Figure 1. The reference frame S' moves in the x-direction with a speed $v(>\text{V})$ with respect to the universal reference frame, i.e., the ultra-referential S_V associated with V. In this figure, we see the two running referentials S and S' with speeds $v = v_S = v_{S/S_V}$ and $v' = v'_S = v_{S'/S_V}$, both of them given in relation to the background frame (ultra-referential S_V), plus two fixed referentials S_0 with speed $v_0 = v_{S_0/S_V} = \sqrt{cV}$ given with respect to the background frame S_V and the own ultra-referential S_V of vacuum associated with the unattainable minimum speed V. Thus, we can find the relative velocity between S' and S, i.e., $v_{\text{rel}} = v_{S'/S}$, which is shown clearly in eqs (13) and (14), thus leading to some important cases, as for instance: (a) If only the running referential S coincides with S_0 ($S \equiv S_0$), we find the relative velocity between S' and S_0, i.e., $v_{\text{rel}} = v_{S'/S_0}$, (b) if only the running referential S' coincides with S_0 ($S' \equiv S_0$), we find the relative velocity between S_0 and S, i.e., $v_{\text{rel}} = v_{S_0/S}$, as it is also indicated in this figure.

frame in relation to which we have the speed v of any particles (figure 1). In view of this, the well-known Lorentz transformations are changed in the presence of the background reference frame S_V (figure 1).

In the case $(1+1)D$ (figure 1), we obtain the space–time transformations given between the running reference frame S' and the background reference frame S_V:

$$\mathrm{d}x' = \frac{\sqrt{1 - V^2/v^2}}{\sqrt{1 - v^2/c^2}}[\mathrm{d}X - v(1-\alpha)\mathrm{d}t] \tag{9}$$

and

$$\mathrm{d}t' = \frac{\sqrt{1 - V^2/v^2}}{\sqrt{1 - v^2/c^2}}\left[\mathrm{d}t - \frac{v(1-\alpha)\mathrm{d}X}{c^2}\right], \tag{10}$$

with

$$\alpha = V/v$$

and

$$\Psi = \theta\gamma = \sqrt{1 - V^2/v^2}/\sqrt{1 - v^2/c^2},$$

where

$$\theta = \sqrt{1 - V^2/v^2}$$

and

$$\gamma = 1/\sqrt{1 - v^2/c^2}.$$

The coordinates X shown in the transformations $(1+1)$ D above, Y and Z (figure 1) form the ultra-referential S_V connected to the vacuum energy.

The inverse transformations for this special case $(1+1)$ D (figure 1) were demonstrated in [18]. Of course, if $V \to 0$, we recover the Lorentz transformations.

The general transformations in the space–time $(3+1)$ D of SSR were also shown in [18]. In [19], it was first shown that SSR transformations break down the Lorentz and Poincaré's groups.

This new causal structure of space–time, i.e., the symmetrical special relativity (SSR) presents energy E and momentum P for a particle as follows: $E = m_0c^2\Psi = m_0c^2\sqrt{1 - V^2/v^2}/\sqrt{1 - v^2/c^2}$ [18,22], in such a way that $E \to 0$ when $v \to V$, and $P = m_0v\Psi = m_0v\sqrt{1 - V^2/v^2}/\sqrt{1 - v^2/c^2}$ [18,22], such that $P \to 0$ when $v \to V$.

It is important to notice that both momentum and energy of a particle in SSR are: $P_0 = m_0v_0 = m_0\sqrt{cV}$ and $E_0 = E(v_0) = m_0c^2$ for $v = v_0 = \sqrt{cV} \neq 0 (> V)$, as we find $\Psi(v_0) = \Psi(\sqrt{cV}) = 1$, where the energy m_0c^2 is exactly equivalent to the rest energy in SR, as there is no rest in SSR. This means that the momentum of a particle never vanishes in SSR due to the invariant minimum speed V, as there is an intermediary speed $v_0(= \sqrt{cV})$ given with respect to the preferred frame S_V (figure1), so that the momentum is non-null (P_0) and the energy is equivalent to the rest energy m_0c^2 in SR with $p = 0$ for $v = 0$. However, in SSR, E_0 is associated with the speed v_0 (reference frame S_0) with respect to the ultra-referential S_V, as there is no rest in the space–time of SSR, where the reference frames S, S' and specially S_0 for $v = v_0(= \sqrt{cV})$ are shown in figure 1.

The SSR metric is a deformed Minkowski metric in the presence of the multiplicative factor $\Theta = \Theta(v) = 1/(1 - V^2/v^2)$ [23], which plays the role of a conformal factor as already shown in [23], thus leading to a conformal special relativity represented by SSR due to the presence of the invariant minimum speed V, as follows:

$$\mathrm{d}\mathcal{S}^2 = \frac{1}{\left(1 - V^2/v^2\right)}[c^2(\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2], \tag{11}$$

or simply $\mathrm{d}\mathcal{S}^2 = \Theta\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$, where $\Theta = 1/(1 - V^2/v^2)$ and $\eta_{\mu\nu}$ is the Minkowski metric.

By dividing eq. (9) by eq. (10), we obtain the following velocity transformation in SSR:

$$v_{rel} = v_{S'/S} = \frac{v' - v + V}{1 - \frac{v'v}{c^2} + \frac{v'V}{c^2}} = \frac{v' - v(1-\alpha)}{1 - \frac{v'v(1-\alpha)}{c^2}}, \quad (12)$$

where $\alpha = V/v$.

We have considered $v_{rel} = v_{relative} = v_{S'/S} \equiv dx'/dt'$ and $v' \equiv dX/dt$ when dividing eq. (9) by eq. (10).

We should stress that $v' = v_{S'} \equiv dX/dt$ is the motion of the referential S' (figure 1) with respect to the background reference frame S_V connected to the unattainable minimum speed V, i.e., we can write the notation $v_{S'} = v_{S'/S_V}$ for representing the absolute motion of S', which is observer-independent, as S_V is absolute for being the preferred reference frame.

The speed v shown in figure 1 represents the motion of the referential S with respect to the background reference frame S_V, i.e., we can write the notation $v = v_S = v_{S/S_V}$ for representing the absolute motion of S (figure 1), which is also observer-independent.

The speed v_{rel} is the relative speed between the absolute speeds $v_{S'}$ and v_S, both are given in relation to the background framework S_V, i.e., we have $v_{rel} = v_{S'/S}$ (figure 1). So we can rewrite the speed transformation in eq. (12) by using the notations in the presence of the background frame S_V, i.e.,

$$v_{rel} = v_{S'/S} = \frac{v_{S'/S_V} - v_{S/S_V} + V}{1 - \frac{(v_{S'/S_V})(v_{S/S_V})}{c^2} + \frac{(v_{S'/S_V})V}{c^2}}, \quad (13)$$

where $v = v_S = v_{S/S_V}$ (speed v of the reference frame S in relation to S_V) and $v' = v_{S'} = v_{S'/S_V}$ (speed v' of the reference frame S' in relation to S_V).

Figure 1 also shows the reference frame S_0, whose speed $v_0 (= \sqrt{cV})$ is also given in relation S_V.

As v_0 is the intermediary speed, such that $V \ll v_0 \ll c$ with $\Psi(v_0) = \Psi(\sqrt{cV}) = 1$, all the speeds v not so far from v_0, where $E \approx E_0 = m_0c^2$ represents the Newtonian approximation within the scenario of SSR, as we get $\Psi\,(V \ll v \ll c) \approx 1$.

If $V \to 0$, eq. (13) would recover the Lorentz velocity transformation, where both speeds $v_{S'}$ and v_S would be given simply in relation to a certain Galilean frame at rest in the lab, such that the background frame S_V would vanish and thus v_0 would also be zero, i.e., the reference frame S_0 would become simply a certain Galilean reference frame at rest in the lab.

In the transformation by eq. (13), let us just consider the important cases, where we must consider $v_{S'} \geq v_S$ (figure 1):

(a) $v_{S'} = c$ (photon) and $v_S \leq c$. This implies $v_{rel} = c$. Such result just verifies the invariance of c.

(b) $v_{S'} > v_S (= V)$. This implies $v_{rel} =$ "$v_{S'} - V$" $= v_{S'}$. For example, $v_{S'} = 2V$ and $v_S = V$, this leads to $v_{rel} =$ "$2V - V$" $= 2V$, which means that V (S_V) really has no influence on the speed of any particle. Thus, V works as if it were an 'absolute zero of motion', being invariant and having the same value at all directions of space 3D of the isotropic background field associated with S_V.

(c) If $v_{S'} = v_S$, then

$$v_{rel} = v_{S'/S} = \text{"}v_S - v_S\text{"} = \text{"}v - v\text{"} (\neq 0) = \frac{V}{1 - \frac{v_S^2}{c^2}(1 - \frac{V}{v_S})} = \frac{V}{1 - \frac{v^2}{c^2}(1 - \frac{V}{v})}.$$

From Case (c), let us consider two specific cases, as follows:

(c_1) $v_S = V$. This implies $v_{rel} =$ "$V - V$" $= V$ as verified before. Indeed V is an invariant minimum speed.

(c_2) $v_S = c$ (photon). This implies $v_{rel} = c$, where we have the interval $V \leq v_{rel} \leq c$ given for the interval $V \leq v_S \leq c$. However, it must be stressed that there is no ordinary massive particle exactly at the ultra-referential S_V with $v = v_S = V$. So, this is just a hypothetical condition to verify the consistency of the transformation in eq. (13) with respect to the invariance of the minimum speed V, as already verified in the specific Case (c_1).

Case (c) shows that it is impossible to find the rest for the particle on its own reference frame S, where $v_{rel}(v_S)$ ($\equiv \Delta v(v_S)$) is a function that increases with the increase of $v = v_S$ of the referential S (figure 1). However, if $V \to 0$, we would have $v_{rel} \equiv \Delta v = 0$ and thus it would be possible to find rest for S, which would recover the inertial reference frames of SR.

The inverse transformations of space–time ($x' \to X$) and ($t' \to t$) in SSR for the special case (1 + 1)D and also the general case (3 + 1)D have already been explored in detail in [18]. Thus, from such transformations above, we can obtain the following inverse transformation of velocity:

$$v_{rel} = v_{S'/S} = \frac{v_{S'/S_V} + v_{S/S_V} - V}{1 + \frac{(v_{S'/S_V})(v_{S/S_V})}{c^2} - \frac{(v_{S'/S_V})V}{c^2}}. \quad (14)$$

The velocity transformation given by eq. (14) leads to the following important cases:

(a) $v' = v_{S'} = v_S = v = V$. This implies "$V + V$" $= V$. Once again we verify that the minimum speed V is in fact invariant.

(b) $v' = v_{S'} = c$ (photon) and $v_S \leq c$. This leads to $v_{rel} = v_{S'/S} = c$. This just confirms that c is invariant.

(c) $v' = v_{S'} > V$ and by considering $v_S = V$, this leads to $v_{rel} = v_{S'/S} = v_{S'}$.

From Case (c), let us investigate the following specific cases, namely:

(c_1) If $v' = v_{S'} = 2V$ and by assuming that $v_S = V$, we would obtain $v_{\rm rel} = v_{S'/S} =$ "$2V + V$" $= 2V$.

(c_2) $v' = v_{S'} = v_S = v$. This implies

$$v_{\rm rel} = v_{S'/S} = \text{“}v_S + v_S\text{”} = \text{“}v + v\text{”} = \frac{2v_S - V}{1 + \frac{v_S^2}{c^2}(1 - \frac{V}{v_S})}$$

$$= \frac{2v - V}{1 + \frac{v^2}{c^2}(1 - \frac{V}{v})}.$$

In the Newtonian regime ($V \ll v \ll c$) for Case c_2, we recover the classical transformation, i.e., $v_{\rm rel} =$ "$v + v$" $= 2v$.

In the relativistic regime ($v \to c$), we recover the Lorentz transformation of velocity given for this specific case c_2 ($v' = v_{S'} = v_S = v$), i.e., we find $v_{\rm rel} = v_{S'/S} =$ "$v_S + v_S$" $=$ "$v + v$" $= 2v_S/(1 + v_S^2/c^2) = 2v/(1 + v^2/c^2)$.

4. Equivalence of the SSR metric with a dS metric

As the universal minimum speed related to the ultra-referential S_V should be associated with the cosmological constant, let us show that the metric of SSR [eq. (11)] is equivalent to a dS metric, where a conformal factor emerges depending on Λ [23]. To do that, let us use a toy model by considering a spherical Universe with Hubble radius $R_{\rm H}$ filled with a vacuum energy density ρ.

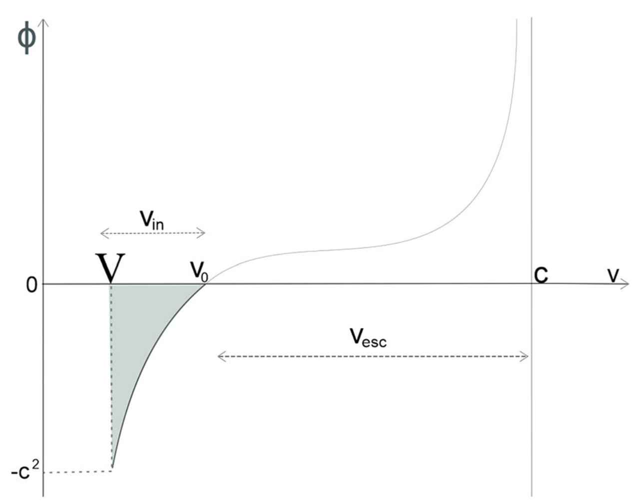

Figure 2. This figure shows the scalar potential $\phi(v) = c^2\left(\sqrt{[(1 - \frac{V^2}{v^2})/(1 - \frac{v^2}{c^2})]} - 1\right)$ given as a function of speed [eq. (16)]. It shows two phases, namely gravity (right side)/antigravity (left side), where the barrier at the right side represents the relativistic limit, i.e., speed of light c with $\phi \to \infty$, and on the other hand, the barrier at the left side is the antigravitational limit only described by SSR with an invariant minimum speed V associated with the potential $\phi_q(V) = \phi(V) = -c^2$. The intermediary region is the Newtonian regime ($V \ll v \ll c$), where a phase transition of gravity/antigravity occurs for $v = v_0 = \sqrt{cV}$.

According to this toy model, on the surface of the sphere which represents the frontier of the observable Universe, objects like galaxies, etc. experience an antigravity given by the accelerated expansion of the Universe. This antigravitational effect is due to the whole vacuum energy (a dark mass) inside such a Hubble sphere. Thus, we think that each galaxy works like a proof body that interacts with this sphere having a dark mass M_Λ ($= M$). Such an interaction can be thought of as being the simple case of interaction between two bodies. In view of this, let us show that there is an antigravitational interaction between the ordinary proof mass m_0 (on the surface of the dark sphere) and the sphere of dark energy with a dark mass M.

Thus, to investigate such antigravitational interaction between the proof mass m_0 and the dark mass M of the Hubble sphere with radius $R_{\rm H}$ in this toy model, let us first consider the model of a proof particle with mass m_0 that escapes from a gravitational potential ϕ on the surface of a certain sphere of matter with mass $M_{\rm matter}$, namely $E = m_0c^2(1 - v^2/c^2)^{-1/2} \equiv m_0c^2(1 + \phi/c^2)$, where E is the escape relativistic energy of the proof particle with mass m_0 and $\phi = GM_{\rm matter}/R$, R being the radius of the sphere of matter. In such a classical case, the interval of escape velocity ($0 \leq v < c$) is associated with the interval of potential ($0 \leq \phi < \infty$), where we define $\phi > 0$ to be the well-known classical (attractive) gravitational potential.

We should notice that the Lorentz symmetry violation in SSR is due to the presence of the ultra-referential S_V (figure 1) connected to the vacuum energy that fills the dark sphere. Such energy has its origin from a non-classical aspect of gravity that leads to a repulsive gravitational potential defined as being negative for representing antigravity, i.e., $\phi = \phi_q < 0$ (figure 2).

In this toy model based on SSR theory, we write the deformed relativistic energy of such a proof particle (m_0), as follows:

$$E = m_0c^2\left(1 + \frac{\phi}{c^2}\right) = m_0c^2\left(\frac{1 - \frac{V^2}{v^2}}{1 - \frac{v^2}{c^2}}\right)^{1/2}, \qquad (15)$$

from which we get

$$\phi = \phi(v) = \left[\left(\frac{1 - \frac{V^2}{v^2}}{1 - \frac{v^2}{c^2}}\right)^{1/2} - 1\right]c^2. \qquad (16)$$

Here, we should realise that eq. (16) reveals two situations, namely,

(i) The well-known Lorentz sector ($\phi = (\gamma - 1)c^2$) represents the gravity sector, as the sphere M is composed of attractive (ordinary) matter. In this case, the speed v is simply the escape velocity (v_{esc}), which is directed away from the sphere.

(ii) The antigravity sector ($\phi = \phi_q = (\theta - 1)c^2$), where $\theta = (1 - V^2/v^2)^{1/2}$ is governed by a dark sphere with mass M. Here, the speed v is the input speed (v_{in}) or the velocity of a proof particle that escapes from antigravity, i.e., $v\,(= v_{\text{in}})$ is directed into the sphere, as antigravity pushes the particle away.

As SSR forbids rest of a particle according to eq. (16), we must notice that v cannot be zero even in the absence of potential ϕ ($\phi = 0$), i.e., we find $v = v_0 = \sqrt{cV}$, so that $\phi(v_0) = 0$ in eq. (16) (figure 2).

Due to the absence of gravitational potential ($\phi = 0$) at the point $v = v_0 (\neq 0)$, this is the only velocity that means both of the escape and input velocities of a particle. Therefore, v_0 is a transition 'zero' point between gravity and antigravity, which highlights the quantum nature of the space–time in SSR, thus leading to the uncertainty principle as shown in [27].

In short, from eq. (16) and figure 2, we can see two regimes of gravitational potential, i.e., the classical (matter) and quantum (vacuum) regimes:

$$\phi = \phi(v) = \begin{cases} \phi_q : \ -c^2 < \phi \leq 0 \text{ for } V < v \leq v_0, \\ \phi_m : 0 \leq \phi < \infty \text{ for } v_0 \leq v < c, \end{cases} \quad (17)$$

where speed v_0 represents the point of transition ($\phi = 0$) between gravity ($\phi_{\text{matter}} = \phi_m = \phi > 0$ for $v > v_0$) and antigravity when the vacuum governs ($\phi_{\text{quantum}} = \phi_q = \phi < 0$ for $v < v_0$).

We must stress that v_0 is given with respect to the preferred reference frame S_V (figure 1). Therefore, it is an observer-independent velocity as well as any velocity v, which is given with respect to S_V, as S_V is related to the unattainable minimum speed V, and thus there is no observer at the ultra-referential S_V.

We realise that the most repulsive potential is $\phi = -c^2$, which is associated with the fundamental vacuum energy of the ultra-referential S_V by imposing $v = v_{\text{in}} = V$ in eq. (16), i.e., $\phi(V) = -c^2$ (figure 2).

So, by taking into account this model of a spherical Universe with a Hubble radius $R_{\text{H}}\,(= R_u)$ and a vacuum energy density ρ, we obtain the total vacuum energy inside the sphere, i.e., $E_{\text{dark}} = \rho V_u = Mc^2$, where V_u is the spherical volume of the Universe and M is the total dark mass associated with the vacuum energy inside the sphere.

As the vacuum energy density ρ is very low and the big sphere with Hubble radius $R_{\text{H}}\,(= R_u)$ presents a dark mass M, but having a very low dark mass density, then the Newtonian gravitational potential is a very good approximation that represents this toy model for the Universe. So, in view of this, we get the following repulsive gravitational potential $\phi (= \phi_q < 0)$ on the surface of such Hubble sphere (Universe):

$$\phi = \phi_q = -\frac{GM}{R_u} = -\frac{4\pi G \rho R_u^2}{3c^2} = -\frac{G\rho V_u}{R_u c^2}, \quad (18)$$

where $M = \rho V_u/c^2$, ρ is the vacuum energy density and $V_u (= 4\pi R_u^3/3)$ is the Hubble volume.

We already know that $\rho = \Lambda c^2/8\pi G$. So, by substituting this relationship into eq. (18), we obtain the repulsive (quantum) potential as follows:

$$\phi = -\frac{\Lambda R_u^2}{6}, \quad (19)$$

where $R_u = R_{\text{H}} = cT_{\text{H}}$ ($\sim 10^{26}$ m) is the Hubble radius. $T_{\text{H}} \cong 13.7$ Gyr is the age of the Universe (Hubble time).

As the whole Universe (a big sphere) is governed by the vacuum energy (a dark mass M), the speed v in eq. (16) is understood as the input speed v_{in} in order to overcome the cosmological antigravity. Thus, the factor $(1 - V^2/v^2)^{1/2}$ [eq. (16)] prevails for determining the potential ϕ. In view of this, we shall neglect the Lorentz factor γ (attractive sector) in eq. (16), and so we should consider only the repulsive sector ($V < v \leq v_0$) for obtaining the non-classical background potential $\phi\,(= \phi_q)$.

Furthermore, after neglecting $\gamma = (1 - v^2/c^2)^{-1/2}$ in eq. (16), we shall just compare its antigravity sector with eq. (19) given for a certain radius $r\,(= ct)$, i.e., $\phi\,(= -\Lambda r^2/6)$, so that we find ϕ/c^2:

$$\frac{\phi}{c^2} = -\frac{\Lambda r^2}{6c^2} = \left(1 - \frac{V^2}{v^2}\right)^{1/2} - 1, \quad (20)$$

where $\phi = \phi_q$ which represents the potentials of antigravity (figure 2), being $0 \leq \phi \leq -c^2$.

By performing the calculations in eq. (20), we rewrite the scale factor $\Theta(v)$ of the SSR metric [eq. (11)] in its equivalent forms, as follows:

$$\Theta(v) = \frac{1}{\left(1 - \frac{V^2}{v^2}\right)} \equiv \frac{1}{\left(1 + \frac{\phi_q}{c^2}\right)^2} \equiv \frac{1}{\left(1 - \frac{\Lambda r^2}{6c^2}\right)^2}, \quad (21)$$

where we realise that there are three equivalent forms for representing $\Theta(v) \equiv \Theta(\phi_q) \equiv \Theta(\Lambda)$ as shown in eq. (21).

By replacing the factor $\Theta(v)$ of eq. (11) (SSR metric) by its equivalent form with dependence of Λ shown in eq. (21), we rewrite the background metric (SSR metric) in its equivalent form within the dS scenario:

$$\mathrm{d}\mathcal{S}^2 = \frac{1}{\left(1-\frac{\Lambda r^2}{6c^2}\right)^2}[c^2(\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2], \tag{22}$$

or simply

$$\mathrm{d}\mathcal{S}^2 = \Theta(\Lambda)\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{23}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $\mathcal{G}_{\mu\nu} = \Theta(\Lambda)\eta_{\mu\nu}$ is the SSR metric with dependence of Λ.

Of course if $\Lambda = 0$ in eq. (22), we get $\Theta = 1$ and so we recover the Minkowski metric $\eta_{\mu\nu}$, where there is no cosmological constant and no antigravitational effect. In other words, as $\Lambda = -6\phi/r^2$ [eq. (19)], for $r \to \infty$ ($\Lambda \to 0$), the interval $\mathrm{d}\mathcal{S}^2$ reduces to the Lorentz-invariant Minkowski interval $\mathrm{d}s^2$, i.e., $\mathrm{d}\mathcal{S}^2 \to \mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ [31].

We should realise that eq. (22) represents a dS metric which presents $\Lambda > 0$, as we must have $\phi < 0$ (antigravity sector) according to eq. (19).

In view of eqs (19) and (22), we first conclude that a cosmological constant Λ emerges from SSR, i.e., $\Lambda = -6\phi/r^2$ [eq. (19)]. We also conclude that there is a correspondence of SSR with the dS relativity [32] shown by eq. (22) that is a dS-metric with a conformal factor [32] given by $\Theta(\Lambda)$.

We finally conclude that a small positive value of Λ is plausible in the context of eqs (19) and (22). So, to estimate the small order of magnitude of Λ, we first consider Λ [eq. (19)] given for the Hubble radius R_H ($\sim 10^{26}$) m, so that we obtain

$$\Lambda = \Lambda(R_\mathrm{H}, \phi) = -\frac{6\phi}{R_\mathrm{H}^2}, \tag{24}$$

where $r = R_\mathrm{H}$ and $-c^2 \leq \phi \leq 0$ (the shaded area in figure 2).

Finally, if we admit that the accelerated expansion of the Universe is governed by the lowest potential $\phi = \phi(V) = -c^2$ associated with the fundamental vacuum energy at the ultra-referential S_V, we find

$$\Lambda = \frac{6c^2}{R_\mathrm{H}^2} \sim 10^{-35}\,\mathrm{s}^{-2}. \tag{25}$$

Such a small Λ may have implications in a realistic cosmological scenario. However, more explorations are required in that respect, which could be taken up as a future work.

5. Conclusions and prospects

We have built an extension of Dirac's large number hypothesis (LNH) and so we have found a new constant of nature, namely a universal minimum speed ($V \sim 10^{-14}$ m/s). This speed is a new kinematic invariant given for lower energies, which led to a new deformed special relativity (DSR), the so-called symmetrical special relativity (SSR), from where emerged the cosmological constant, which allowed us to show the equivalence of SSR metric with a kind of dS metric. The small value of the cosmological constant was estimated.

We have shown that the Bohr velocity ($v_\mathrm{B} = e^2/\hbar \sim 10^5$ m/s) in the fundamental state of the hydrogen atom, the Bohr radius (a_0) of the orbit of an electron in its ground state and the fine-structure constant (α) depend on the Eddginton number N.

The investigation of symmetries of the SSR theory should be done by means of their association with a new kind of electromagnetism when we are in the limit $v \to V$, which could explain the problem of high magnetic fields in magnetars [33], superfluids in the interior of gravastars [34] and other types of black hole mimickers [34–38].

References

[1] F R Klinkhamera and G E Volovik, *Phys. Lett. A* **347**, 8 (2005)
[2] Y B Zeldovich, *JETP Lett.* **6**, 316 (1967); *Sov. Phys. Uspekhi* **11**, 381 (1968)
[3] P A M Dirac, *Nature* **139**, 323 (1937)
[4] P A M Dirac, *Proc. Roy. Soc. London A* **165**, 198 (1938)
[5] Y B Zeldovich, *Usp. Nauk* **95**, 209 (1968)
[6] S C Ulhoa and E P Spaniol, *J. Gravity* **2016**, 1 (2016)
[7] S C Ulhoa, *Ann. Phys.* **524**, 273 (2012)
[8] G Silva, A F Santos and S C Ulhoa, *Eur. Phys. J.* **76**, 167 (2016)
[9] T Padmanabam, arXiv:0802.1798
[10] T Padmanabam, *Adv. Sci. Lett.* **2**, 174 (2009)
[11] T Padmanabhan, *Class. Quant. Grav.* **4**, L107 (1987)
[12] G Amelino-Camelia, *Lect. Notes Phys.* **541**, 1 (2000)
[13] R J Protheroe and H Meyer, *Phys. Lett. B* **493**, 1 (2000)
[14] D V Ahluwalia, *Mod. Phys. Lett. A* **17**, 1135 (2002)
[15] T Jacobson, S Liberati and D Mattingly, *Phys. Rev. D* **66**, 081302 (2002)
[16] R H Brandenberger and J Martin, *Int. J. Mod. Phys. A* **17**, 3663 (2002)
[17] R C Myers and M Pospelov, *Phys. Rev. Lett.* **90**, 211601 (2003)
[18] C Nassif, *Int. J. Mod. Phys. D* **25**, 1650096 (2016)
[19] C Nassif, *Gen. Rel. Grav.* **47**, I.9 (2015)
[20] C Nassif, *Int. J. Mod. Phys. D* **21**, 1 (2012)
[21] C Nassif, *Int. J. Mod. Phys. D* **19**, 539 (2010)

[22] C Nassif, *Pramana – J. Phys.* **71**, 1 (2008)
[23] C Nassif, R F dos Santos and A C Amaro de Faria Jr., *Int. J. Mod. Phys. D* **27**, 1850011 (2018)
[24] C Nassif, A C Amaro de Faria Jr. and R F dos Santos, *Mod. Phys. Lett. A* **33**, 1850148 (2018)
[25] C Nassif, R F dos Santos and A C Amaro de Faria Jr., *Phys. Dark Universe* **22**, 116 (2018)
[26] C Nassif and F A Silva, *Phys. Dark Universe* **22**, 127 (2018)
[27] C Nassif, *Mod. Phys. Lett. A* **34**, 1950212 (2019)
[28] K G Zloshchastiev, *Acta Phys. Polon. B* **42**, 261 (2011)
[29] R G Vishwakarma, *Class. Quant. Grav.* **19**, 4747 (2002)
[30] N Namavarian and M Farhoudi, *Gen. Rel. Grav.* **48**, 140 (2016)
[31] R Aldrovandi, J P Beltran Almeida and J G Pereira, *J. Geom. Phys.* **56**, 1042 (2006)
[32] R Aldrovandi, J P Beltrán Almeida and J G Pereira, *Class. Quant. Grav.* **24**, 1385 (2007)
[33] J Greiner *et al*, *Nature* **523**, 189 (2015)
[34] C Nassif, R F dos Santos and A C Amaro de Faria Jr., *Phys. Dark Universe* **27**, 100454 (2020)
[35] E Mottola, *Acta Phys. Polon. B* **41**, 2031 (2010)
[36] S W Hawking, *Commun. Math. Phys.* **43**, 199 (1975)
[37] H Y Guo, C G Huang and B Zhou, *Europhys. Lett.* **72**, 1045 (2005)
[38] G W Gibbons and S W Hawking, *Phys. Rev. D* **15**, 2738 (1977)