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Exact solution of the semiconfined harmonic oscillator model with a position-dependent effective mass in an external homogeneous field

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Abstract. We extend exactly solvable model of a one-dimensional non-relativistic canonical semiconfined quantum harmonic oscillator with a mass that varies with position to the case where an external homogeneous field is applied. The problem is still exactly solvable and the analytic expression of the wave functions of the stationary states is expressed by means of generalised Laguerre polynomials, too. Unlike the case without any external field, when the energy spectrum completely overlaps with the energy spectrum of the standard non-relativistic canonical quantum harmonic oscillator, the energy spectrum is now still equidistant but depends on the semiconfinement parameter a. We also compute probabilities of the transitions for the model under the external field and discuss limit cases for the energy spectrum, wave functions and probabilities of transitions, when the semiconfinement parameter a goes to infinity.

Keywords. Semiconfined harmonic oscillator; external homogeneous field; probabilities of the transitions.

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1. Introduction

Exact solutions of the Schrödinger equation describing certain non-relativistic quantum system are always attractive due to their huge potential in explaning an enormous number of phenomena in quantum physics and related areas. The problem of an external field applied to the quantum system is one such interesting quantum mechanical problem. The following examples can be considered as evidences for its importance: the effect of differently oriented external electric fields on the velocity of Rayleigh surface acoustic waves in lithium niobate crystal was studied both experimentally and theoretically in [1,2]. The response of a single cell to an external electric field was investigated because of its possible relevance to the mechanism of defibrillation [3]. The effect of an external electric field on the crystallisation of certain proteins have been studied in [4]. Direct impact of external electric fields on the chemical structure of molecular systems and their unprecedented control over chemical reactivity have been discussed thoroughly in [5]. The possibility of construction and diagonalisation of the perturbed Hamiltonian matrix at a relatively less computational cost is demonstrated in [6] for three different sample molecules *in vacuo* under an external field, when the perturbing external field is a homogeneous static electric field. Powerful computational methods of second- and third-order non-linear optical properties of the quantum well structures are discussed in [7–9] by breaking their symmetry via an external electric field. Ganesan and Gebarowski [10] discussed the chaotic dynamics of a hydrogen atom interacting with time-independent and time-dependent external fields of statics and combined electrical and magnetic type. Hosseini *et al* [11] considered a massless spinor Dirac particle in the presence of an external electromagnetic field in the cosmic string space–time and found that the degeneracy of the Minkowski space spectral becomes broken in the transition from Minkowski to cosmic string space. Recently, several methods have been developed to make permanent string-like cluster structures of colloidal particles acquiring a dipole

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moment in a homogeneous external electric field [12]. Also, one needs to highlight here recent developments in the field of econophysics, where by defining wave functions and operators of the stock market, it was possible to establish Schrödinger equation for stock price and then to study the change of the stock price under an external field appearing as certain market information affecting this price [13–15].

Recently, we presented a new model of a one-dimensional non-relativistic canonical quantum harmonic oscillator exhibiting semiconfinement [16]. This was achieved by replacing constant effective mass with a mass that depends on the position. We were able to solve the problem exactly and obtained the analytic expression of the wave functions of the stationary states by means of generalised Laguerre polynomials. We also observed a surprising phenomenon regarding the energy spectrum of this new model: there was complete overlap with the energy spectrum of the standard non-relativistic canonical quantum harmonic oscillator. We also demonstrated that in the limit when the semiconfinement parameter a goes to infinity, the wave functions of this new model tend to the wave functions of the standard non-relativistic oscillator in terms of Hermite polynomials. Here, we shall extend our study of this model to the case when an external homogeneous force $F_{\text{ext}} = -g$ ($g \geq 0$) is applied. The exact solution for the non-relativistic canonical quantum harmonic oscillator under the influence of such an external force is well-known. Its behaviour is like the non-relativistic canonical quantum harmonic oscillator but with a shifted equilibrium position x. Therefore, the wave functions and energy spectrum preserve their general mathematical expressions [17]. For us it was interesting to explore the analogue of this model but with a position-dependent effective mass. We shall show that this model is still analytically solvable: we obtain exact solutions of the wave functions and the energy spectrum. The wave functions display again a shifted equilibrium position compared to the non-relativistic canonical quantum harmonic oscillator. The energy spectrum has interesting properties: it is again equidistant, but the energy gap now depends both on the semiconfinement parameter a and the external force g.

The structure of the present paper is as follows: in §2, we present some basic information about the exact solution for the non-relativistic canonical quantum harmonic oscillator with and without an external homogeneous field. We provide exact expressions of the wave functions and energy spectrum for both cases. Then, §3 includes basic information about the exact expressions of wave functions and energy spectrum of the semiconfined oscillator model developed in our paper [16], and then we solve exactly the semiconfined harmonic oscillator problem with position-dependent effective mass in the presence of an external homogeneous field. The final section includes some discussions regarding the obtained solutions. In order to understand better the main differences of the models under construction, we also compute probabilities of transitions to excited states under the action of the external field.

2. Non-relativistic harmonic oscillator without and with an external field

As we mentioned in the Introduction, this section is informative, because all results and expressions below are already well known in non-relativistic quantum mechanics. We are dealing with a one-dimensional time-independent non-relativistic quantum system, for which the Schrödinger equation reads [17,18] as

$$\hat{H}\psi(x) = E\psi(x), \tag{1}$$

where the Hamiltonian $\hat{H}$ is the sum of the kinetic and potential terms

$$\hat{H} = \hat{H}_0 + V(x), \tag{2}$$

with the kinetic energy operator

$$\hat{H}_0 = \frac{\hat{p}_x^2}{2m_0}, \tag{3}$$

where m_0 is a constant effective mass of the quantum system. In general, the momentum operator $\hat{p}_x$ can be represented according to two different approaches. The first one follows the canonical approach [17]:

$$\hat{p}_x = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}. \tag{4}$$

It is worth mentioning that there exists also another non-canonical approach due to Wigner [19,20]:

$$\hat{p}_x = -i\hbar\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\gamma - 1/2}{x}\hat{R}\right), \tag{5}$$

where $\hat{R}$ is the parity operator and $\gamma > 0$ is a positive constant. One can easily observe that for $\gamma = 1/2$ one completely recovers the canonical form. For simplicity, we shall follow here the canonical case (4). However, it is noteworthy that computations performed by employing the non-canonical definition (5) of the momentum operator can also lead to attractive results [21–23].

For the quantum harmonic oscillator, the potential in the absence of an external homogeneous field is given by

$$V(x) \equiv V^{\text{ho}}(x) = \frac{m_0\omega^2x^2}{2}, \quad -\infty < x < +\infty, \tag{6}$$

where ω is the constant angular frequency of the quantum harmonic oscillator. Substitution of (2) in (1) by taking into account (4) and (6) leads to the following second-order differential equation:

$$\frac{\hbar^2}{2m_0}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+\left(E-\frac{m_0\omega^2x^2}{2}\right)\psi=0. \tag{7}$$

Its exact solution under the condition $\psi(x\to\pm\infty)\to 0$ leads to the discrete energy spectrum $E\equiv E_{\mathrm{n}}^{\mathrm{ho}}$ and wave functions $\psi(x)\equiv\psi_{\mathrm{n}}^{\mathrm{ho}}(x)$ as follows [17,18]:

$$E_n^{\mathrm{ho}}=\hbar\omega\left(n+\frac{1}{2}\right),\quad n=0,1,2,\ldots. \tag{8}$$

$$\psi_n^{\mathrm{ho}}(x)=\mathrm{e}^{-\frac{m_0\omega}{2\hbar}x^2}H_n\left(\sqrt{\frac{m_0\omega}{\hbar}}x\right). \tag{9}$$

Here, $H_n(x)$ is a Hermite polynomial, defined in terms of the ${}_2F_0$ hypergeometric function as follows [24]:

$$H_n(x)=(2x)^n\,{}_2F_0\left(\begin{matrix}-n/2,-(n-1)/2\\ -\end{matrix};-\frac{1}{x^2}\right). \tag{10}$$

The orthogonality relation for Hermite polynomials on the whole interval $(-\infty,+\infty)$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-x^2}H_m(x)H_n(x)\mathrm{d}x=2^nn!\delta_{mn} \tag{11}$$

yields the following orthonormal wave functions $\tilde{\psi}_n(x)$:

$$\tilde{\psi}_n^{\mathrm{ho}}(x)=\frac{1}{\sqrt{2^nn!}}\left(\frac{m_0\omega}{\pi\hbar}\right)^{\frac{1}{4}}\mathrm{e}^{-\frac{m_0\omega x^2}{2\hbar}}H_n\left(\sqrt{\frac{m_0\omega}{\hbar}}x\right). \tag{12}$$

Next, let us assume that an external homogeneous field $V^{\mathrm{ext}}(x)=gx$ is applied to the non-relativistic quantum harmonic oscillator system (6). Then, it is obvious that the resulting potential will be changed as follows [17]:

$$V(x)\equiv V^{\mathrm{ho}}(x)+V^{\mathrm{ext}}(x)=\frac{m_0\omega^2x^2}{2}+gx,\quad -\infty<x<+\infty. \tag{13}$$

The following Schrödinger equation corresponding to this extended potential is still exactly solvable:

$$\frac{\hbar^2}{2m_0}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+\left(E-\frac{m_0\omega^2x^2}{2}-gx\right)\psi=0. \tag{14}$$

Its solutions lead to the discrete energy spectrum $E\equiv E_n^g$ and wave functions $\psi(x)\equiv\psi_n^g(x)$ as follows [17]:

$$E_n^g=\hbar\omega\left(n+\frac{1}{2}\right)-\frac{g^2}{2m_0\omega^2},\quad n=0,1,2,\ldots, \tag{15}$$

$$\begin{aligned}\tilde{\psi}_n^g(x)&=\tilde{\psi}_n^{\mathrm{ho}}\left(x+\frac{g}{m_0\omega^2}\right)\\ &=\frac{1}{\sqrt{2^nn!}}\left(\frac{m_0\omega}{\pi\hbar}\right)^{\frac{1}{4}}\mathrm{e}^{-\frac{m_0\omega^2\left(x+\frac{g}{m_0\omega^2}\right)^2}{2\hbar}}\\ &\quad\times H_n\left(\sqrt{\frac{m_0\omega}{\hbar}}\left(x+\frac{g}{m_0\omega^2}\right)\right).\end{aligned} \tag{16}$$

One easily observes that analytical expression (15) of the energy spectrum of the oscillator under the external field differs from (8) by an additional term $-(g^2/2m_0\omega^2)$. Analytical expression (16) of the wave function of the oscillator under the external field coincides with (12) up to a shift

$$x\to x+\frac{g}{m_0\omega^2}.$$

Both energy spectrum (15) and wave functions (16) easily yield energy spectrum (8) and wave function (12) by putting $g=0$.

3. Semiconfined harmonic oscillator model with a position-dependent effective mass under an external field

In our recent paper [16], we constructed a non-relativistic quantum harmonic oscillator in the canonical approach with the wave functions tending to zero at the right side at position $x\to+\infty$, but from the left side already at some finite value of the position x, i.e. $\psi(x)=0$ for $-\infty<x\le -a$ with a a positive constant ($a>0$). Therefore, we called this a semiconfined harmonic oscillator model. The vanishing of the wave functions of the oscillator for $x\le a$ implied that the potential $V(x)|_{x=-a}$ tends to $+\infty$. We achieved this effect of an 'infinite high wall' by replacing the constant effective mass m_0 of the oscillator by a position-dependent effective mass $M(x)$. Taking into account this property of the effective mass, the following Hermitian version of the kinetic energy operator with position-dependent effective mass (also called as BenDaniel–Duke kinetic energy operator) was chosen for our further computations [25]:

$$\hat{H}_0\equiv\hat{H}_0^{\mathrm{BD}}=-\frac{\hbar^2}{2}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{M(x)}\frac{\mathrm{d}}{\mathrm{d}x}. \tag{17}$$

Here again one needs to mention that different versions of non-relativistic kinetic energy operators (3) exist for the case of effective mass changing with position. There are Gora–Williams, Zhu–Kroemer, von Roos kinetic energy operators [26–28] as well as kinetic energy operators based on the contact point transformation method [29–32] and non-Hermitian PT-symmetric kinetic energy operators [33–38].

Rewriting the harmonic oscillator potential $V(x)$ via modification of (6) under the replacement $m_0 \to M(x)$

$$V(x) \equiv V^{\text{ho}}(x) = \frac{M(x)\,\omega^2 x^2}{2}, \quad -a < x < +\infty, \tag{18}$$

we solved exactly the following Schrödinger equation corresponding to this potential

$$\frac{\hbar^2}{2m_0}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{1}{a+x}\frac{\mathrm{d}\psi}{\mathrm{d}x}\right) + \frac{aE(a+x) - \frac{m_0\omega^2 a^2}{2}x^2}{(a+x)^2} \times \psi = 0, \tag{19}$$

by using the following simple analytic expression for the position-dependent effective mass $M(x)$:

$$M(x) = \begin{cases} \dfrac{am_0}{a+x}, & \text{for } -a < x < +\infty \\ +\infty, & \text{for } x \le -a \end{cases} \quad (a > 0). \tag{20}$$

We found that the energy spectrum E of this oscillator model completely overlaps with the energy spectrum of the non-relativistic quantum harmonic oscillator (8), i.e.

$$E \equiv E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots, \tag{21}$$

but the orthonormalised wave functions of the stationary states are expressed in terms of the generalised Laguerre polynomials as follows:

$$\tilde{\psi}_n(x) = C_n \cdot \left(1 + \frac{x}{a}\right)^{\frac{m_0\omega}{\hbar}a^2} \mathrm{e}^{-\frac{m_0\omega}{\hbar}a(x+a)} L_n^{\left(2\frac{m_0\omega}{\hbar}a^2\right)} \times \left(2\frac{m_0\omega}{\hbar}a(x+a)\right), \quad -a < x < +\infty, \tag{22}$$

where C_n is a normalisation constant that can be extracted from the orthogonality relation for the wave functions (22):

$$\int_{-\infty}^{+\infty} \tilde{\psi}_m(x)\tilde{\psi}_n(x)\,\mathrm{d}x = \int_{-a}^{+\infty} \tilde{\psi}_m(x)\tilde{\psi}_n(x)\,\mathrm{d}x = \delta_{mn}.$$

The exact expression of the normalisation constant is

$$C_n = (-1)^n \left(2\frac{m_0\omega}{\hbar}a^2\right)^{\frac{m_0\omega}{\hbar}a^2 + \frac{1}{2}} \times \sqrt{\frac{n!}{a\Gamma\left(n + 2\frac{m_0\omega}{\hbar}a^2 + 1\right)}}. \tag{23}$$

Also, it was shown that wave function (22) tends to the Hermite oscillator wave function (12) when $a \to +\infty$. Its proof was based on the following known limit relation between the Laguerre and Hermite polynomials [24]:

$$\lim_{\alpha\to+\infty}\left(\frac{2}{\alpha}\right)^{\frac{1}{2}n} L_n^{(\alpha)}\left((2\alpha)^{\frac{1}{2}}x + \alpha\right) = \frac{(-1)^n}{n!}H_n(x)$$

and another simple limit relation

$$\lim_{a\to+\infty}\left(1 + \frac{x}{a}\right)^{\lambda_0^2 a^2} \mathrm{e}^{-\lambda_0^2 a(x+a)} = \mathrm{e}^{-\frac{\lambda_0^2 x^2}{2}},$$

as well as application of Stirling's approximation for the gamma function. Our solution was similar to the known Liénard-type nonlinear one-dimensional quantum oscillator model solved in momentum space [39,40].

Now, let us assume that an external homogeneous field $V^{\text{ext}}(x)$ is applied to the non-relativistic semiconfined harmonic oscillator model with a position-dependent effective mass (18). Then, it is obvious that the resulting potential will be changed as follows:

$$V(x) \equiv V^{\text{ho}}(x) + V^{\text{ext}}(x) = \frac{M(x)\,\omega^2 x^2}{2} + gx, \quad -a < x < +\infty. \tag{24}$$

Then, the Schrödinger equation corresponding to eqs (17), (20) and (24) can be written as follows:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{1}{a+x}\frac{\mathrm{d}\psi}{\mathrm{d}x} - \left(\frac{\frac{m_0^2\omega^2 a^2}{\hbar^2}x^2 + \frac{2m_0 ga}{\hbar^2}x(a+x) - \frac{2m_0 aE}{\hbar^2}(a+x)}{(a+x)^2}\right) \times \psi = 0. \tag{25}$$

In order to solve this, let us apply the following transformation to a dimensionless variable ξ:

$$\xi = \frac{x}{a}, \quad \frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}\psi}{\mathrm{d}\xi} = \frac{1}{a}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}, \quad \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \frac{1}{a^2}\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2}.$$

Then, introducing also the notations

$$\lambda_0 = \sqrt{\frac{m_0\omega}{\hbar}}$$

and

$$c_0 = \frac{2m_0 a^2 E}{\hbar^2}, \quad c_1 = \frac{2m_0 a^3 g}{\hbar^2},$$
$$c_2 = c_0 + \frac{m_0^2 \omega^2 a^4}{\hbar^2} = c_0 + \lambda_0^4 a^4 \tag{26}$$

one arrives at the following second-order differential equation:

$$\psi'' + \frac{1}{1+\xi}\psi' + \frac{c_0 - (c_1 - c_0)\,\xi - (c_2 + c_1 - c_0)\,\xi^2}{(1+\xi)^2} \times \psi = 0, \tag{27}$$

where

$$\psi'' \equiv \frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2}$$

and

$$\psi' \equiv \frac{\mathrm{d}\psi}{\mathrm{d}\xi}.$$

Since this is a second-order differential equations of the type

$$\psi'' + \frac{\tilde{\tau}}{\sigma}\psi' + \frac{\tilde{\sigma}}{\sigma^2}\psi = 0,$$

with σ and $\tilde{\sigma}$ being polynomials of at most second degree and $\tilde{\tau}$ being a polynomial of at most first degree, with

$$\tilde{\tau} = 1, \quad \sigma = 1 + \xi,$$
$$\tilde{\sigma} = c_0 - (c_1 - c_0)\,\xi - (c_2 + c_1 - c_0)\,\xi^2,$$

allows us to apply the Nikiforov–Uvarov method [41] to solve eq. (27) exactly. We write the solution for ψ as

$$\psi = \varphi(\xi)\, y, \tag{28}$$

where $\varphi(\xi)$ is defined as a result of straightforward computations as follows:

$$\varphi(\xi) = (\xi + 1)^{\lambda_0^2 a^2}\, \mathrm{e}^{-\sqrt{\lambda_0^4 a^4 + c_1}\,\xi}. \tag{29}$$

The necessary boundary conditions for $\varphi(\xi)$ are satisfied:

$$\lim_{\xi \to -1} \varphi(\xi) = 0, \quad \lim_{\xi \to +\infty} \varphi(\xi) = 0.$$

The substitution of ψ in eq. (27) leads to the following second-order differential equation for y:

$$(\xi + 1)\, y'' + (2\lambda_0^2 a^2 + 1 - 2\sqrt{\lambda_0^4 a^4 + c_1}\,(\xi + 1))y' = ((2\lambda_0^2 a^2 + 1)\sqrt{\lambda_0^4 a^4 + c_1} - 2\lambda_0^4 a^4 - c_1 - c_0)y. \tag{30}$$

In order to have polynomial solutions, compare with the following equation for the generalised Laguerre polynomials [24]:

$$(x - d)\, y_n''(x) + \{2\varepsilon(x - d) + \alpha + 1\}\, y_n'(x) = 2\varepsilon n y_n(x),$$

where $d < x$, $\varepsilon < 0$ and $\alpha + 1 > 0$. Then

$$y_n(x) = L_n^{(\alpha)}(2\varepsilon(d - x))$$

and hence the energy spectrum of the model under consideration is

$$E \equiv E_n^g = \hbar\omega\sqrt{1 + \frac{2g}{m_0\omega^2 a}}\left(n + \frac{1}{2} + \frac{m_0\omega}{\hbar}a^2\right) - m_0\omega^2 a^2 - ag, \quad n = 0, 1, 2, \ldots. \tag{31}$$

The orthonormal wave functions have the following exact expression:

$$\tilde{\psi}_n^g(x) = C_n^g\left(1 + \frac{x}{a}\right)^{\frac{m_0\omega}{\hbar}a^2} \mathrm{e}^{-\frac{m_0\omega}{\hbar}a\sqrt{1+\frac{2g}{m_0\omega^2 a}}(x+a)} \times L_n^{\left(2\frac{m_0\omega}{\hbar}a^2\right)}\left(2\frac{m_0\omega}{\hbar}a\sqrt{1 + \frac{2g}{m_0\omega^2 a}}\,(x + a)\right). \tag{32}$$

Here, the normalisation constant is determined in similar manner to (23) as follows:

$$C_n^g = \left(\sqrt{1 + \frac{2g}{m_0\omega^2 a}}\right)^{\frac{m_0\omega}{\hbar}a^2 + \frac{1}{2}} C_n = (-1)^n\left(2\frac{m_0\omega}{\hbar}a^2\sqrt{1 + \frac{2g}{m_0\omega^2 a}}\right)^{\frac{m_0\omega}{\hbar}a^2 + \frac{1}{2}} \times \sqrt{\frac{n!}{a\Gamma\left(n + 2\frac{m_0\omega}{\hbar}a^2 + 1\right)}}. \tag{33}$$

Our main goal was to show that the semiconfined quantum harmonic oscillator with position-dependent effective mass is exactly solvable even if it is under an external homogeneous field. We achieved this goal by obtaining analytical expressions of energy spectrum (31) and normalised wave functions (32). In the following section, we are going to discuss some important properties of this model.

4. Discussion and conclusion

First of all, we note an important property of energy spectrum (31). Whereas in all previous models – canonical without external field (8), canonical with external

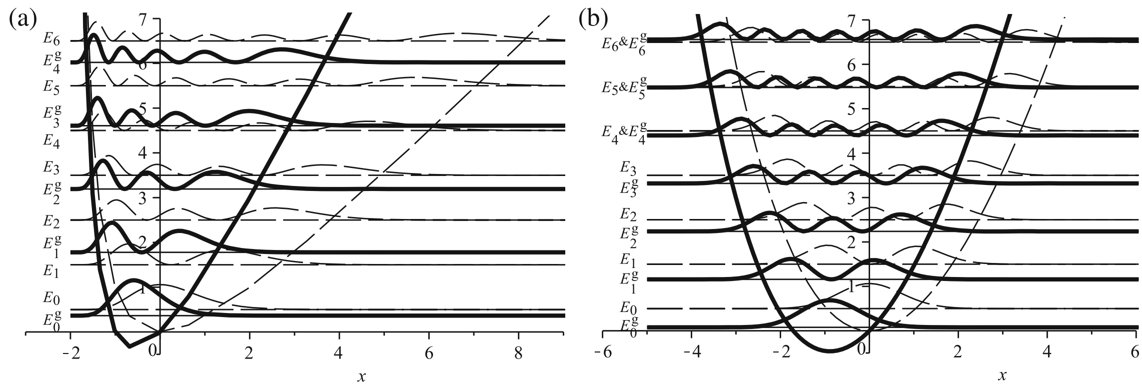

Figure 1. Comparative plot of the semiconfined quantum harmonic oscillator potential without an external field (18) (dashed line) and with an external field (24) (solid line). Also given are the corresponding energy levels (21) and (31) and the probability densities $|\tilde{\psi}_n(x)|^2$ of the wave functions of stationary states (22) and (32) for $g = 1$ and for the ground state and six excited states. (**a**) is for the confinement parameter $a = 2$ and (**b**) is for the confinement parameter $a = 12$ ($m_0 = \omega = \hbar = 1$).

field (15) and semiconfined without external field (21) – the energy gap ΔE of the equidistant spectrum is always given by

$$\Delta E = \hbar\omega,$$

the energy gap of the equidistant spectrum of the current semiconfined model with external field (31) is

$$\Delta E = \hbar\omega\sqrt{1 + \frac{2g}{m_0\omega^2 a}}. \tag{34}$$

Thus, the energy levels are wider apart, and this gap tends to the standard gap when g goes to 0 or when a tends to infinity.

Observe also that energy levels (31) tend to energy levels (15) when a goes to infinity. In order to see this, expand the square root in (31) as follows:

$$\sqrt{1 + \frac{2g}{m_0\omega^2 a}} = 1 + \frac{g}{m_0\omega^2 a} - \frac{g^2}{2m_0^2\omega^4 a^2} + \cdots .$$

Substitution in (31) leads to the following expansion for the energy spectrum:

$$E_n^g = \hbar\omega\left(1 + \frac{g}{m_0\omega^2 a} - \frac{g^2}{2m_0^2\omega^4 a^2} + \cdots\right) \times\left(n + \frac{1}{2} + \frac{m_0\omega}{\hbar}a^2\right) - m_0\omega^2 a^2 - ag$$

and from here it is easy to see that it reduces to (15) when $a \to \infty$. This statement is also true for the corresponding wave functions (32) and (16).

In order to better understand the impact of the external homogeneous field on the behaviour of the semiconfined oscillator model under study, we present some plots in figure 1. We plot the semiconfined quantum harmonic oscillator potential without (18) and with an applied external field (24), the corresponding energy levels (21) and (31), and the probability densities $|\tilde{\psi}_n(x)|^2$ of the wave functions of stationary states (22) and (32). We choose $g = 1$, and make the plots for the ground state and six excited states. We made these plots for a small value of the semiconfinement parameter a, $a = 2$, in figure 1a, and for a large value of a, $a = 12$, in figure 1b ($m_0 = \omega = \hbar = 1$).

In figure 1a, one observes that the location of ground-state energy level E_0^g is lower than the location of the ground-state energy level E_0. However, all excited energy levels E_n^g ($n > 0$) are higher than the energy levels E_n ($n > 0$). A similar feature can be observed in figure 1b, where the behaviour of the semiconfined oscillator becomes closer to the Hermite oscillator due to the fact that a is bigger (thus closer to infinity). There, the ground and first five excited energy levels E_n^g ($n = 0, 1, \ldots, 5$) are below the energy levels E_n. Higher up, the energy levels of E_n^g are greater than E_n. Such a behaviour can be explained by the computation of the ratio E_n^g/E_n. From this ratio, one obtains that $E_n^g \geq E_n$ only if

$$n \geq -\frac{1}{2} + \frac{1}{2}\frac{m\omega a^2}{\hbar}\left(\sqrt{1 + \frac{2g}{m\omega^2 a}} - 1\right).$$

In order to observe the impact of external field under semiconfinement effect, we decided to study also the probabilities of transitions. Let us imagine the situation where an external homogeneous field is suddenly applied to the semiconfined oscillator with a position-dependent effective mass in the ground state. The determination of the probabilities of transitions of the non-relativistic harmonic oscillator wave functions (12) from ground to excited states under the action of such a perturbation are described in [17]. These probabilities are defined as follows:

$$w_{0k} = \left| \int_{-\infty}^{\infty} \tilde{\psi}_0^{\text{ho}}(x)\, \tilde{\psi}_k^{g}(x)\, \mathrm{d}x \right|^2. \tag{35}$$

Taking into account the expression of $\tilde{\psi}_k^{g}(x)$ (16) and the wave functions $\tilde{\psi}_k^{\text{ho}}(x)$ (12), one has

$$w_{0k} = \left| \int_{-\infty}^{\infty} \tilde{\psi}_0^{\text{ho}}(x)\, \tilde{\psi}_k^{\text{ho}}\left(x + \frac{g}{m_0\omega^2}\right) \mathrm{d}x \right|^2. \tag{36}$$

The exact computation of this transition probability is a Poisson distribution, i.e.

$$w_{0k} = \frac{\bar{\kappa}^2}{k!} \mathrm{e}^{-\bar{\kappa}^2}, \qquad \bar{\kappa} = (2m_0\hbar\omega)^{-1/2} \frac{g}{\omega}. \tag{37}$$

In order to explore the semiconfinement oscillator model under study, note that the wave functions $\tilde{\psi}_k^{g}(x)$ (32) formally can be expressed through the wave functions $\tilde{\psi}_k^{\text{ho}}(x)$ (22):

$$\tilde{\psi}_n^{g}(x) = \left(\frac{m_0\omega^2 a}{2g + m_0\omega^2 a} \right)^{\frac{1}{4}} \tilde{\psi}_n \left(\sqrt{1 + \frac{2g}{m_0\omega^2 a}}\, x \right). \tag{38}$$

Then, the computation of the transition probabilities of the semiconfined harmonic oscillator wave functions (22) from ground to excited states $\tilde{\psi}_k^{g}(x)$ (32) under an external homogeneous field leads to the following expression:

$$w_{0k} = \frac{1}{\sqrt{1 + \frac{2g}{m_0\omega^2 a}}} \times \left| \int_{-a}^{\infty} \tilde{\psi}_0(x)\, \tilde{\psi}_k \left(\sqrt{1 + \frac{2g}{m_0\omega^2 a}}\, x \right) \mathrm{d}x \right|^2. \tag{39}$$

This can be computed exactly using the following known integral relation for the generalised Laguerre polynomials [42, eq. (2.19.3.3)]:

$$\int_0^{\infty} \zeta^{\lambda} \mathrm{e}^{-p\zeta} L_n^{(\lambda)}(c\zeta)\, \mathrm{d}\zeta = \frac{\Gamma(\lambda + n + 1)(p - c)^n}{n!\, p^{\lambda+n+1}}, \quad \Re(p) > 0,\ \Re(\lambda) > -1.$$

Without going into the details of this technical calculation, one obtains for (39)

$$w_{0k} = \frac{\left(2\lambda_0^2 a^2 + 1\right)_k}{k!} \left(\frac{1 - \sqrt{1 + 2\sqrt{2}\lambda_0^{-1}\bar{\kappa}}}{1 + \sqrt{1 + 2\sqrt{2}\lambda_0^{-1}\bar{\kappa}}} \right)^{2k} \times \left(\frac{2\sqrt{1+2\sqrt{2}\lambda_0^{-1}\bar{\kappa}}}{1+2\sqrt{2}\lambda_0^{-1}\bar{\kappa}+\sqrt{1+2\sqrt{2}\lambda_0^{-1}\bar{\kappa}}} \right)^{2\lambda_0^2 a^2+1}. \tag{40}$$

It can be shown that (40) tends to (37) for $a \to \infty$.

We leave it to the reader as an exercise to compute the transition probabilities of the non-relativistic harmonic oscillator wave functions from any excited state s to k under an external homogeneous field. Such a probability w_{sk} is also exactly computable by applying the following integral relation involving two generalised Laguerre polynomials [42, eq. (2.19.14.6)]:

$$\int_0^{\infty} \zeta^{\lambda} \mathrm{e}^{-p\zeta} L_s^{(\lambda)}(b\zeta)\, L_k^{(\lambda)}(c\zeta)\, \mathrm{d}\zeta = \frac{(\lambda+1)_s (\lambda+1)_k}{s!k!p^{s+k+\lambda+1}} \Gamma(\lambda+1)(p-b)^s (p-c)^k \times {}_2F_1\left(\begin{matrix} -s, -k \\ \lambda + 1 \end{matrix}; \frac{bc}{(p-b)(p-c)} \right). \tag{41}$$

Then, w_{sk} will be expressed through Meixner polynomials. Under the limit $a \to \infty$ the probability of the corresponding transition for the non-relativistic quantum harmonic oscillator under the external homogeneous field will be recovered by applying the known limit relation between Meixner and Charlier polynomials [24].

Finally, note that we considered here only the case when $g \geq 0$. The same problem can also be studied for negative values of g. However, then it is necessary to extend the computations done here to both continuous and discrete spectrum states.

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