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A higher-order efficient approach to numerical simulations of the RLW equation

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Abstract. A new numerical algorithm for solving the regularised long wave (RLW) equation is proposed using predictor–corrector method in time and quartic B-spline collocation procedure in space. Accuracy of algorithm is shown by studying the motion of a single solitary wave. Interaction of two solitary waves and three solitary waves, and wave generation from Maxwellian condition are performed to illustrate the suitability of the proposed scheme. Results of the present method are compared with the outcomes of some prosperous numerical algorithms.

Keywords. Regularised long wave equation; Adams–Bashforth–Moulton method; predictor–corrector method; quartic B-splines; interaction of solitary waves.

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1. Introduction

The regularised long wave equation (RLWE) was derived by Peregrine for modelling the propagation of the (multidirectional) weakly nonlinear and dispersive waves [1]. RLWE can be effectively employed to express a large class of wave behaviours such as undular bore evolution, ion-acoustic plasma waves, shallow water waves and pressure wave propagation in the liquid–gas bubble mixture [2–4]. Although special analytical solutions of RLWE have been investigated, they only form some parts of the solutions to the RLWE. Therefore, computational studies for showing additional wave behaviours of the RLWE have been increased over the last several years. These include the difference methods, Fourier pseudospectral methods, meshless methods, differential quadrature method and finite element methods based on Galerkin, least squares and collocation techniques [6–23]. The trial function consisting of a combination of the B-splines is adapted to finite element method to solve partial differential equations successfully. It is known that B-spline finite element methods are easy to apply in solving partial differential equations. Especially, collocation procedure of the finite element method is handy to integrate partial differential equations over the finite intervals. Good choice of trial functions, for instance, B-splines, helps collocation method to increase the accuracy of the numerical solution of differential equations. The B-spline collocation method produces smooth, global approximation for the solution of the given differential equation in the form of algebraic polynomial and is preferred because its application is easy for integrating differential equations.

The B-spline collocation method based on various base functions have been established to solve the RLWE. Most researchers used the Crank–Nicolson method to integrate in time and some used variants of Runge–Kutta method for time discretisation. Recently, two-step Adams–Moulton method of order 3 and Simpson's method of order 4 were used to manage the time discretisation and the quadratic/quartic trigonometric B-spline Galerkin algorithm is adapted to integrate time-discretised RLWE [22,23]. Considerable improvements in accuracy have been observed when time integration of higher order is performed. In this work, RLWE is fully discretised by first applying the predictor–corrector method for time discretisation of unknown variable and then the quartic B-spline collocation scheme is used for

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the discretisation of the spatial part of the unknown variable. Solution of the resulting system provides values of unknown parameters from which the approximate solution can be obtained.

2. Numerical method

The solution interval $[x_0 = a, x_N = b]$ is partitioned at equally distributed grid points $x_j = x_0 + jh$, $j = 0, \ldots, N$ and the length of subinterval $[x_j, x_{j+1}]$, $j = 0, \ldots, N-1$ is $h = (b-a)/N$.

Approximate solution can be expressed in terms of the quartic B-spline functions forming the basis for functions defined in the interval. Thus, at grid points x_j, $j = 0, \ldots, N$ and grid points $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{N+1}, x_{N+2}, x_{N+3}, x_{N+4}$ outside the interval $[a, b]$, quartic B-splines $Q_m(x)$, $m = -2, \ldots, N+1$ are defined as

$$Q_m(x) = \frac{1}{h^4}\begin{cases} (x-x_{m-2})^4, & \text{if } x \in [x_{m-2}, x_{m-1}] \\ (x-x_{m-2})^4 - 5(x-x_{m-1})^4, & \text{if } x \in [x_{m-1}, x_m] \\ (x-x_{m-2})^4 - 5(x-x_{m-1})^4 + 10(x-x_m)^4, & \text{if } x \in [x_m, x_{m+1}] \\ (x_{m+3}-x)^4 - 5(x-x_{m+2})^4, & \text{if } x \in [x_{m+1}, x_{m+2}] \\ (x_{m+3}-x)^4, & \text{if } x \in [x_{m+2}, x_{m+3}] \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

which are piecewise functions satisfying continuities up to order 3 at the connecting points x_i, $i = m-1, \ldots, m+2$. Five sequential B-splines cover five subintervals. Consider the initial and boundary value problem defined as RLWE:

$$u_t + u_x + \varepsilon u u_x - \mu u_{xxt} = 0,$$
$$x \in [a, b], \quad t \in (0, T], \tag{2}$$
$$u(x, 0) = f(x), \tag{3}$$
$$u(a, t) = \sigma_1, \quad u(b, t) = \sigma_2,$$
$$u_x(a, t) = u_x(b, t) = 0. \tag{4}$$

Here, u is the wave amplitude, t is the time, x is the space coordinate and μ, ε are positive parameters. We are going to use three boundary conditions and initial condition to determine initial parameters when time $t = 0$. Besides, the three boundary conditions will be employed to obtain unknown parameters when $t > 0$, coming from the application of the quartic B-spline collocation method. Two boundary conditions from the left boundary conditions and one boundary condition from the right boundary conditions are used.

Let us describe the finite element procedure for the RLWE. Solution of $u(x, t)$ is approximated by expression of the form over the interval $[a, b]$

$$U(x, t) = \sum_{m=-2}^{N+1} Q_m(x)\,\delta_m(t), \tag{5}$$

where the time parameters $\delta_m(t)$ are going to be computed using the collocation method. The approximate solution can be written in terms of sequential five B-splines staying within the subinterval $[x_m, x_{m+1}]$, $m = 0, \ldots, N-1$:

$$\begin{aligned} U^e = {} & Q_{m-2}(x)\,\delta_{m-2}(t) + Q_{m-1}(x)\,\delta_{m-1}(t) \\ & + Q_m(x)\,\delta_m(t) + Q_{m+1}(x)\,\delta_{m+1}(t) \\ & + Q_{m+2}(x)\,\delta_{m+2}(t), \end{aligned} \tag{6}$$

where quantities $\delta_j(t)$, $j = m-2, m-1, m, m+1, m+2$ are known as element parameters. Approximate solution (5) and its first three derivatives at the grid points can be expressed in terms of the time variables as

$$\begin{aligned} U^e &= a_1\delta_{m-2}^n + a_2\delta_{m-1}^n + a_3\delta_m^n + a_4\delta_{m+1}^n, \\ (U^e)' &= b_1\delta_{m-2}^n + b_2\delta_{m-1}^n + b_3\delta_m^n + b_4\delta_{m+1}^n, \\ (U^e)'' &= c_1\delta_{m-2}^n + c_2\delta_{m-1}^n + c_3\delta_m^n + c_4\delta_{m+1}^n, \\ (U^e)''' &= d_1\delta_{m-2}^n + d_2\delta_{m-1}^n + d_3\delta_m^n + d_4\delta_{m+1}^n, \end{aligned} \tag{7}$$

where

$$\begin{aligned} &a_1 = 1, && a_2 = 11, && a_3 = 11, && a_4 = 1, \\ &b_1 = -\frac{4}{h}, && b_2 = -\frac{12}{h}, && b_3 = \frac{12}{h}, && b_4 = \frac{4}{h}, \\ &c_1 = \frac{12}{h^2}, && c_2 = -\frac{12}{h^2}, && c_3 = -\frac{12}{h^2}, && c_4 = \frac{12}{h^2}, \\ &d_1 = -\frac{24}{h^3}, && d_2 = \frac{72}{h^3}, && d_3 = -\frac{72}{h^3}, && d_4 = \frac{24}{h^3}. \end{aligned}$$

The time discretisation is managed by using the following expansion:

$$\begin{aligned} u^{n+1} = {} & \theta_1 u^n + \theta_2 u_t^n + \theta_3 u_t^{n-1} + \theta_4 u_t^{n-2} \\ & + \theta_5 u_t^{n-3} + \theta_6 u_t^{n+1}. \end{aligned} \tag{8}$$

Predictor–corrector equation is derived by choosing the sets of parameters: $\theta_1 = 1,\ \theta_2 = 55\Delta t/24,\ \theta_3 = -59\Delta t/24,\ \theta_4 = 37\Delta t/24,\ \theta_5 = -9\Delta t/24,\ \theta_6 = 0$ gives predictor equation of the fourth-order explicit Adams–Bashforth which is going to be used to guess for the solution U. The selection of the parameters $\theta_1 = 1,\ \theta_2 = 646\Delta t/720,\ \theta_3 = -264\Delta t/720,\ \theta_4 = 106\Delta t/720,\ \theta_5 = -19\Delta t/720,\ \theta_6 = 251\Delta t/720$ produces the corrector equation of the fifth-order Adams–Moulton method. Having found guess solution U from the predictor, the solution will be put in corrector formula to have more accurate solution. Now applying time discretisation on RLWE yields

$$\begin{aligned} u^{n+1} - \mu u_{xx}^{n+1} = \theta_1\left(u^n - \mu u_{xx}^n\right) - \theta_2\left(u_x^n + \varepsilon u^n u_x^n\right) \\ - \theta_3(u_x^{n-1} + \varepsilon u^{n-1}u_x^{n-1}) \\ - \theta_4(u_x^{n-2} + \varepsilon u^{n-2}u_x^{n-2}) \\ - \theta_5(u_x^{n-3} + \varepsilon u^{n-3}u_x^{n-3}) \\ - \theta_6(u_x^{n+1} + \varepsilon u^{n+1}u_x^{n+1}) \end{aligned} \tag{9}$$

from which the four-step explicit Adams–Bashforth–Moulton method is derived and the value $(u^{n+1})^*$ can be computed as the predicted value for u^{n+1}

$$\begin{aligned} (u^{n+1})^* - \mu(u_{xx}^{n+1})^* = \theta_1\left(u^n - \mu u_{xx}^n\right) \\ -\theta_2\left(u_x^n + \varepsilon u^n u_x^n\right) \\ -\theta_3(u_x^{n-1} + \varepsilon u^{n-1}u_x^{n-1}) \\ -\theta_4(u_x^{n-2} + \varepsilon u^{n-2}u_x^{n-2}) \\ -\theta_5(u_x^{n-3} + \varepsilon u^{n-3}u_x^{n-3}), \end{aligned} \tag{10}$$

and the corrector formula is defined to be Adams–Moulton formula having the form

$$\begin{aligned} u^{n+1} - \mu u_{xx}^{n+1} = \theta_1\left(u^n - \mu u_{xx}^n\right) - \theta_2\left(u_x^n + \varepsilon u^n u_x^n\right) \\ - \theta_3(u_x^{n-1} + \varepsilon u^{n-1}u_x^{n-1}) \\ - \theta_4(u_x^{n-2} + \varepsilon u^{n-2}u_x^{n-2}) \\ - \theta_5(u_x^{n-3} + \varepsilon u^{n-3}u_x^{n-3}) \\ - \theta_6((u_x^{n+1})^* + \varepsilon(u^{n+1})^*(u_x^{n+1})^*). \end{aligned} \tag{11}$$

The semidiscrete predictor–corrector form of the RLWE is fully discretised by replacing U and its space derivatives U_x, U_{xx} with approximate function (7) and collocating at the grid points yields a recurrence relationship among successive unknown element parameters δ_i^{n+1}, δ_i^n, δ_i^{n-1}, δ_i^{n-2} and δ_i^{n-3}, $i = m-2, m-1, m, m+1$ at successive time levels $k\Delta t$, $k = n-3, \ldots, n+1$:

$$\begin{aligned} & a_1(\delta_{m-2}^{n+1})^* + a_2(\delta_{m-1}^{n+1})^* + a_3(\delta_m^{n+1})^* + a_4(\delta_{m+1}^{n+1})^* \\ & -\mu(c_1(\delta_{m-2}^{n+1})^* + c_2(\delta_{m-1}^{n+1})^* \\ & +c_3(\delta_m^{n+1})^* + c_4(\delta_{m+1}^{n+1})^*) \\ & = \theta_1(a_1\delta_{m-2}^n + a_2\delta_{m-1}^n + a_3\delta_m^n + a_4\delta_{m+1}^n \\ & -\mu(c_1\delta_{m-2}^n + c_2\delta_{m-1}^n + c_3\delta_m^n + c_4\delta_{m+1}^n)) \\ & -\theta_2 s_1(b_1\delta_{m-2}^n + b_2\delta_{m-1}^n + b_3\delta_m^n \\ & +b_4\delta_{m+1}^n) - \theta_3 s_2(b_1\delta_{m-2}^{n-1} \\ & +b_2\delta_{m-1}^{n-1} + b_3\delta_m^{n-1} + b_4\delta_{m+1}^{n-1}) \\ & -\theta_4 s_3(b_1\delta_{m-2}^{n-2} + b_2\delta_{m-1}^{n-2} + b_3\delta_m^{n-2} + b_4\delta_{m+1}^{n-2}) \\ & -\theta_5 s_4(b_1\delta_{m-2}^{n-3} + b_2\delta_{m-1}^{n-3} + b_3\delta_m^{n-3} + b_4\delta_{m+1}^{n-3}), \end{aligned} \tag{12}$$

$$\begin{aligned} & a_1\delta_{m-2}^{n+1} + a_2\delta_{m-1}^{n+1} + a_3\delta_m^{n+1} + a_4\delta_{m+1}^{n+1} - \mu(c_1\delta_{m-2}^{n+1} \\ & +c_2\delta_{m-1}^{n+1} + c_3\delta_m^{n+1} + c_4\delta_{m+1}^{n+1}) \\ & = \theta_1(a_1\delta_{m-2}^n + a_2\delta_{m-1}^n + a_3\delta_m^n + a_4\delta_{m+1}^n \\ & -\mu(c_1\delta_{m-2}^n + c_2\delta_{m-1}^n + c_3\delta_m^n + c_4\delta_{m+1}^n)) \\ & -\theta_2 s_1(b_1\delta_{m-2}^n + b_2\delta_{m-1}^n + b_3\delta_m^n + b_4\delta_{m+1}^n) \\ & -\theta_3 s_2(b_1\delta_{m-2}^{n-1} + b_2\delta_{m-1}^{n-1} + b_3\delta_m^{n-1} + b_4\delta_{m+1}^{n-1}) \\ & -\theta_4 s_3(b_1\delta_{m-2}^{n-2} + b_2\delta_{m-1}^{n-2} + b_3\delta_m^{n-2} + b_4\delta_{m+1}^{n-2}) \\ & -\theta_5 s_4(b_1\delta_{m-2}^{n-3} + b_2\delta_{m-1}^{n-3} + b_3\delta_m^{n-3} + b_4\delta_{m+1}^{n-3}) \\ & -\theta_6 s_5(b_1(\delta_{m-2}^{n+1})^* + b_2(\delta_{m-1}^{n+1})^* \\ & +b_3(\delta_m^{n+1})^* + b_4(\delta_{m+1}^{n+1})^*), \end{aligned} \tag{13}$$

where

$$\begin{aligned} s_1 &= 1 + \varepsilon(a_1\delta_{m-2}^n + a_2\delta_{m-1}^n + a_3\delta_m^n + a_4\delta_{m+1}^n), \\ s_2 &= 1 + \varepsilon(a_1\delta_{m-2}^{n-1} + a_2\delta_{m-1}^{n-1} + a_3\delta_m^{n-1} + a_4\delta_{m+1}^{n-1}), \\ s_3 &= 1 + \varepsilon(a_1\delta_{m-2}^{n-2} + a_2\delta_{m-1}^{n-2} + a_3\delta_m^{n-2} + a_4\delta_{m+1}^{n-2}), \\ s_4 &= 1 + \varepsilon(a_1\delta_{m-2}^{n-3} + a_2\delta_{m-1}^{n-3} + a_3\delta_m^{n-3} + a_4\delta_{m+1}^{n-3}), \\ s_5 &= 1 + \varepsilon(a_1(\delta_{m-2}^{n+1})^* + a_2(\delta_{m-1}^{n+1})^* + a_3(\delta_m^{n+1})^* \\ &\quad +a_4(\delta_{m+1}^{n+1})^*). \end{aligned} \tag{14}$$

Here, the following procedure is applied:

(1) $(\delta_m^{n+1})^*$ is the predicted value of δ_m^{n+1} computed by using formula (12) known as explicit Adams–Bashforth method,

(2) then in the corrector of the implicit Adams–Moulton method (13), $(\delta_m^{n+1})^*$ is placed to calculate (δ_m^{n+1}),

Table 1. Error norms L_2, L_∞ and invariants at time $t = 20$ for $c = 0.1$.

h	Δt	L_2	L_∞	I_1	I_2	I_3
0.5	0.10	5.168×10^{-6}	2.036×10^{-6}	3.9799497	0.8104625	2.5790075
	0.05	5.794×10^{-6}	2.344×10^{-6}	3.9799497	0.8104625	2.5790074
	0.02	5.884×10^{-6}	2.387×10^{-6}	3.9799497	0.8104625	2.5790074
	0.01	5.890×10^{-6}	2.389×10^{-6}	3.9799497	0.8104625	2.5790074
0.2	0.10	1.039×10^{-6}	3.641×10^{-7}	3.9799497	0.8104625	2.5790075
	0.05	1.149×10^{-7}	4.308×10^{-8}	3.9799497	0.8104625	2.5790074
	0.02	1.427×10^{-7}	5.772×10^{-8}	3.9799497	0.8104625	2.5790074
	0.01	1.479×10^{-7}	6.019×10^{-8}	3.9799497	0.8104625	2.5790074
0.1	0.10	1.130×10^{-6}	4.036×10^{-7}	3.9799497	0.8104625	2.5790075
	0.05	1.355×10^{-7}	4.780×10^{-8}	3.9799497	0.8104625	2.5790074
	0.02	7.301×10^{-9}	2.748×10^{-9}	3.9799497	0.8104625	2.5790074
	0.01	8.546×10^{-9}	3.432×10^{-9}	3.9799497	0.8104625	2.5790074
0.05	0.10	1.136×10^{-6}	4.063×10^{-7}	3.9799497	0.8104625	2.5790075
	0.05	1.413×10^{-7}	5.037×10^{-8}	3.9799497	0.8104625	2.5790074
	0.02	8.681×10^{-9}	3.059×10^{-9}	3.9799497	0.8104625	2.5790074
	0.01	8.51×10^{-10}	3.05×10^{-10}	3.9799497	0.8104625	2.5790074
0.02	0.10	1.137×10^{-6}	4.065×10^{-7}	3.9799497	0.8104625	2.5790075
	0.05	1.417×10^{-7}	5.054×10^{-8}	3.9799497	0.8104625	2.5790074
	0.02	9.058×10^{-9}	3.223×10^{-9}	3.9799497	0.8104625	2.5790074
	0.01	1.127×10^{-9}	3.968×10^{-10}	3.9799497	0.8104625	2.5790074

Table 2. Error norms L_2, L_∞ and invariants at time $t = 20$ for $c = 0.03$.

h	Δt	L_2	L_∞	I_1	I_2	I_3
0.10	0.10	7.144×10^{-8}	2.263×10^{-8}	2.1094075	0.1273017	0.3888060
	0.05	8.778×10^{-9}	2.769×10^{-9}	2.1094075	0.1273017	0.3888060
	0.02	4.233×10^{-10}	1.280×10^{-10}	2.1094075	0.1273017	0.3888060
	0.01	1.715×10^{-11}	5.981×10^{-11}	2.1094075	0.1273017	0.3888060

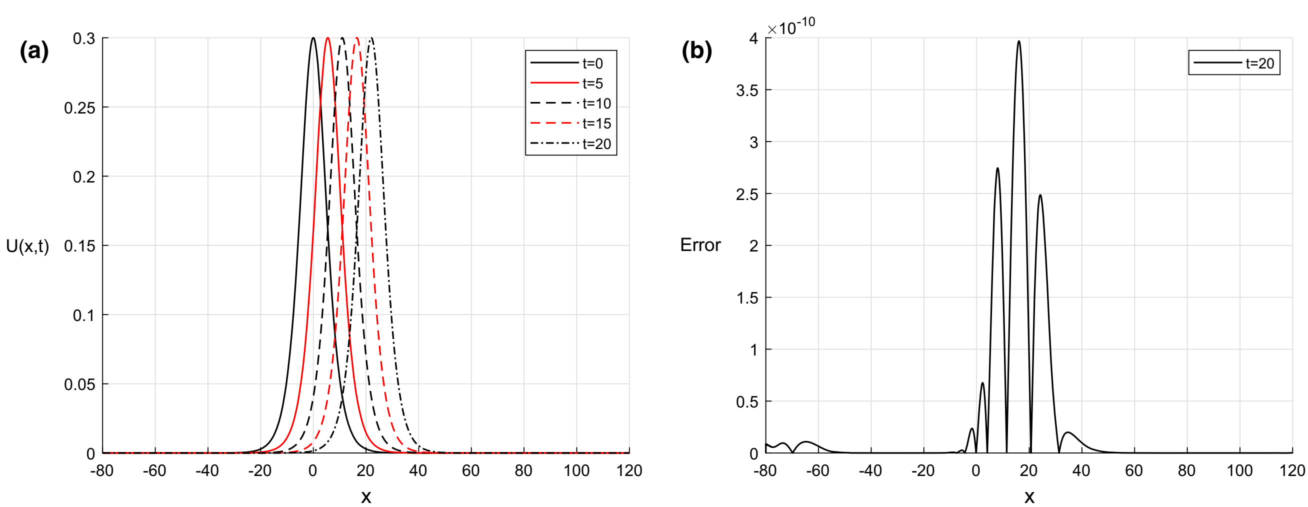

Figure 1. (**a**) $U(x, t)$ for various values of t and (**b**) absolute error.

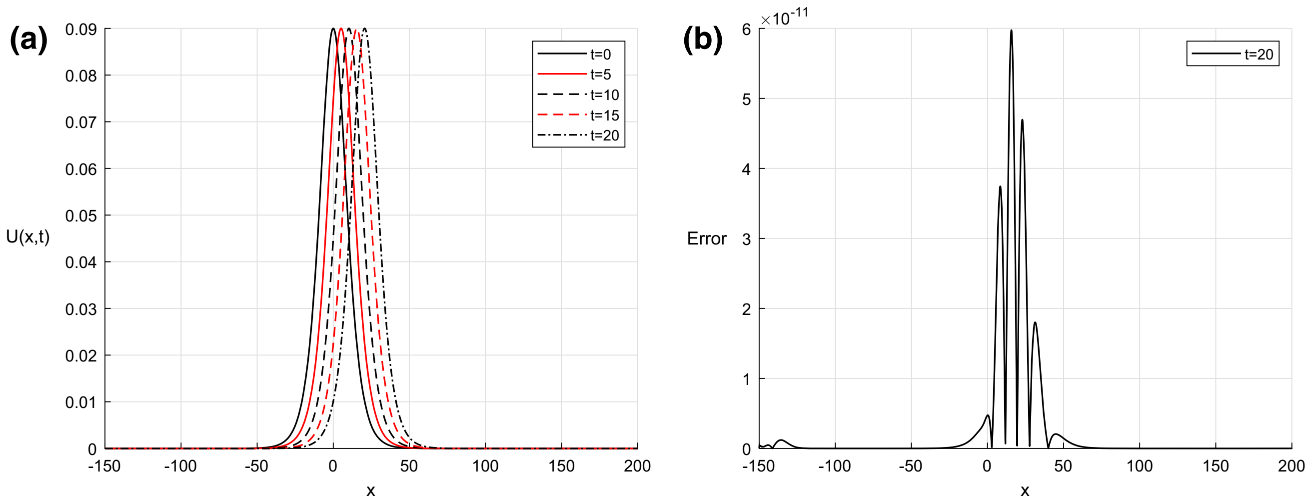

Figure 2. (**a**) $U(x, t)$ for various values of t and (**b**) absolute error.

Table 3. Maximum error norm L_∞ at time $t = 20$ for $c = 0.1$.

	L_∞	h	Δt	
Present	4.04×10^{-7}	0.1	0.1	$-80 \leq x \leq 120$
Present	3.05×10^{-10}	0.05	0.01	$-80 \leq x \leq 120$
[15]	7.28×10^{-8}	0.1	0.1	$-40 \leq x \leq 60$
[18]	6.78×10^{-11}	$N = 512$	0.1	$-80 \leq x \leq 100$
[19]	7.71×10^{-6}	$N = 5000$	0.0001	$-50 \leq x \leq 70$
[22]AM1	2.03×10^{-9}	0.01	0.01	$-80 \leq x \leq 100$
[23]M2	2.22×10^{-11}	0.02	0.02	$-80 \leq x \leq 120$
[24]	2.34×10^{-12}	$N = 257$	0.01	$-100 \leq x \leq 100$
Exact				

(3) we can repeat predictor–corrector again and again until the difference of the last two sequential approximations $(\delta_m^{n+1} - \delta_m^n)$ remains within the given tolerance.

Thus, we obtain a system of $N + 1$ algebraic equations in $N + 4$ unknowns. Before starting to solve the system of equations of $(N + 1) \times (N + 4)$ dimension, three boundary conditions are imposed to the system. Boundary conditions provide three equations,

$$U(a, t^n) = a_1\delta_{-2}^n + a_2\delta_{-1}^n + a_3\delta_0^n + a_4\delta_1^n = \sigma_1, \tag{15}$$

$$U_x(a, t^n) = b_1\delta_{-2}^n + b_2\delta_{-1}^n + b_3\delta_0^n + b_4\delta_1^n = 0, \tag{16}$$

$$U(b, t^n) = a_1\delta_{N-2}^n + a_2\delta_{N-1}^n + a_3\delta_N^n + a_4\delta_{N+1}^n = \sigma_2 \tag{17}$$

from which we get parameters $\delta_{-2}^n, \delta_{-1}^n$ and δ_{N+1}^n so that the elimination of these parameters from systems (12) and (13) gives us solvable system equations of $(N + 1) \times (N + 1)$ dimension. But, initial parameters $\delta^0 = (\delta_{-2}^0, \delta_{-1}^0, \ldots, \delta_{N+1}^0)$ must be computed to commence iteration defined by eqs (12) and (13) using the initial and boundary conditions:

$$\begin{aligned}
U(a, 0) &= a_1\delta_{-2}^0 + a_2\delta_{-1}^0 + a_3\delta_0^0 + a_4\delta_1^0 = \sigma_1, \\
U'(a, 0) &= b_1\delta_{-2}^0 + b_2\delta_{-1}^0 + b_3\delta_0^0 + b_4\delta_1^0 = 0, \\
U(x_m, 0) &= a_1\delta_{m-2}^0 + a_2\delta_{m-1}^0 + a_3\delta_m^0 + a_4\delta_{m+1}^0 \\
&= f(x_m), \quad m = 0, \ldots, N \\
U(b, 0) &= a_1\delta_{N-2}^0 + a_2\delta_{N-1}^0 + a_3\delta_N^0 + a_4\delta_{N+1}^0 = \sigma_2.
\end{aligned} \tag{18}$$

In addition, the predictor equations need three initial vectors of parameters $\delta^1, \delta^2, \delta^3$ to calculate parameters δ^n, $n \geq 4$. Predictor–corrector couple of the Euler-modified Euler method, two-step Adams–Bashforth–Moulton formula, three-step Adams–Bashforth–Moulton formula are implemented by a suitable selection parameter sets for time discretisation respectively to calculate vector of parameter $\delta^1, \delta^2, \delta^3$.

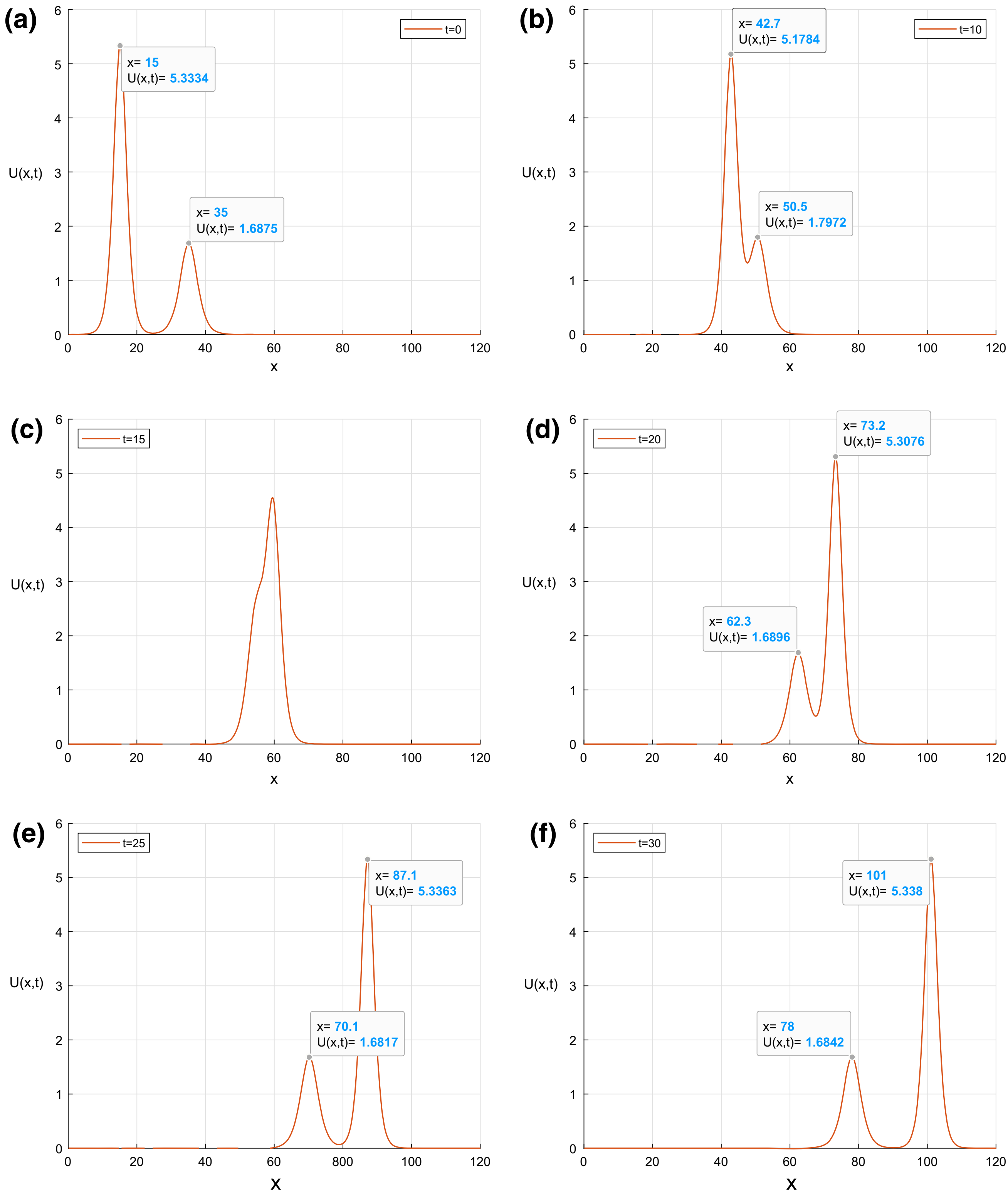

Figure 3. Interaction profiles for various values of t for two solitary waves.

Table 4. Invariants for various values of t for two positive solitary waves.

t	I_1^t	I_2^t	I_3^t
0	37.9165093	120.5232369	744.0812089
5	37.9172711	120.5588354	744.3776286
10	37.9179179	120.5987068	744.7071254
15	37.9185896	120.6142240	744.8357999
20	37.9192320	120.6404150	745.0597257
25	37.9198970	120.6813126	745.4008192
30	37.9205452	120.7229392	745.7477997

3. Numerical results

Accuracy is evaluated using

$$L_\infty = \max_{0\le m\le N} |u_m - U_m|$$

and

$$L_2 = \left[h \sum_{m=0}^{N} |u_m - U_m|^2 \right]^{1/2},$$

where U_m is the approximation solution and u_m is the analytical solution at the grid point x_m. A mass I_1, momentum I_2 and energy I_3 of the RLWE must stay constant at all times, and these constants [5],

$$I_1 = \int_a^b U \,\mathrm{d}x, \tag{19}$$

$$I_2 = \int_a^b (U^2 + \mu(U_x)^2)\,\mathrm{d}x, \tag{20}$$

$$I_3 = \int_a^b (U^3 + 3U^2)\,\mathrm{d}x \tag{21}$$

are recorded to indicate the validity of the numerical simulation of the RLWE.

3.1 *Simulations of the single solitary wave*

Analytical solution of the RLWE is

$$u(x,t) = 3c\,\mathrm{sech}^2(k[x - x_c - vt]). \tag{22}$$

This solution is a model for the propagation of a solitary wave of amplitude $3c$, velocity $v = 1 + \varepsilon c$ and

$$k = \sqrt{\frac{\varepsilon c}{\mu(1+\varepsilon c)}}.$$

Substituting solution (22) in the integrals I_1, I_2, I_3 yields conserved quantities:

$$I_1 = \frac{6c}{k}, \quad I_2 = \frac{12c^2}{k} + \frac{48kc^2}{5},$$
$$I_3 = \frac{36c^2}{k}\left(1 + \frac{4c}{5}\right). \tag{23}$$

Parameters $\varepsilon = \mu = 1$, $x_c = 0$, problem interval $[-80, 120]$ for $c = 0.1$ and $[-150, 200]$ for $c = 0.03$ are taken for the test problem. The problem domain is kept larger to decrease the effect of the boundaries $u(x_0, t) = u(x_N, t) = 0$. Initial solitary wave is determined from the exact solution eq. (22) when $t = 0$. The first set of parameters, $c = 0.1$, time increments $\Delta t = 0.1, 0.05, 0.02, 0.01$ and space increments $h = 0.5, 0.2, 0.1, 0.05, 0.02$ are used to run the program. Analytical invariants are computed from eqs (23) as $I_1 = 3.979949$, $I_2 = 0.810462$ and $I_3 = 2.579007$ for the solitary wave of amplitude $3c = 0.3$. Solitary wave was observed to be propagating to the right within the space interval until $t = 20$. Error norms L_∞, L_2 and invariants are recorded in table 1, from which the algorithm produces lower accuracy by using smaller time-space increments. Conservation laws remain almost constant for all the runs. Further run with the second set of parameters $c = 0.03$, space increment $h = 0.1$ and variable time increments $\Delta t = 0.1, 0.05, 0.02, 0.01$ are carried out and, norms and invariants are illustrated in table 2. Higher accuracy has been achieved as about 10^{-11} by using smaller amplitude in the algorithm. Obtaining higher accuracy depends on taking larger spatial domain so that influence of the boundary conditions on the error can be diminished. Solution profiles for different values of t and error variation at time $t = 20$ are depicted for $h = 0.02$ and $\Delta t = 0.01$ in figure 1 on $[-80, 120]$ for $c = 0.1$. On the other hand, $\Delta t = 0.01$ and $h = 0.1$ are used to obtain numerical graphics for $c = 0.03$ on the interval $[-150, 200]$ in figure 2. Comparison of the presented findings with other existing schemes is shown in table 3. The Hermite-distributed approximation function method provides the least error of 10^{-11} over the problem domain $[-100, 100]$ [24]. Quartic trigonometric B-spline Galerkin method gives the second least error of about 10^{-11} [23]. Another method, which is made up of the discrete Fourier transform in combination with the Runge–Kutta integrating factors to discretise the RLWE fully, achieves an accuracy of the order of 10^{-11} [18]. The presented method achieves an error of 10^{-10}. The quartic B-spline collocation algorithm together with the Adams–Bashforth–Moulton method is more economical and easier than the other methods given in table 3.

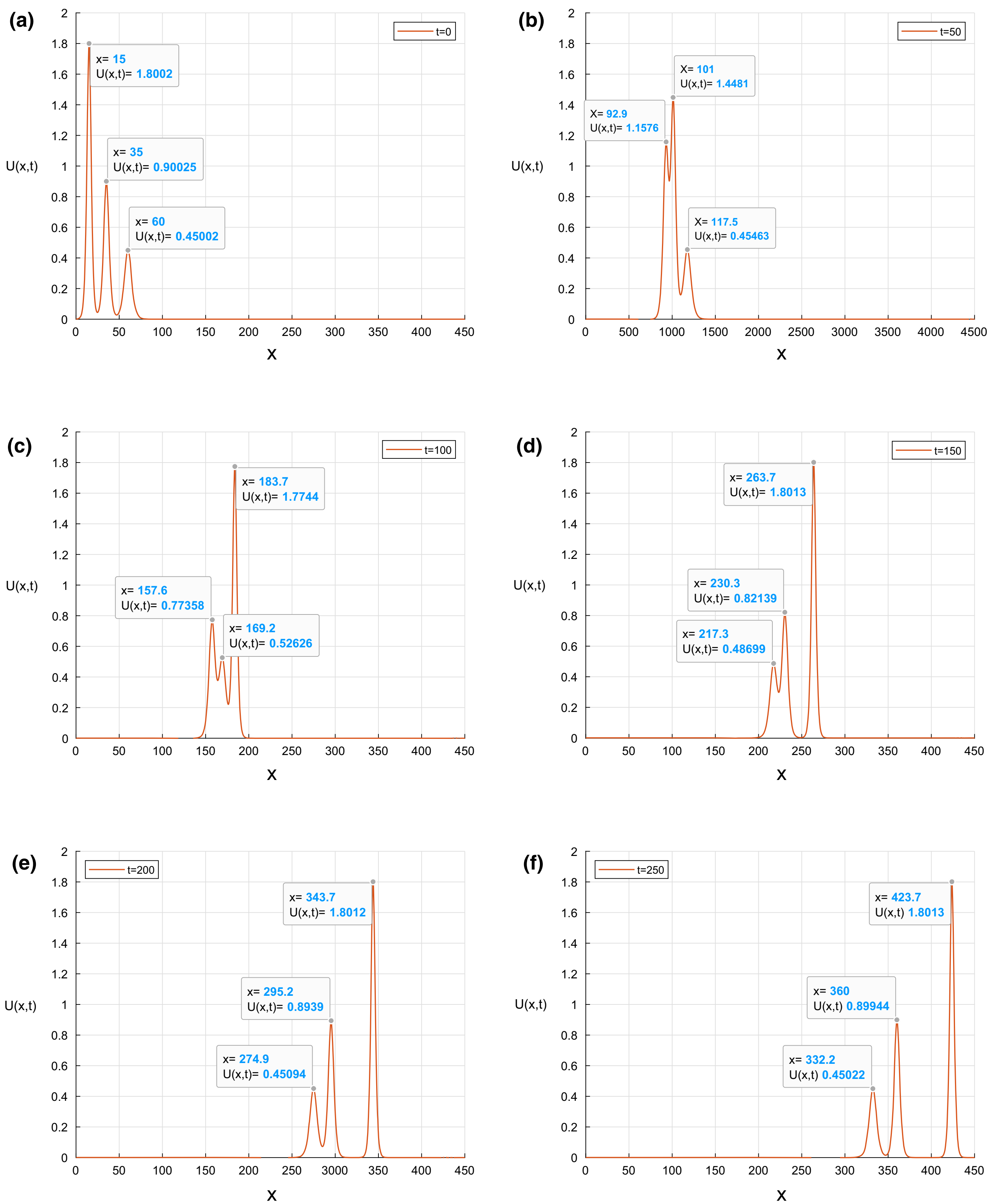

Figure 4. Interaction profiles for various values of t for three solitary waves.

Table 5. Invariants for various values of t for three positive solitary waves.

t	I_1^t	I_2^t	I_3^t
0	24.2343537	21.4296071	84.5047184
50	24.2717170	21.4295194	84.5046596
100	24.3086416	21.4295132	84.5045813
150	24.3455601	21.4294553	84.5042545
200	24.3824753	21.4293951	84.5039179
250	24.4193913	21.4293347	84.5035803

3.2 *Interaction of two solitary waves*

Interaction of two solitary waves is carried out using the initial condition

$$U(x,0) = U_1 + U_2,$$
$$U_j = 3A_j \operatorname{sech}^2(k_j(x - \tilde{x}_j)),$$
$$A_j = \frac{4k_j^2}{1 - 4k_j^2}, \quad j = 1, 2, \tag{24}$$

where parameters $A_1 = 1.7778$, $k_1 = 0.4$, $\tilde{x}_1 = 15$, $A_2 = 0.56199$, $k_2 = 0.3$, $\tilde{x}_2 = 35$ are selected. Thus, initial solitary waves are the sum of two solitary waves of amplitudes 5.33338, 1.68598. The larger wave is situated to the left of the smaller wave so that interaction can be ensured. Peak of the large wave is positioned at $x_1 = 15$ and smaller wave is at $x_2 = 35$. Program run is performed in the space interval [0,120] with time step $\Delta t = 0.1$ and space step $h = 0.1$ until $t = 30$. Interaction process is visualised in figure 3. We observe that the larger wave interacts with and emerges ahead of the smaller wave with the shape and velocity of each wave kept constant. Analytical invariants can be determined as:

$$I_1 = \frac{6c_1}{k_1} + \frac{6c_2}{k_2},$$
$$I_2 = \frac{12c_1^2}{k_1} + \frac{48k_1c_1^2}{5} + \frac{12c_2^2}{k_2} + \frac{48k_2c_2^2}{5},$$
$$I_3 = \frac{36c_1^2}{k_1}\left(1 + \frac{4c_1}{5}\right) + \frac{36c_2^2}{k_2}\left(1 + \frac{4c_2}{5}\right)$$

from which invariants are found to be $I_1 = 37.91660331$, $I_2 = 120.71659130$, $I_3 = 745.69164620$ with the given parameters. Numerical invariants are recorded at various times in table 4 and remain fairly constant until time $t = 30$.

3.3 *Interaction of three solitary waves*

The sum of the well-separated three waves of various amplitudes is used as an initial condition:

$$U(x,0) = U_1 + U_2 + U_3,$$
$$U_j = 3A_j \operatorname{sech}^2(k_j(x - \tilde{x}_j)),$$
$$k_j = \sqrt{\frac{A_j}{4\mu(1 + A_j)}}, \quad j = 1, 2, 3. \tag{25}$$

The interaction of three solitary waves is substantiated by positioning the solitary wave of the largest amplitude to the left of the other two waves and solitary wave of the smallest amplitude to the right of the other two waves so that simultaneous interaction of triple solitary waves can be observed while three solitary waves move to the right. Analytical values of invariants for the case of three solitary waves can be computed using the initial condition

$$I_1 = \frac{6A_1}{k_1} + \frac{6A_2}{k_2} + \frac{6A_3}{k_3},$$
$$I_2 = \frac{12A_1^2}{k_1} + \frac{48k_1A_1^2\mu}{5} + \frac{12A_2^2}{k_2} + \frac{48k_2A_2^2\mu}{5} + \frac{12A_3^2}{k_3} + \frac{48k_3A_3^2\mu}{5},$$
$$I_3 = \frac{36A_1^2}{k_1}\left(1 + \frac{4A_1}{5}\right) + \frac{36A_2^2}{k_2}\left(1 + \frac{4A_2}{5}\right) + \frac{36A_3^2}{k_3}\left(1 + \frac{4A_3}{5}\right). \tag{26}$$

Computation is done over the region $0 \le x \le 450$, subdivided into 4500 subintervals from $t = 0$ to $t = 250$ with time step $\Delta t = 0.1$. How the larger waves overtake and pass through the smaller waves while keeping their original shapes after the interaction is shown. This is seen for various values of t in figure 4. Changes of magnitudes from larger amplitudes to smaller from the initial amplitudes are computed as 0.0011, 0.00081, 0.0002 at $t = 250$ respectively. With the given parameters, invariants are found as $I_1 = 24.2343537$, $I_2 = 21.4296071$, $I_3 = 84.5047184$. In table 5, invariants are documented for three positive solitary waves.

3.4 *The Maxwellian initial condition*

Development of more solitary waves is examined using the Maxwellian initial condition

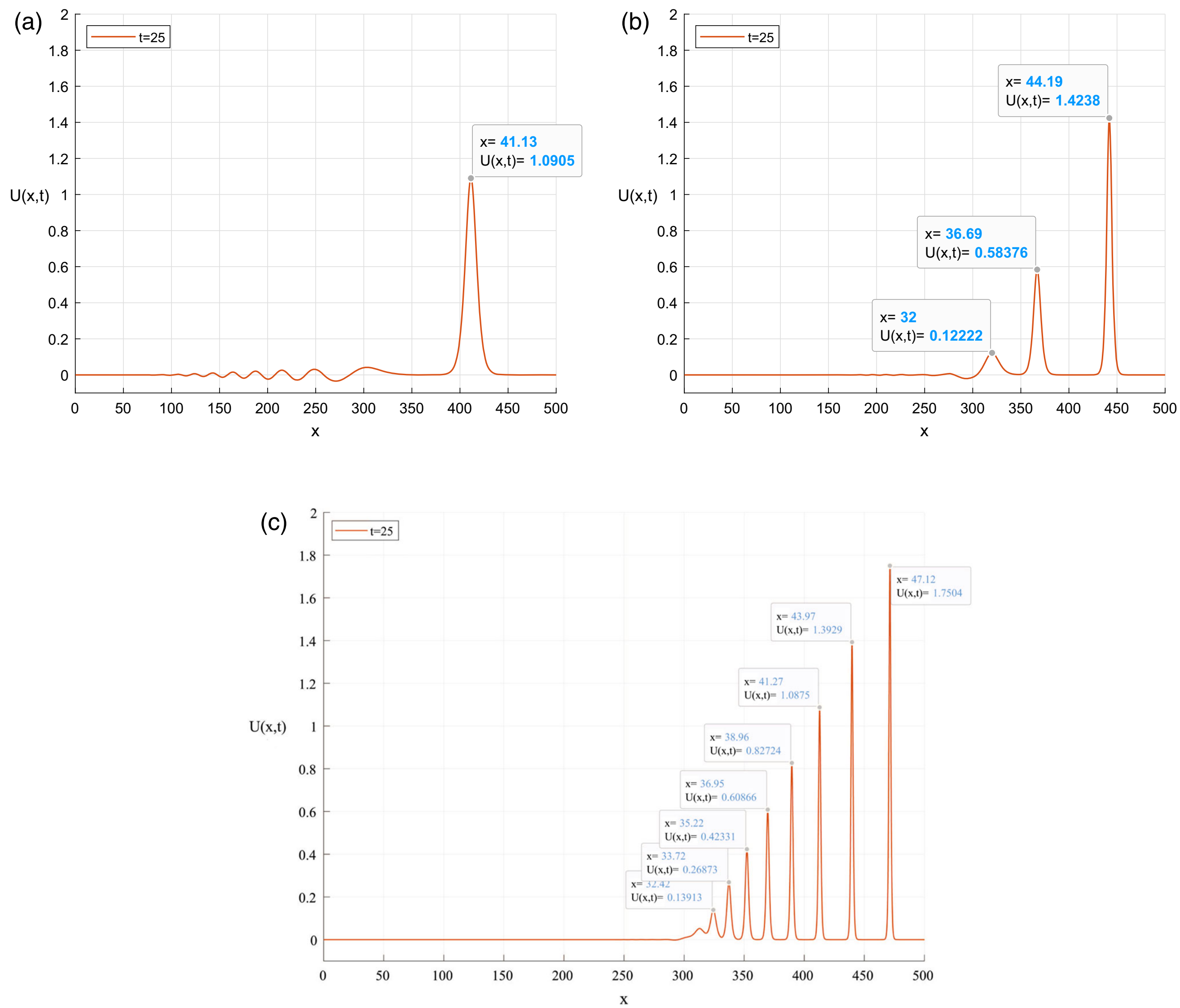

Figure 5. The behaviour of the numerical solution for different values of μ.

$$U(x,0) = \mathrm{e}^{-(x-7)^2}. \tag{27}$$

Maxwellian initial condition breaks up into more solitary waves depending on the dispersion term μ. Computation is implemented within the interval [0, 500] with space step $h = 0.01$, time step $\Delta t = 0.01$ and dispersive terms $\mu = 0.04, 0.01, 0.001$. Numerical solutions with $\mu = 0.04$ result in a single solitary wave with an oscillating tail on the left (figure 5a). Wave peak is positioned at $x = 41.13$ and the amplitude is recorded as $U = 1.0905$ at $t = 25$. The value $\mu = 0.01$ causes three solitary waves having amplitudes 0.122, 0.584, 1.424 positioned at 32, 36.69 and 44.19. Profiles of the three solitary waves are demonstrated in figure 5b. Finally when $\mu = 0.001$, nine waves evolve in different amplitudes occurring in the order of small to large (figure 5c). After the formation of the waves, waves propagate to the right with their forms nearly constant. Invariants are recorded in tables 6–8 and all remain constant substantially.

Table 6. Invariants for $\mu = 0.04$.

t	I_1^t	I_2^t	I_3^t
0	1.7724539	1.3034467	4.7832691
5	1.7724539	1.3034468	4.7832693
10	1.7724539	1.3034468	4.7832697
15	1.7724539	1.3034469	4.7832700
20	1.7724539	1.3034470	4.7832703
25	1.7724539	1.3034471	4.7832706

Table 7. Invariants for $\mu = 0.01$.

t	I_1^t	I_2^t	I_3^t
0	1.7724539	1.2658473	4.7832691
5	1.7724539	1.2658529	4.7832939
10	1.7724539	1.2658640	4.7833431
15	1.7724539	1.2658752	4.7833928
20	1.7724539	1.2658865	4.7834424
25	1.7724539	1.2658977	4.7834921

Table 8. Invariants for $\mu = 0.001$.

t	I_1^t	I_2^t	I_3^t
0	1.7724539	1.3034467	4.7832691
5	1.7724539	1.3034468	4.7832693
10	1.7724539	1.3034468	4.7832697
15	1.7724539	1.3034469	4.7832700
20	1.7724539	1.3034470	4.7832703
25	1.7724539	1.3034471	4.7832706

4. Conclusion

Discretisation using the Adams–Bashforth–Moulton time integrator and the quartic B-spline collocation for spatial integration has proved to give smaller error for solving RLWE. When solving two separate systems of the banded algebraic equations including $(N+1)$ equations in four unknowns given with eqs (12) and (13) using Matlab, the number of basic operations to perform is of order $O(2 \times 4^2 \times (N+1) \times M)$ where M is the iteration number for time discretisation. The advantage is that the presented algorithm is easy to implement and computationally cheaper than either the Galerkin numerical method or compact finite difference schemes and the distributed approximating functional method tabulated in table 3. The work is a contribution to the field because of the accuracy of the results acquired for the low computational cost. It also performs well by keeping the conserved laws and acquire the single solitary wave solution accurately. Interaction of solitary waves is simulated satisfactorily. The proposed numerical scheme can be advised for solving partial differential equations without hesitation.

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