Pramana – J. Phys. (2022) 96:26
https://doi.org/10.1007/s12043-021-02254-2

Traveling-wave solutions of the Klein–Gordon equations with M-fractional derivative

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MS received 9 May 2021; accepted 23 August 2021

Abstract. Based on two algorithm integrations, such as the $\exp(-\Phi(\xi))$-expansion method and the hyperbolic function method, we build dark, bright and trigonometric function solution to the Klein–Gordon equations with M-fractional derivative of order α. By adopting the travelling-wave transformation, the constraint condition between the model coefficients and the travelling-wave frequency coefficient for the existence of soliton solutions is also obtained. Moreover, miscellaneous soliton solutions obtained is depicted in 3D and 2D.

Keywords. Travelling wave solutions; M-fractional derivative; solitons.

PACS Nos 04.20.Jb; 05.45.Yv; 94.05.Fg

1. Introduction

Nowadays, fractional calculus have advanced in analytical solutions of nonlinear partial differential equation. Lots of attention has been given for investigating exact travelling-wave solutions of fractional models which yield fractional differential equations. Fractional calculus can provide us mathematical formulas to transform the nonlinear partial differential equation (PDEs) to the nonlinear ordinary equation to handle them by some tractable integration tools. Also, it is very important to use fractional derivatives which can be used for the describing memory and hereditary properties [1]. Moreover, conformable fractional versions of some nonlinear system were investigated [2–4]. Thus, investigation of optical solitons with fractional time evolution become very important due to their applications in secure communication system of analog and digital signals, and to carry out high speed data transmission over several thousands of kilometres [5–12]. Recently, some effective integration methods have been used to construct exact solutions for PDEs, such as semi-inverse variational principe [13], the simplest equation approach [14], the first integral method [15], ansatz scheme [16], the generalised tanh method [17] and so on. That is why various soliton solutions of physical systems with different nonlinearities were reported in literature [18–20].

We aim to investigate solitary wave solutions to the Klein–Gordon equations with M-fractional derivative [21].

$$D_{M,t}^{2\alpha,\beta}u(x,t) - \lambda D_{M,x}^{2\alpha,\beta}u(x,t) + \mu u(x,t) + \sigma u^2(x,t) = 0, \quad t > 0, \quad 0 < \alpha < 1. \tag{1}$$

Section 2 is dedicated to M-fractional preliminaries. In §3 we apply two integration techniques to retrieve travelling-wave solutions to (1) and the last section concludes the work.

2. Truncated M-fractional derivative-type preliminaries

During the last decade, several definitions of fractional derivatives have been used in literature [22, 23]. Atangana–Baleanu derivative in Caputo direction, Atangana–Baleanu fractional derivative in Riemann–Liouville sense, the new truncated M-fractional derivative of Sousa and Oliveira [24], are just a few definitions.

Published online: 13 January 2022

This section will highlight some basic definitions and theorem of M-derivative.

DEFINITION 1

Let g: $[0, \infty) \to \mathbb{R}$.

$$D_M^{\alpha,\beta} g(t) = \lim_{\varepsilon \to +\infty} \frac{g(t\mathbb{E}_\beta(\varepsilon t^{1-\alpha})) - g(t)}{\varepsilon}. \quad \forall t > 0, \quad \beta > 0. \tag{2}$$

Here $\mathbb{E}_\beta(\cdot)$ is the Mittag–Leffler function of one parameter [25].

Theorem 1. *Let* $0 < \alpha < 1, \beta > 0, a, b \in \mathbb{R}$ *and* g, f α*-differentiable at a point* $t > 0$. *Hence,*

1. $D_M^{\alpha,\beta}[(ag+bf)(t)] = aD_M^{\alpha,\beta}[g(t)]+bD_M^{\alpha,\beta}[f(t)]$.
2. $D_M^{\alpha,\beta}[(g\cdot f)(t)]=g(t)D_M^{\alpha,\beta}[f(t)]+f(t)D_M^{\alpha,\beta}[g(t)]$.
3. $D_M^{\alpha,\beta}[\frac{g}{f}(t)] = \frac{f(t)D_M^{\alpha,\beta}[g(t)]-g(t)D_M^{\alpha,\beta}f(t)}{[f(t)]^2}$.
4. $D_M^{\alpha,\beta}[c] = 0$.
5. *If g is differentiable, then*

$$D_M^{\alpha,\beta}[g(t)] = \frac{t^{1-\alpha}}{\Gamma(\beta+1)} \frac{\mathrm{d}g(t)}{\mathrm{d}t}.$$

3. Analytical solutions of the Klein–Gordon equations with M-fractional derivative

We shall apply two integration algorithms to investigate analytical solutions to (1). To do so, the first step is to adopt the travelling-wave transformation to derive the nonlinear ordinary differential equations.

$$u(x,t) = g(\xi)\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right],$$

$$\xi = \frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) \tag{3}$$

Substitute (3) in (1) to get

$$\left(\frac{\kappa^2\lambda}{\omega^2} - 1\right)\lambda g'' - (\omega^2 - \lambda\kappa^2 - \mu)g + \sigma g^2 = 0 \tag{4}$$

and

$$v = -\frac{\kappa\lambda}{\omega}, \tag{5}$$

where $g(\xi) = g$.

3.1 *The* $\exp(-\Phi(\xi))$*-expansion method*

Assume the solution of (4) as follows [26]:

$$\phi(\xi) = a_0 + \sum_{i=1}^{N} a_i \exp(-\Phi(\xi))^i, \tag{6}$$

where $\Phi(\xi)$ satisfies the following ODE:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + r\exp(\Phi(\xi)) + q \tag{7}$$

and $a_i (i = 1, 2, 3, ...)$, r and q are reals constant to be determined. By using the homogeneous balance principle between g'' and g^2, $N = 2$ is obtained.

Therefore, (6) gives

$$\phi(\xi) = a_0 + a_1\exp(-\Phi(\xi)) + a_2\exp(-2\Phi(\xi)). \tag{8}$$

By substituting (8) and (7) into (6), a set of algebraic equations is obtained. After solving the set of algebraic equations using MAPLE, we get the following results:

Set 1:

$$a_0 = a_2 r, \quad a_1 = a_2 q, \quad a_2 = a_2,$$
$$\mu = -\frac{(\lambda\kappa^2 - \omega^2)(-4r\lambda + q^2\lambda - \omega^2)}{\omega^2},$$
$$\sigma = -\frac{6\lambda(\lambda\kappa^2 - \omega^2)}{a_2\omega^2}.$$

Set 2:

$$a_0 = \frac{1}{6}a_2(q^2 + 2r), \quad a_1 = a_2 q, \quad a_2 = a_2,$$
$$\mu = \frac{(\lambda\kappa^2 + \omega^2)(-4r\lambda + q^2\lambda - \omega^2)}{\omega^2},$$
$$\sigma = -\frac{6\lambda(\lambda\kappa^2 - \omega^2)}{a_2\omega^2}.$$

Using the five solutions of the auxiliary ODE (7) as in [26], the following solutions can be obtained from Set 1:

Case 1: If $-q^2+4r>0$ and $r\neq 0$,

$$u_{1,1}(x,t)=a_2r\left[1+\frac{2q}{-\sqrt{-4\,r+q^2}\tanh\left(\frac{1}{2}\sqrt{-8\,r+2\,q^2}\,(\xi+\xi_0)\right)-q}+\frac{4r}{\left(-\sqrt{-4\,r+q^2}\tanh\left(\frac{1}{2}\sqrt{-8r+2q^2}\,(\xi+\xi_0)\right)-q\right)^2}\right]\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{9}$$

Case 2: If $-q^2+4r<0$ and $r\neq 0$,

$$u_{1,2}(x,t)=a_2r\left[1+\frac{2q}{\sqrt{4r-q^2}\tan\left(\frac{1}{2}\sqrt{-8\,r+2\,q^2}\,(\xi+\xi_0)\right)-q}+\frac{4r}{\left(\sqrt{4r-q^2}\tan\left(\frac{1}{2}\sqrt{-8r+2q^2}\,(\xi+\xi_0)\right)-q\right)^2}\right]\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{10}$$

Case 3: If $-q^2+4r>0$ and $r=0$,

$$u_{1,3}(x,t)=a_2q^2\left[\frac{r}{q^2}+\frac{1}{\cosh\left(q\,(\xi+\xi_0)+\sinh\left(q\,(\xi+\xi_0)\right)-1\right)}+\frac{1}{\left(\cosh\left(q\,(\xi+\xi_0)+\sinh\left(q\,(\xi+\xi_0)\right)-1\right)\right)^2}\right]\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{11}$$

Case 4: If $-q^2+4r=0$ and $r\neq 0$ and $q=0$,

$$u_{1,4}(x,t)=a_2q^2\left[\frac{r}{q^2}+\frac{(\xi+\xi_0)}{-2\,(\xi+\xi_0)+2}+\frac{q^2\,(\xi+\xi_0)^2}{(-2\,q\,(\xi+\xi_0)+2)^2}\right]\times\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{12}$$

Case 5: If $-q^2+4r=0$ and $r=0$ and $q=0$,

$$u_{1,5}(x,t)=a_2\left[r+\frac{q}{\xi+\xi_0}+\frac{1}{(\xi+\xi_0)^2}\right]\times\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{13}$$

From Set 2, we obtain

Case 1: If $-q^2+4r>0$ and $r\neq 0$,

$$u_{1,6}(x,t)=\left[\frac{a_2\left(q^2+2r\right)}{6}+\frac{2a_2qr}{-\sqrt{-4r+q^2}\tanh\left(\frac{1}{2}\sqrt{-8r+2q^2}\,(\xi+\xi_0)\right)-q}+\frac{4a_2r^2}{\left(-\sqrt{-4r+q^2}\tanh\left(\frac{1}{2}\sqrt{-8r+2q^2}\,(\xi+\xi_0)\right)-q\right)^2}\right]\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{14}$$

Case 2: If $-q^2+4r<0$ and $r\neq 0$,

$$u_{1,7}(x,t)=a_2\left[\frac{\left(q^2+2r\right)}{6}+\frac{2qr}{\sqrt{4r-q^2}\tan\left(\frac{1}{2}\sqrt{-8r+2q^2}\,(\xi+\xi_0)\right)-q}+\frac{4r^2}{\left(\sqrt{4r-q^2}\tan\left(\frac{1}{2}\sqrt{-8r+2q^2}\,(\xi+\xi_0)\right)-q\right)^2}\right]\times\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{15}$$

Case 3: If $-q^2+4r>0$ and $r=0$,

$$u_{1,8}(x,t)=a_2\left[\frac{\left(q^2+2r\right)}{6}r+\frac{q^2}{\cosh\left(q\,(\xi+\xi_0)+\sinh\left(q\,(\xi+\xi_0)\right)-1\right)}+\frac{q^2}{\left(\cosh\left(q\,(\xi+\xi_0)+\sinh\left(q\,(\xi+\xi_0)\right)-1\right)\right)^2}\right]\times\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{16}$$

Case 4: If $-q^2+4r=0$ and $r\neq 0$ and $q=0$,

$$u_{1,9}(x,t)=a_2\left[\frac{\left(q^2+2r\right)}{6}r+\frac{q^2\,(\xi+\xi_0)}{-2\,(\xi+\xi_0)+2}+\frac{q^4\,(\xi+\xi_0)^2}{(-2q\,(\xi+\xi_0)+2)^2}\right]\times\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{17}$$

Case 5: If $-q^2+4r=0$ and $r=0$ and $q=0$,

$$u_{1,10}(x,t)=a_2\left[\frac{\left(q^2+2r\right)}{6}r+\frac{q}{\xi+\xi_0}+\frac{1}{(\xi+\xi_0)^2}\right]\times\exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{18}$$

3.2 *The extended hyperbolic function method*

In this section, we apply the extended hyperbolic function method to ODE (4) [27]

$$g(\xi)=\sum_{i=0}^{N}a_i(U(\xi))^i, \tag{19}$$

where $a_i(i=\pm1,\pm2,...)$ are constants to be determined and $U(\xi)$ satisfies the following ODE:

$$\frac{\partial U(\xi)}{\partial \xi} = U\sqrt{b + cU(\xi)^2}, \tag{20}$$

and a, b are real. By using the balance homogeneous principle to (4) we get $N = 2$. Consequently,

$$g(\xi) = a_0 + a_1 U(\xi) + a_2 U(\xi)^2. \tag{21}$$

Inserting (21) and (20) into (4) gives a set of algebraic equations in terms of $(U(\xi))^i$. After solving the set of nonlinear system of equations with the aid of MAPLE, the following results are obtained:

Set 1:

$$a_0 = 0, \quad a_1 = 0,$$
$$a_2 = \frac{3c\left(\omega\left(-\kappa^2 + 4b - b\omega\right) + \sqrt{\kappa^4\omega^2 + 16b^2\omega^2 + 8\kappa^2 b\omega^2 - 16\kappa^2 b^2\mu}\omega + b\mu\right)}{16b^2\sigma},$$
$$\omega = \omega, \quad \lambda = \frac{\left(-\kappa^2\omega + 4b\omega + \sqrt{\kappa^4\omega^2 + 8\kappa^2 b\omega^2 + 16b^2\omega^2 - 16\kappa^2 b\mu}\right)\omega}{8\kappa^2 b}.$$

Set 2:

$$a_0 = \frac{-8b\omega^2 + \left(-\kappa^2\omega - 4b\omega + \sqrt{\kappa^4\omega^2 - 8\kappa^2 b\omega^2 + 16b^2\omega^2 + 16\kappa^2 b\mu}\right)\omega + 8b\mu}{8b\sigma},$$
$$a_1 = 0, \quad \omega = \omega,$$
$$a_2 = \pm\frac{3}{2}\left(\frac{8b\omega^2 + \left(-\kappa^2\omega - 4b\omega + \sqrt{\kappa^4\omega^2 - 8\kappa^2 b\omega^2 + 16b^2\omega^2 + 16\kappa^2 b\mu}\right)\omega - 8b\mu}{8b^2\sigma}\right),$$
$$\lambda = \frac{\left(\kappa^2\omega - 4b\omega + \sqrt{\kappa^4\omega^2 + 8\kappa^2 b\omega^2 + 16b^2\omega^2 - 16\kappa^2 b\mu}\right)\omega}{8\kappa^2 b}.$$

Then, using (5) the speed of the soliton is obtained as

$$v = \frac{\kappa^2\omega - 4b\omega + \sqrt{\kappa^4\omega^2 + 8\kappa^2 b\omega^2 + 16b^2\omega^2 - 16\kappa^2 b\mu}}{8\kappa b}.$$

Consequently, the constraint relation is

$$(\kappa^4\omega^2 + 8\kappa^2 b\omega^2 + 16b^2\omega^2 - 16\kappa^2 b\mu) > 0.$$

Using the general solution of ODE (20), we obtain from Set 1, the following exact travelling-wave solutions to the Klein–Gordon equations:

S1: For $b > 0$ and $c > 0$, the singular soliton solution is obtained as

$$u_{2,1}(x,t) = a_2\frac{a}{b}\left[\operatorname{csch}\sqrt{b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right] \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{22}$$

S2: For $b < 0$ and $c > 0$, the trigonometric function solution is obtained as

$$u_{2,2}(x,t) = -a_2\frac{a}{b}\left[\sec\sqrt{-b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{23}$$

S3: For $b > 0$ and $c < 0$, the bright soliton solution is obtained as

$$u_{2,3}(x,t) = a_2\frac{a}{-b}\left[\operatorname{sech}\sqrt{b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{24}$$

S4: For $b < 0$ and $c > 0$, the following equation is obtained:

$$u_{2,4}(x,t) = a_2 \frac{-a}{b}\left[\csc\sqrt{-b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{25}$$

S5: For $b > 0$ and $c = 0$, the following equation is obtained:

$$u_{2,5}(x,t) = a_2 \exp\left[\sqrt{b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{26}$$

S6: For $b < 0$ and $c = 0$, the following equation is obtained:

$$u_{2,6}(x,t) = a_2 \left[\cos\sqrt{-b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right) + i\sin\sqrt{-b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[i(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)\right]. \tag{27}$$

S7: For $b = 0$ and $c > 0$, the following rational function solution is obtained:

$$u_{2,7}(x,t) = \pm a_2 \left[\frac{1}{\sqrt{b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)}\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{28}$$

S8: For $b = 0$ and $c < 0$, the following rational function solution is obtained:

$$u_{2,8}(x,t) = a_2 \left[\frac{1}{\sqrt{-b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)}\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{29}$$

From Set 2, we obtain

S9: For $b > 0$ and $c > 0$, the following singular soliton solution is obtained:

$$u_{2,9}(x,t) = a_0 \pm \frac{a_2 a}{b}\left[\mathrm{csch}\sqrt{b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right] \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{30}$$

S10: For $b < 0$ and $c > 0$, the following trigonometric function solution is obtained:

$$u_{2,10}(x,t) = a_0 \pm \frac{a_2 a}{b}\left[\sec\sqrt{-b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{31}$$

S11: For $b > 0$ and $c < 0$, the following bright soliton solution is obtained:

$$u_{2,11}(x,t) = a_0 \pm a_2\left(\frac{a}{-b}\right)\left[\mathrm{sech}\sqrt{b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{32}$$

S12 For $b < 0$ and $c > 0$, the following trigonometric function solution is obtained:

$$u_{2,12}(x,t) = a_0 \pm a_2\left(\frac{-a}{b}\right) \times \left[\csc\sqrt{-b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{33}$$

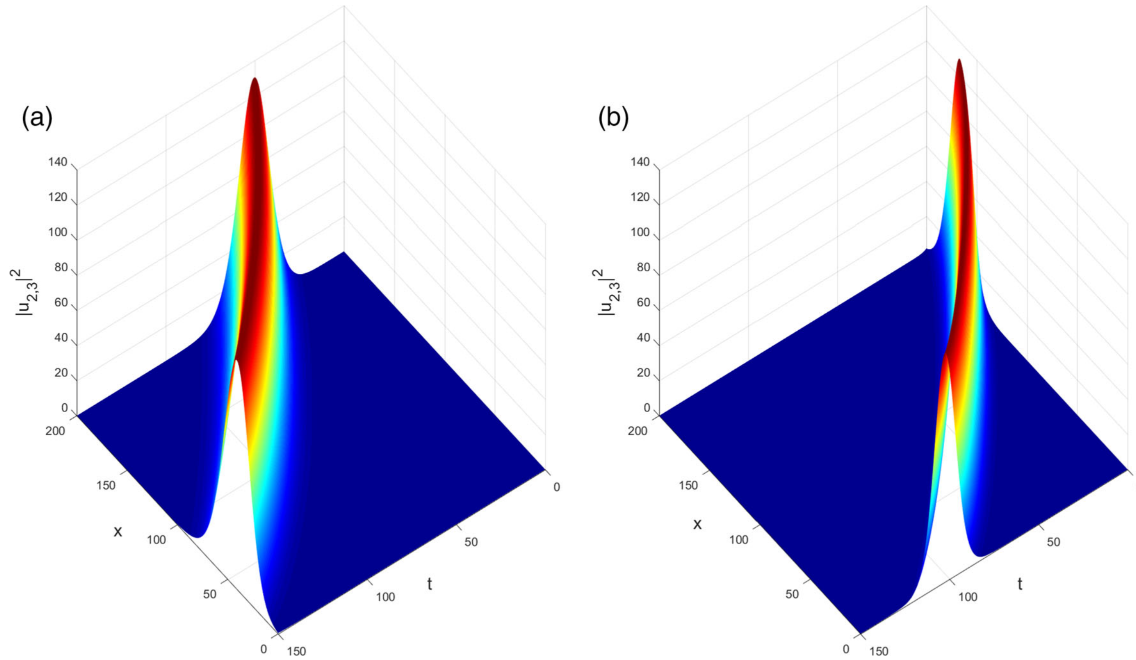

Figure 1. Spatiotemporal evolution of the bright soliton $|u_{2,3}|^2$ of eq. (32) when $a = 20.75, b = 0.0078, \omega = 0.5, \kappa = 0.7, \mu = 0.3, \sigma = 200.52, c = 1.045, v = -1.55$ for (**a**) $\alpha = 0.79$ and (**b**) $\alpha = 0.85$.

S13: For $b > 0$ and $c = 0$, the following rational solution is obtained:

$$u_{2,13}(x,t) = a_0 \pm a_2 \exp\left[\sqrt{b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{34}$$

S14: For $b < 0$ and $c = 0$, the following complex solution is obtained:

$$u_{2,14}(x,t) = a_0 \pm a_2 \left[\cos\sqrt{-b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right) + i\sin\sqrt{-b}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{35}$$

S15: For $b = 0$ and $c > 0$, the following rational solution is obtained:

$$u_{2,15}(x,t) = a_0 \pm a_2 \left[\frac{1}{\sqrt{c}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)}\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right]. \tag{36}$$

S16: For $b = 0$ and $c < 0$, the following rational solution is obtained:

$$u_{2,16}(x,t) = a_0 \pm a_2 \left[\frac{1}{\sqrt{-c}\left(\frac{\Gamma(\beta+1)}{\alpha}(x^\alpha - vt^\alpha) + \xi_0\right)}\right]^2 \times \exp\left[i\left(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha\right)\right], \tag{37}$$

where ξ_0 is an integration constant.

Figures 1–5 show the plots of spatiotemporal evolution in 3D and 2D of the bright soliton solutions $|a_{2,3}|^2$ and $|a_{2,11}|^2$ respectively. It is observed that the evolution plots of the bright soliton solution for $-150 \le x \le 150$, at $t = 0$, $t = 10$, $t = 15$, $t = 20$ shift from left to

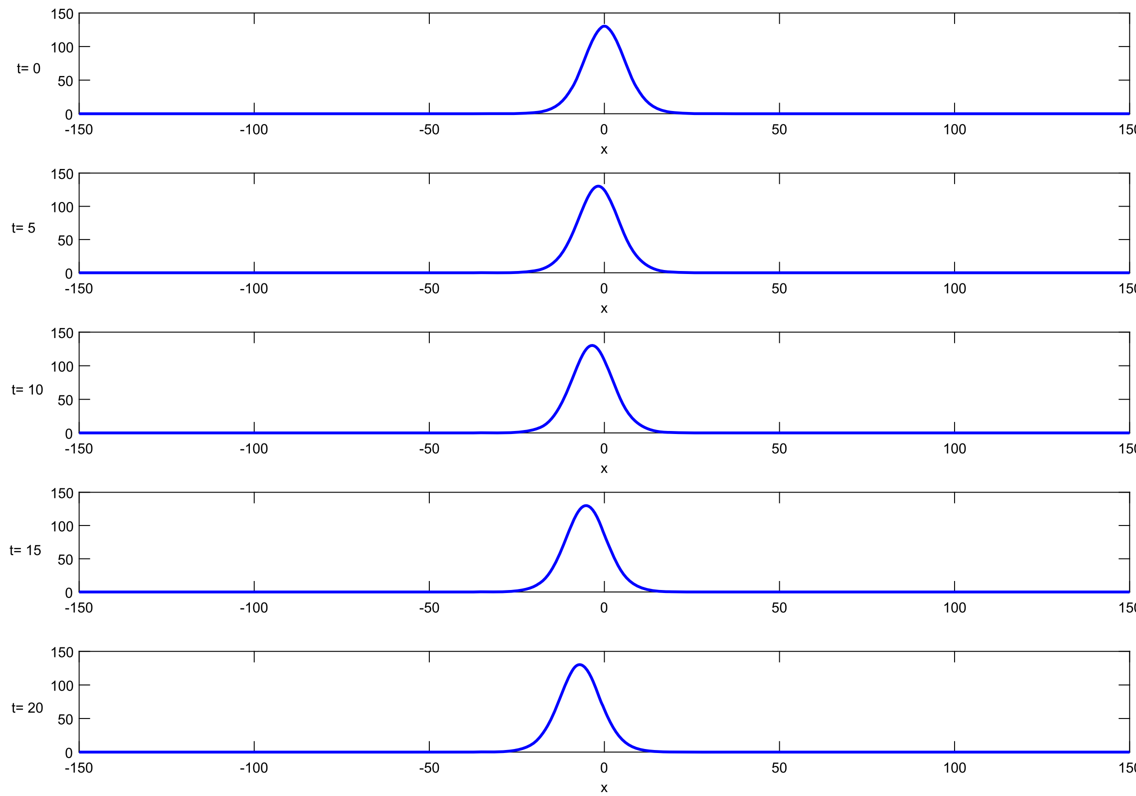

Figure 2. Plot of spatiotemporal evolution of $|a_{2,3}|^2$ of eq. (32) when $a = 20.75, b = 0.0078, \omega = 0.5, \kappa = 0.7, \mu = 0.3, \sigma = 200.52, c = 1.045, \upsilon = -0.45$ and $\alpha = 1$.

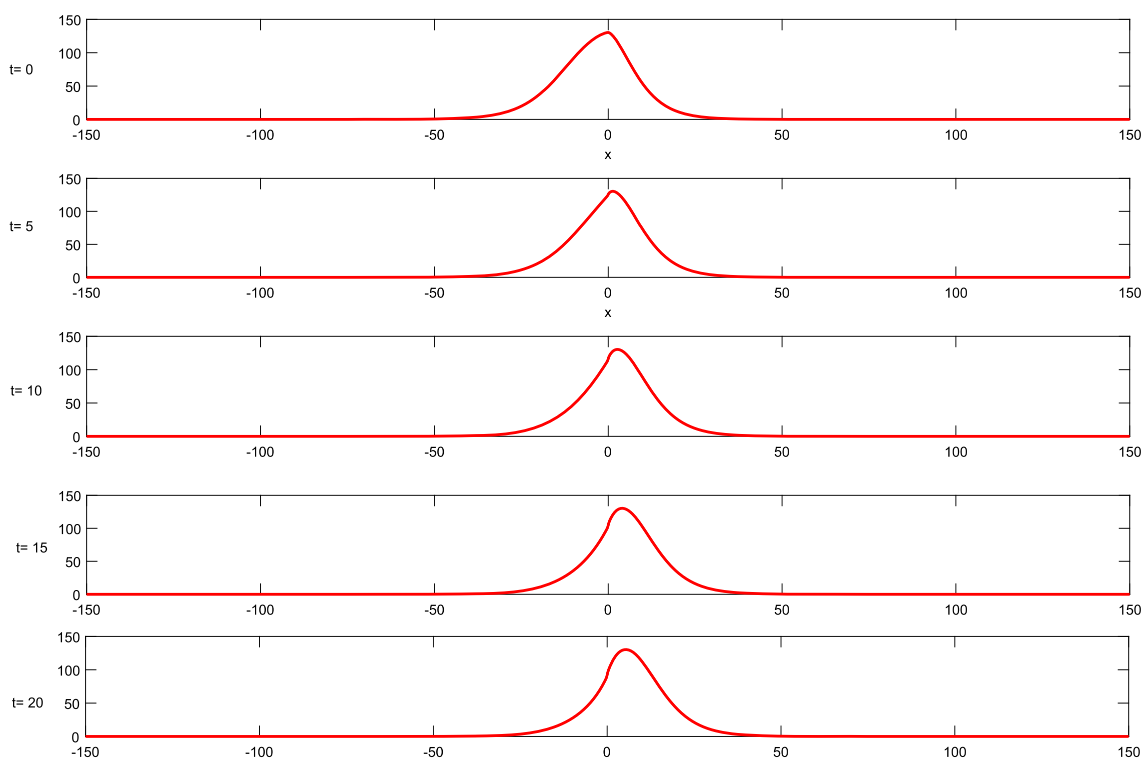

Figure 3. Plot of spatiotemporal evolution of $|a_{2,3}|^2$ of eq. (32) when $a = 20.75, b = 0.0078, \omega = 0.5, \kappa = 0.7, \mu = 0.3, \sigma = 200.52, c = 1.045, \upsilon = 0.35$ and $\alpha = 0.8$.

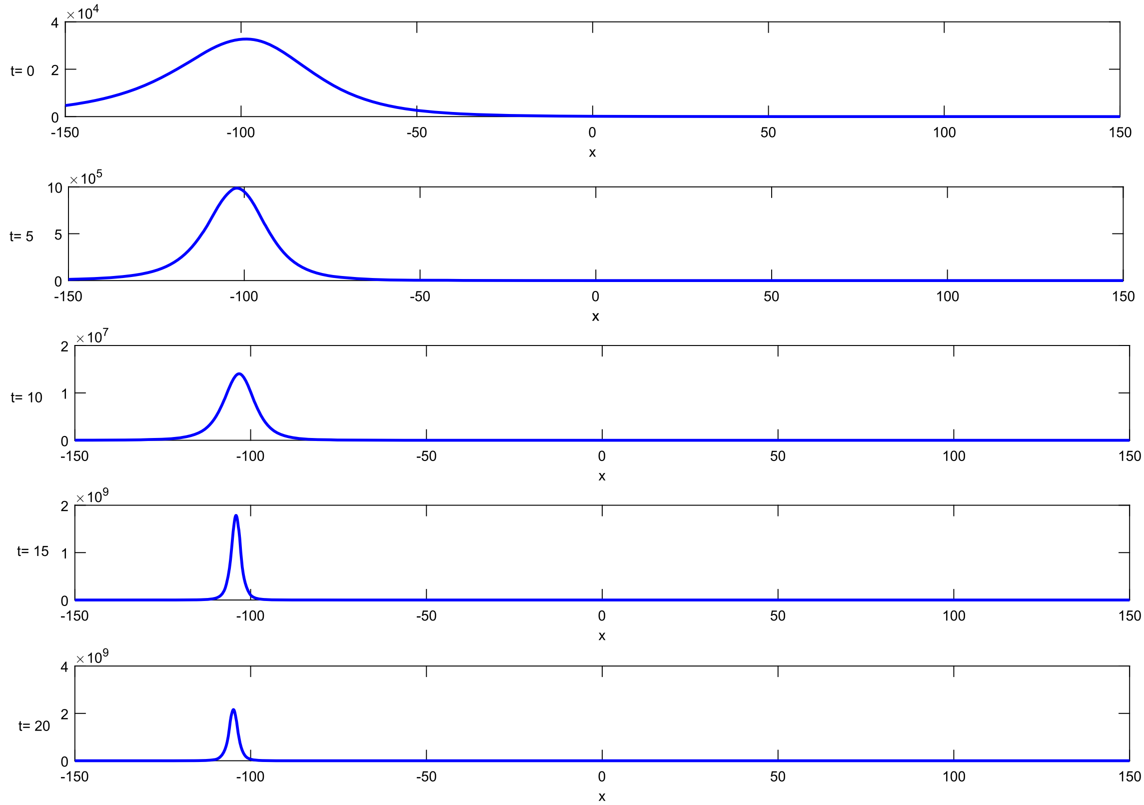

Figure 4. Plot of spatiotemporal evolution of $|a_{2,3}|^2$ of eq. (32) when $a = 20.75$, $b = 0.0078$, $\omega = 0.5$, $\kappa = 0.7$, $\mu = 0.3$, $\sigma = 200.52$, $c = 1.045$, $v = 0.35$ and $\alpha = 0.45$.

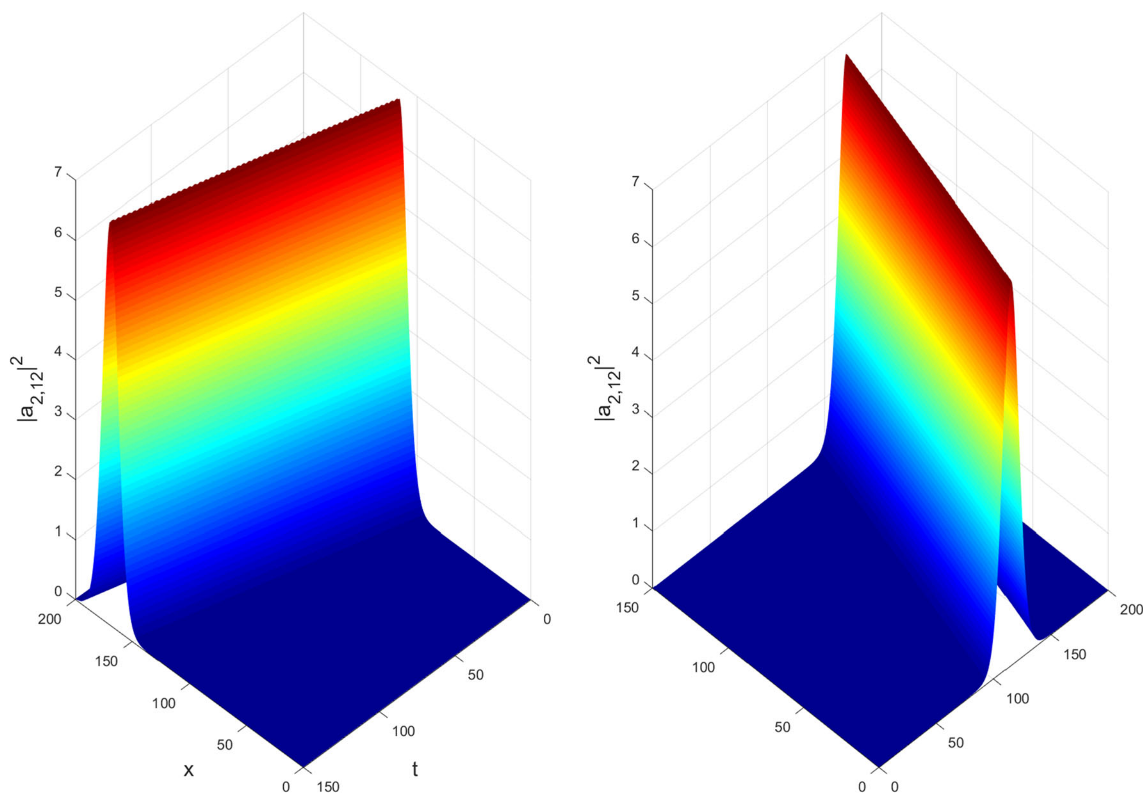

Figure 5. Plot of spatiotemporal evolution of $|a_{2,11}|^2$ of eq. (32) when $a = 20.75$, $b = 0.0078$, $\omega = 0.5$, $\kappa = 0.7$, $\mu = 0.3$, $\sigma = 200.52$, $c = 1.045$, $v = 0.35$ for $\alpha = 0.95$ and $\alpha = 0.98$ respectively.

right when the speed of the solitons increases ($v > 0$) and shift from right to left when the speed of the solitons decreases ($v < 0$) (arbitrarily chosen in this case). The effects of the derivative can also be observed in figures 1b, 3 and 4.

4. Conclusion

In this paper, analytical solutions of the Klein–Gordon equations involving M-fractional derivative has been solved by adopting two relevant integration schemes. The behaviour of the obtained bright soliton solutions was studied. It is observed that the fractional order has an effect on the width of the solitons solutions (see figures 1–4). These results will certainly be helpful to explain physics phenomena in nonlinear complex systems.

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