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Treatment of hadronic systems involving two potentials under a new approximation scheme

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Abstract. In this work, exact analytical expressions for regular solution and on-shell Jost function are calculated for nuclear Hulthén plus atomic Hulthén potential by imposing the same range approximation to both nuclear and atomic potentials. In this context, the regular solution is utilised to find expressions for the off-shell Jost function and half-shell T-matrix. The half-off-shell T-matrix and the elastic scattering phase shifts for the nucleon–nucleon and nucleus–nucleus systems are computed. Our results are found to be in good agreement with the standard data.

Keywords. Atomic and nuclear Hulthén potentials; regular solution; off-shell Jost function; half-off-shell T-matrix; p–p and α–^{12}C Systems.

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1. Introduction

In non-relativistic quantum mechanics for two-body interactions, exact analytical solutions of the Schrödinger equation are important because they provide all the required information about the quantum systems under consideration. The Schrödinger equation can only be solved precisely for a few potentials. The nuclear scattering of two charged particles usually occurs under the combined influence of two potentials, one is electromagnetic in nature, and the other is nuclear in origin. For a short-range local nuclear plus electromagnetic potentials, the Schrödinger equation does not admit an exact analytical solution. In the recent past, one of us [1,2] treated the problems containing nuclear Hulthén plus atomic Hulthén potentials to treat the α–proton and α–α systems by calculating approximate analytical solutions of the concerned Schrödinger equations and the formalism of supersymmetric algebra. The short-range local nuclear potential is usually replaced by a non-local separable one to obtain an analytical solution, and the results for such problems have been advocated by a number of researchers [3–10]. The off-energy-shell prolongation of the T-matrix is fundamental to the description of the multiparticle systems encountered in atomic and nuclear physics [11–19]. The description of the multiparticle systems requires the knowledge of the two-body interaction off energy shell. We study the on- and off-shell Jost functions as well as half-off-shell T-matrix within the Hulthén model of interaction. Here we consider a differential equation approach to the problem by using nuclear Hulthén plus atomic Hulthén potentials and construct the regular solution and in turn off-shell Jost function and half-shell T-matrix. The feasibility of our methodology is examined using some model calculations.

Generally, we consider the screening radius of the atomic Hulthén potential [20] to be very large so that the electromagnetic interaction becomes short-ranged. Usually, one would like to see the effect of the combined electromagnetic and nuclear interactions within the nuclear domain. In the light of this, we consider the same range of parameters for both the nuclear Hulthén plus atomic Hulthén potentials to obtain an exact analytical solution and test the validity of our assumption. This is the main motivation of this text. Here, we consider the s-wave case only because the Hulthén potential is exactly solvable for $\ell = 0$. In § 2 we construct on-shell solutions and identify the Jost function. Section 3 is associated with the derivation of off-shell Jost function and half-shell transition matrix. We discuss our results in §4 and concluding remarks are given in §5.

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2. Regular solution and Jost function

The s-wave nuclear Hulthén potential [1,2] is defined by

$$V_{0N}(s) = -\xi_1 \frac{\exp(-s/d)}{1-\exp(-s/d)}, \tag{1}$$

where $\xi_1 = (\beta^2 - \alpha^2)$, $d = (\beta - \alpha)^{-1}$ and α, β will be provided for nuclear part. As an electromagnetic interaction we adapt the screened atomic Hulthén potential [20]

$$V_{0A}(s) = \xi_0 \frac{\exp(-s/d)}{1-\exp(-s/d)} \tag{2}$$

with ξ_0, the strength and d, the screening radius of the potential. The effective potential reads as

$$\begin{aligned} V_{\mathrm{eff}}(s) &= V_{0N}(s) + V_{0A}(s) \\ &= (\xi_0 - \xi_1)\frac{\exp(-s/d)}{1-\exp(-s/d)}. \end{aligned} \tag{3}$$

The s-wave ($\ell = 0$) Schrödinger equation for the above effective potential is written as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \chi^2 - (\xi_0 - \xi_1)\frac{\exp(-s/d)}{1-\exp(-s/d)}\right]\phi(\chi, s) = 0, \tag{4}$$

where χ stands for the centre of mass momentum and is related to the centre of mass energy

$$E = \frac{\hbar^2\chi^2}{2\mu},$$

where μ is the reduced mass. The wave function $\phi(\chi, s)$ satisfies the regular boundary condition.

Introducing the following transformation

$$\phi(\chi, s) = d\left[1 - \exp(-s/d)\right]\exp(\mathrm{i}\chi s)w(\chi, s) \tag{5}$$

eq. (4) takes the form

$$\begin{aligned} & d^2\exp(s/d)(1-\exp(-s/d))w''(\chi, s) \\ & \quad + \{2d + 2i\chi d^2 \exp(s/d) \\ & \quad \times (1-\exp(-s/d))\}w'(\chi, s) \\ & \quad + \{2i\chi d - 1 + (\xi_1 - \xi_0)d^2\}w(\chi, s) = 0. \end{aligned} \tag{6}$$

If we rewrite eq. (4) by changing a new variable of the form $(1 - \exp(-s/d)) = \eta$, it yields

$$\begin{aligned} & \eta(1-\eta)\frac{\mathrm{d}^2 w}{\mathrm{d}\eta^2} + \{2 - (3 - 2\mathrm{i}\chi d)\eta\}\frac{\mathrm{d}w}{\mathrm{d}\eta} \\ & \quad - \{1 - 2\mathrm{i}\chi d - (\xi_1 - \xi_0)d^2\}w = 0. \end{aligned} \tag{7}$$

To obtain the regular solution, we compare eq. (7) with the following standard differential equation for Gaussian hypergeometric function [21–23]:

$$\begin{aligned} & \eta(1-\eta)\frac{\mathrm{d}^2 w}{\mathrm{d}\eta^2} + \{P - (1 + M + N)\eta\}\frac{\mathrm{d}w}{\mathrm{d}\eta} \\ & \quad - MNw = 0 \end{aligned} \tag{8}$$

to obtain

$$w(\chi, \eta) = {}_2F_1(M, N; P; \eta) \tag{9}$$

with

$$M = 1 - \mathrm{i}\chi d + d\sqrt{(\xi_1 - \xi_0) - \chi^2}, \tag{10}$$

$$N = 1 - \mathrm{i}\chi d - d\sqrt{(\xi_1 - \xi_0) - \chi^2} \tag{11}$$

and

$$P = 2. \tag{12}$$

Substituting eq. (9) into eq. (5) together with the value of η the regular solution reads as

$$\begin{aligned} \phi(\chi, s) &= d\left[1 - \exp(-s/d)\right]\exp(\mathrm{i}\chi s) \\ & \quad \times {}_2F_1(M, N; P; 1 - \exp(-s/d)). \end{aligned} \tag{13}$$

In order to obtain the irregular solution $f(\chi, s)$ from eq. (13) for the potential under consideration we use the transformation formulae of the generalised hypergeometric function [21–23]

$$\begin{aligned} {}_2F_1(A_1, B_1; C_1; Z) &= \frac{\Gamma(C_1)\Gamma(C_1 - A_1 - B_1)}{\Gamma(C_1 - A_1)\Gamma(C_1 - B_1)} \\ & \quad \times {}_2F_1(A_1, B_1; A_1 + B_1 - C_1 + 1; 1 - Z) \\ & \quad + (1-Z)^{C_1 - A_1 - B_1}\frac{\Gamma(C_1)\Gamma(A_1 + B_1 - C_1)}{\Gamma(A_1)\Gamma(B_1)} \\ & \quad \times {}_2F_1(C_1 - A_1, C_1 - B_1; \\ & \quad C_1 - A_1 - B_1 + 1; 1 - Z) \end{aligned} \tag{14}$$

and

$$\begin{aligned} {}_2F_1(A_1, B_1; C_1; Z) &= (1-Z)^{C_1 - A_1 - B_1} \\ & \quad \times {}_2F_1(C_1 - A_1, C_1 - B_1; C_1; Z). \end{aligned} \tag{15}$$

After some algebraic manipulation, eq. (13) leads to

$$\phi(\chi,s)=\frac{1}{2\mathrm{i}\chi}\left[\frac{\Gamma(1+2\mathrm{i}\chi d)}{\Gamma(M^*)\Gamma(N^*)}\exp(\mathrm{i}\chi s)\right.$$
$$\times{}_2F_1(1-2\mathrm{i}\chi d-M,1-2\mathrm{i}\chi d-N;1-2\mathrm{i}\chi d;$$
$$\times\exp(-s/d))-\frac{\Gamma(1-2\mathrm{i}\chi d)}{\Gamma(M)\Gamma(N)}\exp(-\mathrm{i}\chi s)$$
$$\times{}_2F_1(1+2\mathrm{i}\chi d-M^*,$$
$$\left.1+2\mathrm{i}\chi d-N^*;1+2\mathrm{i}\chi d;\exp(-s/d))\right]. \quad (16)$$

The connection between regular and irregular solutions [24,25] is expressed as

$$\phi(\chi,s)=\frac{1}{2\mathrm{i}\chi}\left[\mathcal{J}_-(\chi)f_+(\chi,s)-\mathcal{J}_+(\chi)f_-(\chi,s)\right], \quad (17)$$

where the Jost function $\mathcal{J}_+(\chi)=(\mathcal{J}_-(\chi))^*$ and the Jost solution $f_+(\chi,s)=(f_-(\chi,s))^*$. Now, one can successfully obtain the Jost solution and the Jost function by comparing eqs (16) and (17), which is written as

$$f_{(+)}(\chi,s)=\exp(\mathrm{i}\chi s)\,{}_2F_1(1-2\mathrm{i}\chi d-M,$$
$$1-2\mathrm{i}\chi d-N;1-2\mathrm{i}\chi d;\exp(-s/d)) \quad (18)$$

and

$$\mathcal{J}_+(\chi)=\frac{\Gamma(1-2\mathrm{i}\chi d)}{\Gamma(M)\Gamma(N)}. \quad (19)$$

The Jost function $\mathcal{J}_+(\chi)$ [24–26], which plays a vital role in examining the analytic properties of partial wave scattering amplitudes, is determined from near the origin behaviour of the irregular solution $f_+(\chi,s)$ of the radial Schrödinger equation. The bound-state energies are determined by the zeros of the Jost function in the upper half of the complex χ-plane. In general, the Jost function $\mathcal{J}_+(\chi)$ is a complex quantity and the phase of the Jost function is the negative of the scattering phase shift. The relation between scattering phase shift and Jost function is given by $\tan\delta=-[\mathrm{Im}(\mathcal{J}_+(\chi))/\mathrm{Re}(\mathcal{J}_+(\chi))]$. With the knowledge of the Jost function, one can calculate binding energies and scattering phase shifts for a given potential field. In eq. (19) $\mathcal{J}_+(\chi)$ with $\chi=\mathrm{i}\kappa$ becomes zero at the pole of the gamma function $\Gamma(M)$ or $\Gamma(N)$.

When $M=1-\mathrm{i}\chi d+d\sqrt{(\xi_1-\xi_0)-\chi^2}=-n; n=0,1,2,\ldots$ for $\chi=\mathrm{i}\kappa$ one has

$$\kappa=\frac{d(\xi_1-\xi_0)}{2}-\frac{1}{2d},\ \text{for}\ n=0, \quad (20)$$

where

$$\kappa=\frac{\sqrt{2\mu E_B}}{\hbar}.$$

Here E_B denotes the binding energy of the associated system.

3. Off-shell Jost function and half-shell T-matrix

In order to obtain the off-shell Jost function, we exploit the regular solution together with the following relation [27,28]:

$$f(\chi,q)=(\chi^2-q^2)\int_0^\infty \mathrm{d}s\ \exp(\mathrm{i}qs)\,\phi(\chi,s). \quad (21)$$

Also there exists a relation between the off-shell Jost function and the half-off-shell T-matrix [27,28] which reads as

$$T(\chi,q,\chi^2)=\frac{f(\chi,q)-f(\chi,-q)}{\mathrm{i}\pi q f(\chi)}. \quad (22)$$

Substituting eq. (13) into eq. (21) and using the following integral and standard relation [21–23,29,30]

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}{}_2F_1(\alpha,\beta;\gamma;x)\mathrm{d}x$$
$$=\frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;1) \quad (23)$$

with $\mathrm{Re}\,\rho>0$, $\mathrm{Re}\,\sigma>0$, $\mathrm{Re}\,(\gamma+\sigma-\alpha-\beta)>0$ and

$${}_3F_2(a,b,c;d,e;z)={}_2F_1(a,b;e;z)\ \text{when}\ c=d, \quad (24)$$

we obtain $f(\chi,q)$ as

$$f(\chi,q)=\frac{\Gamma(1-\mathrm{i}(\chi+q)d)\Gamma(1+\mathrm{i}(\chi-q)d)}{\Gamma(2-\mathrm{i}(\chi+q)d-M)\Gamma(2-\mathrm{i}(\chi+q)d-N)}. \quad (25)$$

It is easy to find the expressions for $f(\chi,-q)$ by replacing q by $-q$ in eq. (25). Having inserted the expressions of $f(\chi,q)$ and $f(\chi,-q)$ into eq. (22), one can get the compact expression of half-shell T-matrix $T(\chi,q,\chi^2)$ for the potential under consideration. The calculation of the half-shell T-matrix can be verified by ensuring that it reproduces proper on-shell limit. Under the limit $q=\chi$ the half-shell T-matrix gives proper on-shell value. In §4 we shall apply these relations to compute the half-shell T-matrices and related scattering phase shifts for the proton–proton and alpha–carbon systems.

4. Results and discussion

The $(p–p)$ system is unbound while for the $(\alpha–{}^{12}\mathrm{C})$ system the compound nucleus is the O^{16} nucleus with binding energy 125.1936 MeV. We have used a chi square test to find the best-fitted parameters for the phase shifts and binding energy of the associated systems. We have used $\hbar^2/2\mu=41.47$ MeV fm^2 and $\hbar^2/2\mu=6.964$ MeV fm^2 for $(p–p)$ and $(\alpha–{}^{12}\mathrm{C})$ systems respectively.

For the nuclear part of the interaction, the parameters α and β are found to be $\alpha = -0.0544$ fm^{-1}, β =1.1 fm^{-1} [10,18,19] and the strength of the electromagnetic part $\xi_0 d$ =0.03472 fm^{-1} for the proton–proton system whereas the same for the alpha–carbon system are $\alpha = 5.4964$ fm^{-1}, $\beta = 6.5359$ fm^{-1} and $\xi_0 d = 2.51277$ fm^{-1}. These parameters reproduce the correct binding energy of the O^{16} nucleus as verified by eq. (20). As the phase of the Jost function $f(\chi)$ is negative of the scattering phase shift, it is evident from eq. (22) that the phase of the half-shell T-matrix is equal to the phase parameter. The computed results for the half-shell transition matrices are portrayed in figures 1 and 2 and the corresponding phase shifts are presented in tabular form in tables 1 and 2 for the concerned systems.

The half-shell T-matrices for the $(p$–$p)$ system at E_{Lab} =10 and 20 MeV, as shown in figure 1, start from negative values and gradually increase with the off-shell momentum q towards zero. This is quite expected as the projectile spends less time within the potential field of the target nucleus as energy increases, thereby the scattering amplitude decreases. The results for the $(p$–$p)$ transition matrices are very much similar to our recent observations [31] for the $(n$–$p)$ system with pure nuclear Manning–Rosen, Hulthén and Yamaguchi potentials without any hard core. For the $(\alpha$–^{12}C) system, half-shell transition matrices for E_{Lab} =3.85 and 6.009 MeV have peaks at q =1.32 and 1.4 fm^{-1} respectively. Beyond these points, they gradually approach zero with the off-shell momentum q. The potentials of the associated systems are depicted in figure 3. Our $(p$–$p)$ and $(\alpha$–^{12}C) potentials also do not possess any hard cores. The absence of hard cores in the effective potentials is due to the consideration of small screening radius of the electromagnetic potentials. These small values of the screening radius prevent the reproduction of the proper Coulomb effects when the particles are very close to each other. As observed in figure 3, the repulsive strengths of the atomic Hulthén potentials are much smaller than the attractive nuclear parts of both the systems. This results in an attractive effective potential.

The phase shifts obtained from the transition matrices for the $(p$–$p)$ and $(\alpha$–^{12}C) systems, presented in tables 1 and 2, are consistent with the more sophisticated data of Wiringa *et al* [32], Bystricky *et al* [33], Plaga *et al* [34], Behera *et al* [35] and other researchers [36–38] except at very low energies. The discrepancies in the very low energy phase parameters may possibly point towards the absence of hard cores in the interaction. The disappearance of the hard cores in the effective potentials points towards the electromagnetic interactions with small screening parameters. In particular, the screened Coulomb potential with a large screening

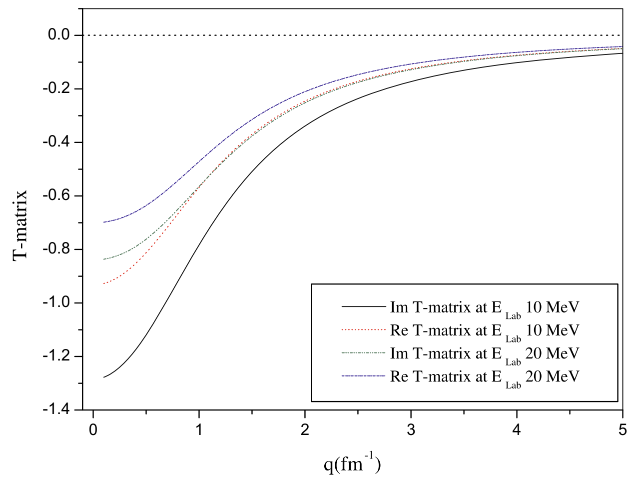

Figure 1. $T(\chi, q, \chi^2)$ as a function of q for the $(p$–$p)$ system.

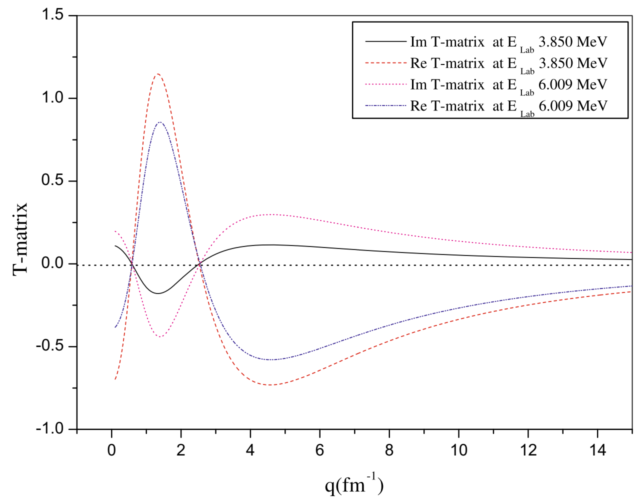

Figure 2. $T(\chi, q, \chi^2)$ as a function of q for the $(\alpha$–^{12}C) system.

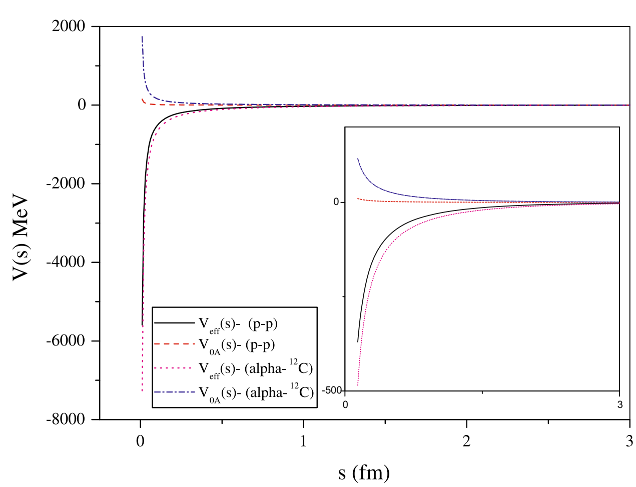

Figure 3. Potentials for the $(p$–$p)$ and $(\alpha$–^{12}C) systems as a function of distance.

Table 1. Scattering phase shifts for the $(p–p)$ system.

E_{Lab} (MeV)	Phase shift $^1S_0$ state (degree) (present work)	Phase shift $^1S_0$ state (degree) Ref. [32]	Phase shift $^1S_0$ state (degree) Ref. [33]
1	48.357365	38.68	–
5	55.623679	54.74	–
10	54.031798	55.09	55.17
15	51.987371	–	52.96
20	50.151009	–	50.60
25	48.556764	48.51	48.36
50	40.024178	38.71	39.20

Table 2. Scattering phase shifts for the $(\alpha–^{12}\text{C})$ system.

E_{Lab} (MeV)	Phase shift $1/2^+$ state (degree) (present work)	Phase shift $1/2^+$ state (degree) Ref. [34]	Phase shift $1/2^+$ state (degree) Ref. [35]
2.971	0.922	−4.1 ± 2.1	−3.549
3.069	−0.227	−4.2±0.8	−4.026
3.070	−0.389	−4.7±1.6	−4.094
3.171	−1.507	−4.6±0.8	−4.532
3.172	−1.665	−6.5±2.1	−4.605
3.850	−8.965	−5.6±2.1	−8.916
4.051	−10.959	−7.8±2.1	−10.485
4.175	−12.119	−12.6±1.7	−11.392
4.451	−14.726	−12.3±2.1	−13.572
4.650	−15.684	−13.4±2.1	−15.060
4.851	−18.117	−16.0±1.1	−16.773
5.050	−19.786	−17.4±1.1	−18.399
5.450	−23.074	−24.3±0.5	−21.657
5.500	−24.092	−24.6±0.5	−22.408
5.550	−24.593	−25.6±0.8	−22.517
5.650	−24.792	−27.1±1.1	−23.276
5.700	−25.091	−26.8±1.1	−23.702
5.720	−25.188	−28.2±0.8	−23.917
5.899	−26.455	−25.9±1.1	−25.308
5.949	−26.838	−26.0±1.1	−25.735
5.999	−27.124	−26.3±1.1	−26.163
6.009	−27.219	−27.3±1.1	−26.269
6.056	−27.503	−26.3±1.6	−26.585
6.109	−28.066	−28.8±1.6	−27.005
6.159	−28.345	−27.5±1.6	−27.322
6.208	−28.623	−26.4±2.5	−27.745
6.258	−28.899	−25.7±2.5	−28.059
6.358	−29.629	−26.7±2.5	−28.891
6.458	−30.260	−27.0±2.5	−29.723
6.558	−30.971	−29.8±2.5	−30.439

radius is used to visualise the effect of electromagnetic force at very low energies. This effect becomes significant for those systems where large Coulomb forces are involved. It is clearly seen in our phase data presented in tables 1 and 2. For the $(p–p)$ system, the discrepancies in phase shifts are noticed up to 2 MeV and beyond that they are in close agreement with those of refs [32] and [33]. On the other hand, for nucleus–nucleus, $(\alpha–^{12}\text{C})$ system, our phase values discern from those of refs [34] and [35] up to 3.2 MeV.

5. Conclusions

Within the approximation scheme we have treated both the elastic and inelastic scattering of the nucleon–nucleon and nucleus–nucleus systems and obtained good agreement in bound-state energy and phase shifts. The disagreement in the very low-energy phase data, particularly for the nucleus–nucleus system, might be due to the improper accountability of the electromagnetic force at very low energies. In practice, the screened Coulomb potential with a large screening radius is considered to account for the Coulomb repulsion at short distances. In contrast to this, relatively smaller values of screening radius are imposed as per the approximation scheme for simplicity of calculation and to observe its impact. In the case of proton–proton scattering, the effect of electromagnetic force is significant within few 100 keV while for the alpha–carbon system, large Coulomb repulsion is involved that plays a dominant role to some greater extent of energy. At relatively higher energies, the electromagnetic part of the potential becomes less significant in the face of the strong nuclear part of the interaction. Thus, our phase parameters agree well with the earlier works at moderate values of laboratory energies. The present method can be applicable to the case of an arbitrary exponential type of nuclear local potential. The computation of the binding energy per particle in nuclear matter and the determination of the shell-model spectrum are carried out in terms of the transition matrices. Also the theoretical investigation of the $(p–p)$ bremsstrahlung is closely related to the study of the half-off-shell nucleon–nucleon T-matrix. Therefore, the expression for the T-matrix facilitates us to make best possible use of the available information about the two-nucleon wave function in coordinate space. The present text deals with two-parameter central nuclear potential while in refs [32] and [33] several parameter interactions are involved with the inclusion of spin–orbit and tensor interactions. The same explanation is also applicable for the $(\alpha–^{12}\text{C})$ system. From the foregoing discussion we conclude that our simple model has the right ability to describe the nucleon–nucleon and nucleus–nucleus systems at moderate energies and the validity of our conjecture is quite established.

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