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On the p–$^{2}H_{1}$ and p–^{16}O elastic scattering

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Abstract. We solve the wave equation with the Manning–Rosen plus rank one separable non-local potential to obtain exact analytical solutions through ordinary differential equation method. The regular, Jost and physical state solutions are found to involve special functions of mathematics. As an application of the Jost function and Fredholm determinant, the bound-state energies and the scattering phase parameters for the p–$^{2}H_{1}$ and p–^{16}O systems are computed. The results achieved are in good agreement with the other methods published earlier.

Keywords. Schrödinger equation; Manning–Rosen plus non-local potential; Jost function; Fredholm determinant; scattering phase shifts.

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1. Introduction

Our understanding of the physical systems in atomic and subatomic world is mostly based on relativistic or non-relativistic quantum scattering theory. For charged hadron systems, the scattering takes place by additive interactions. In such a situation, the scattering takes place under the combined influence of two potentials, one due to electric charges and the other due to nuclear forces. The interaction between the oxygen nucleus ^{16}O and proton can be demonstrated nicely by an optical model potential and has been studied by several researchers [1–6]. The Breit–Wigner resonances were incorporated accurately to the optical amplitude to obtain a fair agreement for the 10.5 MeV proton–oxygen scattering data [1]. More reliable theoretical and experimental data [7–16] at various energies are available with regard to the proton–deuteron (p–$^{2}H_{1}$) elastic scattering to explain different off-shell behaviours in realistic potential models of two-nucleon systems. In general, the nucleon–nucleon interactions are described in a phenomenological way to fit the observed scattering data. In contrast to this, the effective field theory represents a crucial method for constructing nuclear forces together with quantum chromodynamics in a more precise, model-independent way [17,18]. In this paper, we treat the (p–$^{2}H_{1}$) and (p–^{16}O) elastic scattering in terms of the two-body model potential.

To solve the problem exactly, we consider a separable non-local interaction instead of a local nuclear one. Thus, there is no loss of generality if we get exact solution to investigate the system because a local potential can be replaced distinctly by a separable non-local potential. The problems with Coulomb or Hulthén plus rational separable potentials have been studied at length by various groups [19–31]. But the problem related to Manning–Rosen plus separable interaction has not yet been treated in the literature. As the Manning–Rosen potential admits exact solution for $\ell = 0$ state, we use the Manning–Rosen plus Yamaguchi potential to construct the solutions for wave functions and the Fredholm determinants.

In quantum mechanics, exact analytical solutions corresponding to the Schrödinger equation are of great interest for physical potentials. The analytical solutions help to develop theoretical understanding of the physical systems and come up with strong support for correctness of the quantum theory. These expressions have the ability to reproduce scattering or bound-state observables for various complicated nuclear systems. However, the numbers of exactly solvable problems are limited in quantum mechanics. Recently, we have studied nuclear systems considering the Manning–Rosen potential [32] as the nuclear interaction and achieved excellent agreement both in bound and scattering state data [33–36]. Motivated by this, we use the Manning–Rosen potential as a short-range electromagnetic interaction and

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a non-local separable potential as the nuclear part. We construct expressions for the scattering states of the wave equation involving two potentials by adapting ordinary differential equation approach for various boundary conditions. Exploiting these scattering states and their associated Fredholm determinants, we examine the usefulness of our expressions by studying bound and scattering-state observables for some nuclear systems. In §2 we derive exact analytical solutions of the Schrödinger equation related to regular, irregular and physical boundary conditions. Section 3 gives results and discussion and in §4 we present some concluding remarks.

2. Regular–irregular/Jost and physical solutions

The inhomogeneous wave equation for Manning–Rosen plus rank one Yamaguchi non-local separable potential reads as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\chi^2-V_M(s)\right]\phi(\chi,s) = \lambda\,\mathrm{e}^{-\omega s}\int_0^\infty \mathrm{e}^{-\omega s'}\phi(\chi,s')\,\mathrm{d}s', \quad (1)$$

where

$$V_M(s)=d^{-2}\left\{\frac{\alpha(\alpha-1)}{(1-\mathrm{e}^{-s/d})^2}\,\mathrm{e}^{-2s/d}-\frac{A\mathrm{e}^{-s/d}}{1-\mathrm{e}^{-s/d}}\right\}. \quad (2)$$

The quantities A, d and α are three adjustable parameters of the Manning–Rosen potential of which A and α are dimensionless and d possesses the dimension of length. The other quantities λ and ω related to the Yamaguchi potential denote strength and inverse range parameters.

2.1 *Regular solution*

By applying the following transformation for the regular solution

$$\phi(\chi,s) = d^\alpha\,\mathrm{e}^{i\chi s}\left(1-\mathrm{e}^{-s/d}\right)^\alpha g(\chi,s) \quad (3)$$

in eq. (1) we get

$$\begin{aligned}&\mathrm{e}^{s/d}\,d^2\,(1-\mathrm{e}^{-s/d})\,\frac{\mathrm{d}^2g}{\mathrm{d}s^2}\\&\quad+\left[2\alpha d+2i\chi d^2\,\mathrm{e}^{s/d}\,(1-\mathrm{e}^{-s/d})\right]\frac{\mathrm{d}g}{\mathrm{d}s}\\&\quad+(2i\chi d\alpha-\alpha+A)\,g(\chi,s)\\&=\lambda\,d^{2-\alpha}\,\mathrm{e}^{s/d}\,(1-\mathrm{e}^{-s/d})^{1-\alpha}\,\mathrm{e}^{-(\omega+i\chi)s}\,J^{(R)}(\chi),\end{aligned} \quad (4)$$

where

$$J^{(R)}(\chi)=\int_0^\infty \mathrm{d}s'\,\mathrm{e}^{-\omega s'}\phi(\chi,s'). \quad (5)$$

Change of variable $z=(1-\mathrm{e}^{-s/d})$ in eq. (4) yields

$$\begin{aligned}&z\,(1-z)\frac{\mathrm{d}^2g}{\mathrm{d}z^2}+[c_1-(a_1+b_1+1)\,z]\,\frac{\mathrm{d}g}{\mathrm{d}z}-a_1b_1\,g\\&=\lambda\,d^{2-\alpha}J^{(R)}(\chi)\,z^{\sigma-1}\,(1-\rho z)^{\tau-1}\end{aligned} \quad (6)$$

with

$$\begin{aligned}a_1&=\alpha-i\chi d+(\alpha^2-\chi^2d^2-\alpha+A)^{1/2},\\b_1&=\alpha-i\chi d-(\alpha^2-\chi^2d^2-\alpha+A)^{1/2},\\c_1&=2\alpha,\quad \rho=1,\quad \sigma=2-\alpha,\quad \tau=(\omega+i\chi)d.\end{aligned} \quad (7)$$

Equation (6) resembles the inhomogeneous Gaussian hypergeometric equation [37–40] and its complementary functions, namely $u_1(z)$ and $u_2(z)$, read as

$$u_1(z)={}_2F_1\,(a_1,\,b_1;\,c_1;\,z);\; c_1>0 \quad (8)$$

and

$$\begin{aligned}u_2(z)&={}_2F_1\,(a_1,\,b_1;\,a_1+b_1-c_1\\&\quad+1;\,1-z);\;a_1+b_1-c_1+1\neq 0,\,-1,\,-2,\,\ldots.\end{aligned} \quad (9)$$

The particular solution of eq. (6) may be written in comparison with the non-homogeneous Gaussian hypergeometric equation [40] as

$$\begin{aligned}&x\,(1-x)\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+[c_1-(a_1+b_1+1)x]\,\frac{\mathrm{d}y}{\mathrm{d}x}\\&-a_1b_1y=x^{\sigma-1}(1-\rho x)^{\tau-1},\end{aligned} \quad (10)$$

where $a_1, b_1, c_1, \rho, \tau$ and σ are constants, to have

$$\begin{aligned}F_p(z)&=d^{2-\alpha}\,\lambda\,J^{(R)}(\chi)\\&\times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(i\chi+\omega)d)}{\Gamma(1-(i\chi+\omega)d)\,n!}\,f_{n+\sigma}\,(a_1,\,b_1;\,c_1;\,z)\end{aligned} \quad (11)$$

with

$$\begin{aligned}&f_n\,(a,\,b;\,c;\,z)\\&=z^n\sum_{j=0}^{\infty}\frac{\Gamma(n+a+j)\Gamma(n+b+j)\Gamma(n)\Gamma(n+c-1)}{\Gamma(n+a)\Gamma(n+b)\Gamma(n+j+1)\Gamma(n+c+j)}z^j\\&=\frac{z^n}{n(n+c-1)}\\&\quad\times{}_3F_2(1,\,n+a,\,n+b;\,n+1,\,n+c;\,z).\end{aligned} \quad (12)$$

Equation (3) in conjunction with eqs (7)–(9) and (11) yields

$$\phi(\chi,s) = d^{\alpha}e^{i\chi s}(1-e^{-s/d})^{\alpha} \times\Big[D_1\,{}_2F_1\,(a_1,b_1;c_1;1-e^{-s/d}) + D_2\,(1-e^{-s/d})^{-1-2\alpha} \times{}_2F_1(a_1,b_1;1-2i\chi d;e^{-s/d}) + d^{2-\alpha}\lambda J^{(R)}(\chi) \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(i\chi+\omega)d)}{\Gamma(1-(i\chi+\omega)d)\,n!}f_{2-\alpha+n} \times\Big(a_1,b_1;c_1;1-e^{-s/d}\Big)\Big]. \quad (13)$$

The constants D_1 and D_2 in this equation, can easily be determined from the boundary conditions

$$\phi(\chi,0)=0 \text{ and } \lim_{s\to 0}\frac{d}{ds}\phi(\chi,s)=1.$$

From the boundary condition at $s=0$, one gets $D_2 = 0$. Substituting $\alpha=\alpha+1$ in eq. (13) one obtains

$$\phi(\chi,s) = d^{\alpha+1}e^{i\chi s}(1-e^{-s/d})^{\alpha+1} \times\big[D_1\,{}_2F_1\,(M,N;P;1-e^{-s/d}) + d^{1-\alpha}\lambda J^{(R)}(\chi) \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(i\chi+\omega)d)}{\Gamma(1-(i\chi+\omega)d)\,n!}f_{1-\alpha+n} \times(M,N;P;1-e^{-s/d})\big]. \quad (14)$$

Applying the other boundary condition to eq. (14) we get $D_1 = 1$. Inserting the value of D_1 in eq. (14) the exact regular solution is

$$\phi(\chi,s) = d^{\alpha+1}e^{i\chi s}(1-e^{-s/d})^{\alpha+1} \times\big[{}_2F_1\,(M,N;P;1-e^{-s/d}) + d^{1-\alpha}\lambda J^{(R)}(\chi) \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(i\chi+\omega)d)}{\Gamma(1-(i\chi+\omega)d)\,n!}f_{1-\alpha+n} \times(M,N;P;1-e^{-s/d})\big]. \quad (15)$$

To estimate the unknown quantity $J^{(R)}(\chi)$ in this equation, we proceed as follows. The regular solution of eq. (1) may be written as

$$\phi(\chi,s) = \phi_M(\chi,s) + \lambda J^{(R)}(\chi)\int_0^s e^{-\omega s'}G_M^{(R)}(s,s')\,ds'. \quad (16)$$

Here $\phi_M(\chi,s)$ and $G_M^{(R)}(s,s')$ stand for the regular solution and Green's function [33–35,41] for the Manning–Rosen potential,

$$\phi_M(\chi,s) = d^{\alpha+1}(1-e^{-s/d})^{\alpha+1}e^{i\chi s} \times{}_2F_1\big(M,N;\;P;\;1-e^{-s/d}\big) \quad (17)$$

and

$$G_M^{(R)}(s,s') = \frac{1}{f_M(\chi)}\big[\phi_M(\chi,s)\,f_M(\chi,s') - \phi_M(\chi,s')\,f_M(\chi,s)\big]. \quad (18)$$

Here $f_M(\chi,s)$ and $f_M(\chi)$ are given by [33–35]

$$f_M(\chi,s) = (1-e^{-s/d})^{-\alpha}e^{i\chi s} \times{}_2F_1\big(M-1-2\alpha,N-1-2\alpha;\;1-2i\chi d;\;e^{-s/d}\big) \quad (19)$$

and

$$f_M(\chi) = \frac{d^{\alpha}\,\Gamma(2\alpha+2)\,\Gamma(1-2i\chi d)}{\Gamma(M)\,\Gamma(N)}, \quad (20)$$

with

$$M = 1+\alpha-i\chi d + (\alpha^2-\chi^2d^2+\alpha+A)^{1/2}, \quad (21)$$

$$N = 1+\alpha-i\chi d - (\alpha^2-\chi^2d^2+\alpha+A)^{1/2} \quad (22)$$

and

$$P = 2\alpha+2. \quad (23)$$

Multiplying by $e^{-\omega s}$ and integrating both sides of eq. (16) from 0 to infinity $J^{(R)}(\chi)$ is simplified to

$$J^{(R)}(\chi) = \frac{1}{D^{(R)}(\chi)}\int_0^{\infty}e^{-\omega s}\phi_M(\chi,s)\,ds, \quad (24)$$

where $D^{(R)}(\chi)$ is the regular Fredholm determinant,

$$D^{(R)}(\chi) = 1-\lambda\int_0^{\infty}\int_0^{s}e^{-\omega s}e^{-\omega s'}G_M^{(R)}(s,s')\,ds\,ds'. \quad (25)$$

The double Laplace transform involved in eq. (25) cannot be solved in a straightforward way. We first apply the following analytic continuation formula [37–40]:

$$F(a_1,\,b_1;\,c_1;\,x) = \frac{\Gamma(c_1)\Gamma(c_1-a_1-b_1)}{\Gamma(c_1-a_1)\Gamma(c_1-b_1)}F(a_1,\,b_1;\,a_1 + b_1-c_1+1;\,1-x) + (1-x)^{c_1-a_1-b_1} \times\frac{\Gamma(c_1)\Gamma(a_1+b_1-c_1)}{\Gamma(a_1)\Gamma(b_1)}F(c_1 - a_1,\,c_1-b_1;\,c_1-a_1-b_1+1;\,1-x) \quad (26)$$

in eq. (18) along with the standard integrals [37–40]

$$f_{\sigma}\,(a_1,b_1;c_1;x) = \frac{1}{c_1-1}\Bigg[\,{}_2F_1\,(a_1,b_1;c_1;x)$$

$$\times \int_0^x s^{\sigma-1}(1-s)^{a_1+b_1-c_1}{}_2F_1\,(a_1-c_1+1, b_1-c_1+1; 2-c_1; s)\,\mathrm{d}s$$
$$-x^{1-c_1}{}_2F_1\,(a_1-c_1+1, b_1-c_1+1; 2-c_1; x)$$
$$\times \int_0^x s^{\sigma+c_1-2}(1-s)^{a_1+b_1-c_1}{}_2F_1\,(a_1,b_1;c_1;s)\,\mathrm{d}s\Big] \quad (27)$$

and

$$\int_0^s (s-z)^{\nu-1} z^{\rho-1} f_\sigma\,(a_1,b_1;c_1;pz)\,\mathrm{d}z = \frac{\Gamma(\rho+\sigma)\,\Gamma(\nu)}{\sigma\,(\sigma+c-1)\,\Gamma(\rho+\sigma+\nu)}\, p^\sigma\, s^{\rho+\sigma+\nu-1} \times {}_4F_3(1,\sigma+a_1,\sigma+b_1, +\rho\sigma; \sigma+1, \sigma+c_1, \rho+\sigma+\nu; ps)$$
$$Re\,\sigma > 0,\ Re(\sigma+c_1) > 1,\ Re(\nu) > 0,\ Re(\rho) > 0,\ |ps| < 1 \quad (28)$$

to have

$$D^{(R)}(\chi) = 1 - \lambda\, d^3 \sum_{n=0}^{\infty} \frac{\Gamma(n+1-(\omega+i\chi)d)}{\Gamma(1-(\omega+i\chi)d)}\,\frac{1}{n!} \times \frac{\Gamma(n+3)\,\Gamma((\omega-i\chi)d)}{(n-\alpha+1)\,(n+\alpha+2)\Gamma(n+3+(\omega-i\chi)d)} \times {}_4F_3\,(1, n-\alpha+1+M, n-\alpha+1+N, n+3; n+2-\alpha, n+3+\alpha, n+3+(\omega-i\chi)d; 1). \quad (29)$$

The definite integral in eq. (24) leads to

$$\int_0^\infty \mathrm{e}^{-\omega s}\phi_M(\chi,s)\,\mathrm{d}s = d^{\alpha+2}\frac{\Gamma(\alpha+2)\,\Gamma((\omega-i\chi)d)}{\Gamma(\alpha+2+(\omega-i\chi)d)} \times {}_3F_2(M,N,\alpha+2;P,\alpha+2+(\omega-i\chi)d;1). \quad (30)$$

In evaluating the above integral we have used [37–40]

$$\int_0^1 (1-x)^{\sigma-1}x^{\rho-1}{}_2F_1(\alpha,\beta;\nu;x)\,\mathrm{d}x = \frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\,{}_3F_2(\alpha,\beta,\rho;\nu,\rho+\sigma,1)$$

with

$$Re\,\rho > 0,\ Re\,\sigma > 0,\ Re(\nu+\sigma-\alpha-\beta) > 0. \quad (31)$$

Equation (15) in conjunction with eqs (24), (29) and (30) gives the regular solution $\phi(\chi,s)$ for Manning–Rosen plus Yamaguchi potential

$$\phi(\chi,s) = \phi_M(\chi,s) + \frac{\lambda\,d^{\alpha+2}}{D^{(R)}(\chi)} \times \frac{\Gamma(\alpha+2)\,\Gamma((\omega-i\chi)d)}{\Gamma(\alpha+2+(\omega-i\chi)d)} \times {}_3F_2\,(M,N,\alpha+2;P,\alpha+2+(\omega-i\chi)d;1) \times d^2\,\mathrm{e}^{i\chi s}\,(1-\mathrm{e}^{-s/d})^{\alpha+1} \times \sum_{n=0}^{\infty}\frac{\Gamma\,(n+1-(\omega+i\chi)d)}{\Gamma(1-(\omega+i\chi)d)}\,\frac{1}{n!} \times f_{n-\alpha+1}(M,N;P;1-\mathrm{e}^{-s/d})\,. \quad (32)$$

2.2 *Irregular solution*

The irregular solution for Manning–Rosen-modified Yamaguchi potential satisfies

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\chi^2-V_M(s)\right] f(\chi,s) = \lambda\,\mathrm{e}^{-\omega s}\int_0^\infty \mathrm{e}^{-\omega s'} f(\chi,s')\,\mathrm{d}s'. \quad (33)$$

The solution of eq. (33) is written as

$$f(\chi,s) = f_M(\chi,s) + \lambda J^{(I)}(\chi)\int_s^\infty \mathrm{e}^{-\omega s'}G_M^{(I)}(s,s')\,\mathrm{d}s' \quad (34)$$

with $G_M^{(I)}(s,s')$, the irregular Green's function for Manning–Rosen potential [33–35,41]

$$G_M^{(I)}(s,s') = \frac{1}{f_M\,(\chi)}\left[\phi_M(\chi,s')\,f_M(\chi,s) - \phi_M(\chi,s)\,f_M(\chi,s')\right] \quad (35)$$

and

$$J^{(I)}(\chi) = \int_0^\infty \mathrm{e}^{-\omega s'} f(\chi,s')\,\mathrm{d}s'. \quad (36)$$

Multiplying $\mathrm{e}^{-\omega s}$ and integrating over the limit 0 to ∞ on both sides of eq. (34) the quantity $J^{(I)}(\chi)$ can be simplified to

$$J^{(I)}(\chi) = \frac{1}{D^{(I)}(\chi)}\int_0^\infty \mathrm{e}^{-\omega s} f_M(\chi,s)\,\mathrm{d}s, \quad (37)$$

where $D^{(I)}(\chi)$ is the irregular Fredholm determinant which reads as

$$D^{(I)}(\chi) = 1-\lambda\int_0^\infty\int_s^\infty \mathrm{e}^{-\omega s}\mathrm{e}^{-\omega s'}G_M^{(I)}(s,s')\,\mathrm{d}s\,\mathrm{d}s'. \quad (38)$$

The Fredholm determinants $D^{(I)}(\chi)$ and $D^{(R)}(\chi)$ are real quantities and have the same values. Applying eqs (19) and (24) with the exploitation of standard integral [37–40,42], one has

$$\int_0^\infty \mathrm{e}^{-\omega s} f_M(\chi,s)\,\mathrm{d}s = d\,\frac{\Gamma(1-\alpha)\,\Gamma((\omega-i\chi)d)}{\Gamma(1-\alpha+(\omega-i\chi)d)} \times {}_3F_2\,(M-1-2\alpha, N-1-2\alpha, (\omega-i\chi)d; 1-2i\chi d, 1-\alpha+(\omega-i\chi)d; 1). \quad (39)$$

Following the method of in §2.1, the complete irregular solution becomes

$$f(\chi,s) = d^{\alpha+1}e^{i\chi s}(1-e^{-s/d})^{\alpha+1} \times \Big[R_1\,{}_2F_1\,(M,N;P;1-e^{-s/d}) + R_2\,(1-e^{-s/d})^{-1-2\alpha} \times {}_2F_1(M-1-2\alpha,N-1-2\alpha\,;1-2i\chi d;e^{-s/d}) + d^{1-\alpha}\lambda\,J^{(I)}(\chi) \times f_{1-\alpha+n}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(i\chi+\omega)d)}{\Gamma(1-(i\chi+\omega)d)\,n!} \times \left(M,N;P;1-e^{-s/d}\right)\Big]. \tag{40}$$

The two unknown constants R_1 and R_2 will be calculated by applying the boundary conditions.

As $s\to 0$, the solution $f(\chi,s)$ produces the function $f(\chi)$ for the Manning–Rosen-modified Yamaguchi potential. For $s=0$, eqs (20) and (40) give

$$R_2 = \frac{f(\chi)}{d^{\alpha+1}f_M(\chi)}. \tag{41}$$

To calculate the limit $s\to\infty$ in eq. (36) we apply the following identity. The term involving infinite sum on the RHS of eq. (40) is equal to the Laplace transform of the regular Manning–Rosen Green's function $G_M^{(R)}(s,s')$

$$\lambda\,J^{(I)}(\chi)\int_0^s G_M^{(R)}(s,s')\,e^{-\omega s'}ds' = d^2\,e^{i\chi s}\,(1-e^{-s/d})^{\alpha+1}\,\lambda\,J^{(I)}(\chi) \times\sum_{n=0}^{\infty}\frac{\Gamma\,(n+1-(\omega+i\chi)d)}{\Gamma(1-(\omega+i\chi)d)}\,\frac{1}{n!} \times f_{n-\alpha+1}(M,N;P;1-e^{-s/d}). \tag{42}$$

When $s\to\infty$, eq. (40) yields

$$R_1 = -\frac{\lambda\,J^{(I)}(\chi)}{f_M(\chi)\,(\omega-i\chi)} \times\frac{\Gamma(1-\alpha)\,\Gamma(1+(\omega-i\chi)d)}{\Gamma(1-\alpha+(\omega-i\chi)d)} \times{}_3F_2\,(M,N,\alpha+2;P,\alpha+2+(\omega-i\chi)d;1). \tag{43}$$

Equation (40) along with eqs (41) and (43) produces the Jost solution for Manning–Rosen plus Yamaguchi potential,

$$f(\chi,s) = d^{\alpha+1}e^{i\chi s}(1-e^{-s/d})^{\alpha+1} \times\Big[-\frac{\lambda\,J^{(I)}(\chi)}{f_M(\chi)\,(\omega-i\chi)} \times\frac{\Gamma(1-\alpha)\,\Gamma(1+(\omega-i\chi)d)}{\Gamma(1-\alpha+(\omega-i\chi)d)} \times{}_3F_2\,(M-1-2\alpha,N-1-2\alpha,(\omega-i\chi)d;1-2i\chi d,1-\alpha+(\omega-i\chi)d;1) \times{}_2F_1\,(M,N;P;1-e^{-s/d}) +\frac{f(\chi)}{d^{\alpha+1}f_M(\chi)}\,(1-e^{-s/d})^{-1-2\alpha} \times{}_2F_1\,(M-1-2\alpha,N-1-2\alpha\,;1-2i\chi d;e^{-s/d}) + d^{1-\alpha}\lambda\,J^{(I)}(\chi) \times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(i\chi+\omega)d)}{\Gamma(1-(i\chi+\omega)d)\,n!} \times f_{1-\alpha+n}\left(M,N;P;1-e^{-s/d}\right)\Big]. \tag{44}$$

When $s\to 0$ the Jost solution $f(\chi,s)$ goes to the Jost function $f(\chi)$ [34,41]. Thus, the Jost function for the combined potential is

$$f(\chi) = f_M(\chi)+\lambda\,d^{\alpha+2} \times\frac{\Gamma(1-\alpha)\,\Gamma(1+(\omega-i\chi)d)}{D^{(I)}(\chi)\,(\omega-i\chi)\times\Gamma(1-\alpha+(\omega-i\chi)d)} \times{}_3F_2\,(M-1-2\alpha,N-1-2\alpha,(\omega-i\chi)d;1-2i\chi d,1-\alpha+(\omega-i\chi)d;1) \times\frac{\Gamma(\alpha+2)\,\Gamma((\omega-i\chi)d)}{\Gamma(\alpha+2+(\omega-i\chi)d)} \times{}_3F_2\,(M,N,\alpha+2;P,\alpha+2+(\omega-i\chi)d;1). \tag{45}$$

2.3 *Physical solution*

With the physical boundary condition, the solution of eq. (1) is given by

$$\psi^{(+)}(\chi,s) = \psi_M^{(+)}(\chi,s) +\lambda J^{(+)}(\chi)\int_0^{\infty}e^{-\omega s'}G_M^{(+)}(s,s')\,ds'. \tag{46}$$

Here $G_M^{(+)}(s,s')$ and $\psi^{(+)}(\chi,s)$ are the Green's function and the solution. The other quantity $J^{(+)}(\chi)$ is

$$J^{(+)}(\chi) = \frac{1}{D^{(+)}(\chi)}\int_0^{\infty}e^{-\omega s}\psi_M^{(+)}(\chi,s)\,ds, \tag{47}$$

where $D^{(+)}(\chi)$ is the Fredholm determinant related to the physical boundary condition

$$D^{(+)}(\chi)=1-\lambda\int_0^{\infty}\int_0^{\infty}e^{-\omega s}\,e^{-\omega s'}G_M^{(+)}(s,s')\,ds\,ds'. \tag{48}$$

Equation (48) involves the double Laplace transform of $G_M^{(+)}(s,s')$. We adapt the differential equation approach to evaluate the single Laplace transform of the pure Manning–Rosen Green's function. The Green's function $G_M^{(+)}(s,s')$ satisfies [27–29,41],

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \chi^2 - V_M(s)\right) G_M^{(+)}(s,s') = \delta(s-s'). \tag{49}$$

Taking the Laplace transform of the above equation, we have

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \chi^2 - V_M(s)\right) \bar{G}_M^{(+)}(s,\omega) = \mathrm{e}^{-\omega s}. \tag{50}$$

Using the following transformation

$$\bar{G}_M^{(+)}(s,\omega) = d^{\alpha+1}\mathrm{e}^{i\chi s}(1-\mathrm{e}^{-s/d})^{\alpha+1}\bar{F}(s,\omega) \tag{51}$$

in eq. (50) gives

$$\begin{aligned}&\mathrm{e}^{s/d}\, d^2(1-\mathrm{e}^{-s/d})\frac{\mathrm{d}^2\bar{F}}{\mathrm{d}s^2}\\ &\quad+\left[2(\alpha+1)d + 2i\chi d^2\mathrm{e}^{s/d}(1-\mathrm{e}^{-s/d})\right]\frac{\mathrm{d}\bar{F}}{\mathrm{d}s}\\ &\quad+[2i\chi d(\alpha+1)-(\alpha+1)+A]\,\bar{F}\\ &= \mathrm{e}^{s/d}\, d^{1-\alpha}(1-\mathrm{e}^{-s/d})^{-\alpha}\,\mathrm{e}^{-(\omega+i\chi)s}.\end{aligned} \tag{52}$$

Making change of variable $z=(1-\mathrm{e}^{-s/d})$ in the above equation, one gets

$$\begin{aligned}&z(1-z)\frac{\mathrm{d}^2\bar{F}}{\mathrm{d}z^2} + [2(\alpha+1)(1-z)+2i\chi d\, z - z]\frac{\mathrm{d}\bar{F}}{\mathrm{d}z}\\ &\quad-[(\alpha+1)-A-2i\chi d(\alpha+1)]\,\bar{F}(s,\omega)\\ &= d^{1-\alpha}(1-z)^{(\omega+i\chi)d-1}z^{-\alpha}.\end{aligned} \tag{53}$$

Combining eqs (10), (51) and (53) leads to

$$\begin{aligned}\bar{G}_M^{(+)}(s,\omega) &= \int_0^\infty \mathrm{e}^{-\omega s'}G_M^{(+)}(s,s')\,\mathrm{d}s'\\ &= d^{\alpha+1}\mathrm{e}^{i\chi s}(1-\mathrm{e}^{-s/d})^{\alpha+1}\\ &\quad\times\Big[T_1\,{}_2F_1(M,N;P;1-\mathrm{e}^{-s/d})\\ &\quad+T_2(1-\mathrm{e}^{-s/d})^{-1-2\alpha}{}_2F_1(M-1-2\alpha,N-1\\ &\quad-2\alpha;1-2i\chi d;\mathrm{e}^{-s/d})\\ &\quad+d^{1-\alpha}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(i\chi+\omega)d)}{\Gamma(1-(i\chi+\omega)d)\,n!}\\ &\quad\times f_{1-\alpha+n}\big(M,N;P;1-\mathrm{e}^{-s/d}\big)\Big].\end{aligned} \tag{54}$$

In eq. (54) the constants T_1 and T_2 will be found from the boundary conditions at $s=0$ and $s=\infty$. When $s=0$, $\bar{G}_M^{(+)}(s,\omega)=0$ and we obtain $T_2=0$.

The physical solution of the pure Manning–Rosen potential [27–29] reads as

$$\begin{aligned}\psi_M^{(+)}(\chi,s) &= \frac{d^{\alpha+1}\chi\,\mathrm{e}^{i\chi s}}{f_M(\chi)}(1-\mathrm{e}^{-s/d})^{\alpha+1}\\ &\quad\times{}_2F_1(M,N;P;1-\mathrm{e}^{-s/d}).\end{aligned} \tag{55}$$

The associated Green's function is [41]

$$G_M^{(+)}(s,s') = -\chi^{-1}\psi_M^{(+)}(\chi,s_<)\,f_M(\chi,s_>), \tag{56}$$

where $s_<$ and $s_>$ have standard meanings. The Green's function $G_M^{(+)}(s,s')$ is related to $G_M^{(R)}(s,s')$ as

$$\begin{aligned}G_M^{(+)}(s,s') &= G_M^{(R)}(s,s')\\ &\quad-\frac{1}{f_M(\chi)}\phi_M(\chi,s)\,f_M(\chi,s').\end{aligned} \tag{57}$$

Under the limit $s\to\infty$, eq. (54) along with eq. (42) yields

$$\begin{aligned}T_1 &= -\frac{1}{f_M(\chi)(\omega-i\chi)}\\ &\quad\times\frac{\Gamma(1-\alpha)\,\Gamma(1+(\omega-i\chi)d)}{\Gamma(1-\alpha+(\omega-i\chi)d)}\\ &\quad\times{}_3F_2(M-1-2\alpha,N-1\\ &\quad-2\alpha,(\omega-i\chi)d;\\ &\quad1-2i\chi d,1-\alpha+(\omega-i\chi)d;1).\end{aligned} \tag{58}$$

Hence, from eqs (54) and (58), along with $T_2=0$ we have

$$\begin{aligned}\bar{G}_M^{(+)}(s,\omega) &= d^{\alpha+1}\mathrm{e}^{i\chi s}(1-\mathrm{e}^{-s/d})^{\alpha+1}\Bigg[-\frac{\Gamma(1-\alpha)\,\Gamma(1+(\omega-i\chi)d)\,{}_2F_1(M,N;P;1-\mathrm{e}^{-s/d})}{f_M(\chi)(\omega-i\chi)\,\Gamma(1-\alpha+(\omega-i\chi)d)}\\ &\quad\times{}_3F_2(M-1-2\alpha,N-1-2\alpha,(\omega-i\chi)d;1-2i\chi d,1-\alpha+(\omega-i\chi)d;1)\\ &\quad+d^{1-\alpha}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(i\chi+\omega)d)}{\Gamma(1-(i\chi+\omega)d)\,n!}f_{1-\alpha+n}\big(M,N;P;1-\mathrm{e}^{-s/d}\big)\Bigg].\end{aligned} \tag{59}$$

Taking the Laplace transform of eq. (59) once again with respect to r one gets the double Laplace transform of $G_M^{(+)}(s, s')$. The double Laplace transform of $G_M^{(+)}(s, s')$ can be evaluated by using eq. (28) and the recursion formulae of the generalised hypergeometric functions [40,43]

$$\begin{aligned}{}_3F_2\,(a_1,\ b_1,\ c_1;\ e_1,\ f_1;\ 1) &= \frac{\Gamma(S)\Gamma(f_1)}{\Gamma(S+a_1)\Gamma(f_1-a_1)} \\ &\quad\times {}_3F_2\,(a_1,\ e_1-b_1,\ e_1-c_1;\ S+a_1,\ e_1;\ 1); \\ S &= e_1+f_1-a_1-b_1-c_1 \end{aligned} \tag{60}$$

to have

$$\begin{aligned}&\int_0^\infty\int_0^\infty \mathrm{e}^{-\omega s}\,\mathrm{e}^{-\omega s'}\,G_M^{(+)}(s,s')\,\mathrm{d}s\,\mathrm{d}s' \\ &= d^3\sum_{n=0}^\infty \frac{\Gamma(n+1-(\omega+i\chi)d)}{\Gamma(1-(\omega+i\chi)d)}\,\frac{1}{n!} \\ &\times\frac{\Gamma(n+3)\,\Gamma((\omega-i\chi)d)}{(n-\alpha+1)\,(n+\alpha+2)\,\Gamma(n+3+(\omega-i\chi)d)} \\ &\times{}_4F_3\,(1, n-\alpha+1+M, n-\alpha+1+N, n+3; \\ &n+2-\alpha, n+3+\alpha, n+3+(\omega-i\chi)d;\ 1) \\ &-\frac{d^{\alpha+2}}{f_M(\chi)}\,\frac{\Gamma(1-\alpha)\,\Gamma(1+(\omega-i\chi)d)}{(\omega-i\chi)\,\Gamma(1-\alpha+(\omega-i\chi)d)} \\ &\times{}_3F_2\,(M-1-2\alpha, N-1-2\alpha, (\omega-i\chi)d; \\ &1-2i\chi d, 1-\alpha+(\omega-i\chi)d;\ 1) \\ &\times\frac{\Gamma(\alpha+2)\,\Gamma((\omega-i\chi)d)}{\Gamma(\alpha+2+(\omega-i\chi)d)} \\ &\times{}_3F_2(M, N, \alpha+2;\ P, \alpha+2+(\omega-i\chi)d;\ 1).\end{aligned} \tag{61}$$

The integral involved in eq. (47) can easily be calculated by using eq. (31) to get

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\omega s}\psi_M^{(+)}(\chi,s)\,\mathrm{d}s = \frac{\chi\,d^{\alpha+2}}{f_M(\chi)} \\ &\times\frac{\Gamma(\alpha+2)\,\Gamma((\omega-i\chi)d)}{\Gamma(\alpha+2+(\omega-i\chi)d)} \\ &\times{}_3F_2\,(M, N, \alpha+2;\ P, \alpha+2+(\omega-i\chi)d;\ 1).\end{aligned} \tag{62}$$

The complete expression for on-shell physical solution in eq. (46) together with eqs (47), (48), (55), (59), (61) and (62) reads as

$$\begin{aligned}&\psi^{(+)}(\chi,s) = \psi_M^{(+)}(\chi,s) + \lambda\,J^{(+)}(\chi) \\ &\times\left\{d^{\alpha+1}\left(1-\mathrm{e}^{-s/d}\right)^{\alpha+1}\mathrm{e}^{i\chi s}\left[-\frac{1}{f_M(\chi)\,(\omega-i\chi)}\right.\right. \\ &\frac{\Gamma(1-\alpha)\,\Gamma(1+(\omega-i\chi)d)}{\Gamma(1-\alpha+(\omega-i\chi)d)} \\ &\times{}_3F_2\,(M-1-2\alpha, N-1-2\alpha, (\omega-i\chi)d; \\ &1-2i\chi d, 1-\alpha+(\omega-i\chi)d;\ 1) \\ &\times{}_2F_1(M,\ N;\ P;\ 1-\mathrm{e}^{-s/d}) \\ &+d^{1-\alpha}\sum_{n=0}^\infty\frac{\Gamma(n+1\ -\ (i\chi+\omega)d)}{\Gamma(1\ -\ (i\chi+\omega)d)\,n!} \\ &\left.\left.\times f_{1-\alpha+n}\left(M, N;\ P; 1-\mathrm{e}^{-s/d}\right)\right]\right\}.\end{aligned} \tag{63}$$

We have checked that our eqs (32), (44) and (63) reproduce proper Hulthén limits when $\alpha = 0$ [31,44] and Manning–Rosen one for $\lambda = 0$ [33,34]. Also, when the Manning–Rosen interaction is turned off, i.e. A and $\alpha = 0$ they reduce to pure Yamaguchi case [45].

3. Results and discussion

The bound-state energies and scattering phase shifts are computed by considering $\hbar^2/2\mu = 31.1025$ MeV fm^2, $A = -0.01185$, $d = 0.256$ fm, $\alpha = 0.05$, $\lambda = -1.022801$ fm^{-3} and $\omega = 0.1855$ fm^{-1} for $(\mathrm{p}\text{–}{}^2\mathrm{H}_1)$ and $\hbar^2/2\mu = 22.03$ MeV fm^2, $A = -0.214384$, $d = 0.40999$ fm, $\alpha = 0.08$, $\lambda = -22.9855$ fm^{-3}and $\omega = 0.199\ fm^{-1}$ for the $(\mathrm{p}\text{–}{}^{16}\mathrm{O})$ systems. These parameters of the potential for both the systems reproduce their binding energies. As the zeros of the Jost function $f(\chi)$ or $D_M^{(+)}(\chi)$ represent the bound states for $Im\chi > 0$ one expects $D_M^{(+)}(\chi) = 0$ or, $f(\chi) = 0$ at $\chi = i\kappa$, where

$$\kappa = \sqrt{\frac{2\mu E_B}{\hbar^2}}.$$

We obtain $E_B = 7.718$ MeV for He^3 and $E_B = 128.22075$ MeV for the ^{17}F nucleus. Our phase shift values for both the systems are computed by exploiting the expression for $D_M^{(+)}(\chi)$ and are depicted in tables 1 and 2 along with the standard data [6,13,14]. Firstly, table 1 shows the variation of phase shifts δ with the labor atory energy E_{Lab} for the $(\mathrm{p}\text{–}{}^2\mathrm{H}_1)$ system. Looking at table 1, it is easily observed that our computed phase shift results for the $(\mathrm{p}\text{–}{}^2\mathrm{H}_1)$ systems agree well with the experimental values [13,14]. The discrepancy in our computed phase shift values with the experimental results [13,14] is less than 1% except for $E_{\mathrm{Lab}} = 2$ and 3 MeV.

In table 2 it is noticed that for the $(\mathrm{p}\text{–}{}^{16}\mathrm{O})$ system the phase shifts obtained in the present work are in good agreement with the experimental results [6]. The accuracy of our phase shift analysis is within 2%. The best-fitted values of the parameters are obtained by considering the binding energy of the ^{17}F nucleus. Thus, our two-body potential model has the ability to describe the nucleon–nucleus scattering quite efficiently.

Table 1. Scattering phase parameters for the $\left(p-{}^{2}H_1\right)$ system.

E_{Lab} (MeV)	$\delta_{1/2^+}(^\circ)$ (present work)	$\delta_{1/2^+}(^\circ)$ Ref. [13]	$\delta_{1/2^+}(^\circ)$ Ref. [14]
0.075	− 0.216	− 0.113	
0.15	− 0.553	− 0.537	− 0.527
0.3	− 2.573	− 1.96	
0.45	− 4.665	− 3.73	
0.6	− 6.638	− 5.62	
0.75	− 8.448	− 7.53	
0.9	− 10.09	− 9.40	
1.0	− 10.82	− 10.6	− 10.5
1.05	− 11.58	− 11.2	
1.2	− 13.07	− 13.0	
1.35	− 14.29	− 14.6	
1.5	− 15.82	− 16.2	− 16.2
2.0	− 18.68	− 21.1	− 21.0
3	− 22.66	− 28.8	− 28.7

Table 2. Scattering phase parameters for the $(p-{}^{16}O)$ system.

E_{Lab} (MeV)	Phase shift $\delta^{+}_{1/2}(^\circ)$	Phase shift $\delta^{+}_{1/2}(^\circ)$ Ref. [6]
0.3855	183.27	179.59
0.4871	181.29	179.74
0.6162	177.82	179.65
0.6631	176.66	178.11
0.7162	175.30	177.96
0.759	174.25	176.46
0.8108	173.02	174.55
0.8612	171.86	173.37
0.9058	170.97	174.19
0.979	169.50	172.17
1.1063	167.21	169.82
1.2508	165.01	166.43
1.3704	163.19	163.61
1.5898	160.55	158.42
1.7903	158.28	155.78
1.9909	154.41	151.09

4. Conclusion

The present paper deals with numerical results of the $\left(p-{}^{2}H_1\right)$ and $(p-{}^{16}O)$ elastic scattering for the realistic two-body model interaction. The separable non-local potential represents strong interaction and the Manning–Rosen one denotes the electromagnetic part. In our method, we include the electromagnetic effect rigorously to the separable nucleon–nucleon interaction. Results of the phase-shift parameters agree well with the available accurate results obtained using sophisticated approaches to the problems. Thus, separable non-local potential model which admits exact solutions of the quantum mechanical wave equation with the inclusion of electromagnetic force may provide excellent explanation of the nucleon–nucleus elastic scattering. Our method is equally applicable for higher rank potential and there is no restriction on the form factors of the potential with respect to its symmetry.

References

[1] P E Hodgson, *The optical model of elastic scattering* (Clarendon Press, Oxford, 1963)
[2] H C Chow, G M Gtiffits and T H Hall, *Can. J. Phys.* **53**, 1672 (1975)
[3] Y Guohua, Z Dezhang, X Hoogjie and P Haochang, *Nucl. Instrum. Methods B* **61**, 175 (1991)
[4] R Amirikas, D N Jamiesoo and S P Dooley, *Nucl. Instrum. Methods B* **77**, 110 (1993)
[5] A F Gurbich, *Nucl. Instrum. Methods Phys. Res. B* **129**, 311 (1997)
[6] S Dubovichenko *et al*, *Chin. Phys. C* **41**, 014001 (2017)
[7] G H Berthold, A Stadler and H Zankel, *Phys. Rev. C* **41**, 1365 (1990)
[8] L D Knutson, L O Lamm and J E McAninch, *Phys. Rev. Lett.* **71**, 3762 (1993)
[9] P A Schmelzbach *et al*, *Nucl. Phys. A* **197**, 273 (1972)
[10] S König, *J. Phys. G: Nucl. Part. Phys.* **44**, 064007 (2017)
[11] S König and H W Hammer, *Phys. Rev. C* **83**, 064001 (2011)
[12] E Huttel *et al*, *Nucl. Phys. A* **406**, 443 (1983)
[13] C R Chen, G L Payne, J L Friar and B F Gibson, *Phys. Rev. C* **39**, 1261 (1989)
[14] S Ishikawa, *Few-Body Syst.* **32**, 229 (2003)
[15] A Kievsky, M Viviani and S Rosati, *Phys. Rev. C* **52**, R15 (1995)
[16] K Sagara, H Oguri, S Shimizu, K Maeda, H Nakamura, T Nakashima and S Morinobu, *Phys. Rev.* **50**, 576 (1994)
[17] P F Bedaque and UvanKolck, *Annu. Rev. Nucl. Part. Sci.* **52**, 339 (2002)
[18] E Epelbaum, H W Hammer and U G Meißner, *Rev. Mod. Phys.* **81**, 1773 (2009)
[19] W Plessas, L Streit and H Zingl, *Acta. Phys. Austriaca* **40**, 272 (1974)
[20] L Crepinsek, C B Lang, H Oberhumer, W Plessas and H Zingl, *Acta Phys. Austriaca* **42**, 139 (1975)
[21] H van Haeringen and R van Wageningen, *J. Math. Phys.* **16**, 1441 (1975)
[22] H Zankel, W Plessas and J Haidenbauer, *Phys. Rev. C* **28**, 538 (1983)
[23] W Schwinger, W Plessas, L P Kok and H van Haeringen, *Phys. Rev. C* **27**, 515 (1983)
[24] B Talukdar, U Laha and T Sasakawa, *J. Math. Phys.* **27**, 2080 (1986)

[25] U Laha, B J Roy and B Talukdar, *J. Phys. A* **22**, 3597 (1989)
[26] U Laha and B Talukdar, *Pramana – J. Phys.* **36**, 289 (1991)
[27] U Laha, *Phys. Rev. A* **74**, 012710 (2006)
[28] U Laha, *Pramana – J. Phys.* **72**, 457 (2009)
[29] U Laha and J Bhoi, *J. Math. Phys*. **54**, 013514 (2013)
[30] U Laha, A K Behera, M Majumder and J Bhoi, *Pramana – J. Phys.* **94**: 144 (2020)
[31] P Sahoo, U Laha and A K Behera, *Phys. At. Nucl.* **83**, 802 (2020)
[32] M F Manning and N Rosen, *Phys. Rev* **44**, 953 (1933)
[33] B Khirali, A K Behera, J Bhoi and U Laha, *J. Phys. G: Nucl. Part. Phys.* **46**, 115104 (2019)
[34] B Khirali, A K Behera, J Bhoi and U Laha, *Ann. Phys.* **412**, 168044 (2020)
[35] B Khirali, A K Behera, J Bhoi and U Laha, *Phys. Scr.* **95**, 075308 (2020)
[36] A K Behera, J Bhoi, U Laha and B Khirali, *Commun. Theor. Phys.* **72**, 075301 (2020)
[37] A Erdeyli, *Higher transcendental functions* (McGraw-Hill, New York, 1953) Vol. 1
[38] L J Slater, *Generalized hypergeometric functions* (Cambridge University Press, London, 1966)
[39] W Magnus and F Oberhettinger, *Formulae and theorems for the special functions of mathematical physics* (Chelsea, New York, 1949)
[40] A W Babister, *Transcendental functions satisfying nonhomogeneous linear differential equations* (MacMillan, New York, 1967)
[41] R G Newton, *Scattering theory of waves and particles* (McGraw-Hill, New York, 1982)
[42] I S Gradshteyn and I M Ryzhik, *Tables of integrals, series and products* (Academic Press, London, 2000)
[43] W N Bailey, *Generalized hypergeometric series* (Cambridge University Press, London, 1935)
[44] U Laha, S Ray, S Panda and J Bhoi, *Theor. Math. Phys.* **193**, 1498 (2017)
[45] D K Ghosh, S Saha, K Niyogi and B Talukdar, *Czech. J Phys. B* **33**, 528 (1983)