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Algebraic structures on the flows of dispersionless modified KP equation

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Abstract. In this paper, we derive the non-isospectral flows of dispersionless modified Kadomtsev–Petviashvili (dmKP) hierarchies by applying quasiclassical limit in the associated Lax equations of the mKP system. Along with the isospectral flows, we investigate the underlying infinite-dimensional Lie algebraic structure of the dmKP system through the construction of implicit flow representations. In addition to this, we also discuss the correspondence between the non-isospectral flows of dKP and dmKP hierarchies by the dispersionless Miura map.

Keywords. Non-isopectral flows; dispersionless equations; Lie algebraic structures.

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1. Introduction

In the past three decades, the dispersionless KP (dKP) hierarchy [1–6] and its various reductions have gained lots of interest because they are linked with topological field theory, Whitham hierarchy, its connections to string theory and two-dimensional gravity [1,7–11]. Many properties of this equation, such as twistor construction of dispersionless systems using Orolov functions [5], solutions through hodograph transformations [12,13], tau function theory, dispersionless analogue of Virasoro constrains [10], etc. were discussed in the recent past. As in the case of dKP hierarchy, the study of dispersionless modified KP (dmKP) hierarchy and its extensions also share many richness related in the fields of mathematical physics [14–19]. However, identifying the Lie algebraic structures by deriving iso and non-isospectral flows of dmKP hierarchy is still in the initial stage. Moreover, the derivation of non-isospectral flows not only plays a crucial in deducting the underlying algebraic structures of a given system, but also quite important from the physical point of view. In the case of soliton systems, the solutions of the isospectral equations explain the behaviour of solitary waves in the lossless and uniform media whereas the solutions of the non-isospectral equations reveal the nature of solitary waves in a certain type of non-uniform media. In addition, composing the isospectral flows with time-dependant symmetries of an integrable system is also often related with an infinite-dimensional Lie algebraic structure.

Recently, Fu *et al* studied the algebraic properties of dKP equation such as iso and non-isospectral flows, underlying infinite-dimensional Lie algebraic structures and the Hamiltonian formalisms of the dKP hierarchies using Lax triad approach [20]. On the other hand, Chang and Tu [14] studied the canonical property (preservation of bi-Hamiltonian structure) of the dispersionless Miura map between the dmKP and dKP hierarchies. Using this, they derived the Orlov function of the dmKP hierarchy, which was useful to establish its twistor construction. Throughout this paper, we follow the field variables $u = u(x, y, t)$ and $v = v(x, y, t)$ for dispersionful hierarchies and $U = U(X, Y, T)$ and $V = V(X, Y, T)$ for dispersionless hierarchies. The dispersionless KP and mKP equations are given as

$$U_T = 3UU_X + \frac{3}{4}\partial^{-1}U_{YY}(\text{dKP}), \tag{1}$$

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$$V_T = \frac{3}{2}V_X\partial^{-1}V_Y - \frac{3}{2}V_XV^2 + \frac{3}{4}\partial^{-1}V_{YY}(\text{dmKP}), \tag{2}$$

where $U = U(X, Y, T)$ and $V = V(X, Y, T)$. Equations (1) and (2) are connected by dispersionless Miura transformation [14],

$$U = \frac{1}{2}(-V^2 + \partial_X^{-1}V_Y). \tag{3}$$

Motivated by these works [14,20], in this paper, we construct the non-isospectral flows of dmKP and obtain the associated infinite-dimensional Lie algebraic structure. Furthermore, we extend the analysis of dispersionless Miura map between the non-isospectral flows of dKP and dmKP hierarchies.

This paper is organised as follows: in §2, we discuss the basic results such as the iso and non-isospectral flows of the dKP and mKP equations. In §3, we derive the iso and non-isospectral flows of the dmKP equation. In §4, we investigate the underlying algebraic structure of the dmKP hierarchies. In §5, through dispersionless Miura map we identify the correspondence between the non-isospectral flows of the dKP and dmKP hierarchies. Section 6 gives the conclusion.

2. Preliminaries

Throughout this paper, we restrict with following notions to study the iso and non-isospectral flows of evolution equations and obtain their underlying Lie algebraic structures.

Consider an evolution equation

$$u_t = K(u), \ u \in \mathcal{M}, \tag{4}$$

where K is a vector field on a linear manifold $\mathcal{M}$ of infinite dimension consisting of C^∞ functions $u(x, y)$ defined on $\mathbb{R}^2$, vanishing rapidly at $\pm\infty$. Here, one need the solution of (4) to be depending in C^∞ way on the time parameter t. This happens because $\mathcal{M}$ is linear and all the fibres of the tangent bundle $T\mathcal{M}$ are replications of the same vector space S, i.e. $\mathcal{M} = S$.

The Gâteaux derivative of a vector field $f(u) \in \mathcal{M}$ in the direction of another vector field $g(u) \in \mathcal{M}$ is defined as

$$f(u)'[g(u)] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} f\left(u + \epsilon g(u)\right)|_{\epsilon=0}. \tag{5}$$

Using this, S forms a Lie algebra of C^∞ vector fields on $\mathcal{M}$ with Lie commutator $[\![\cdot,\cdot]\!]$, defined as

$$[\![f, g]\!](u) = f(u)'[g(u)] - g(u)'[f(u)]. \tag{6}$$

For convenience, hereafter we use $f'[g]$ instead of $f(u)'[g(u)]$ and $[\![f, g]\!]$ instead of $[\![f, g]\!](u)$.

A vector field $F : \mathcal{M}\times\mathbb{R} \to S$, i.e. $(u, t) \to F(u, t)$ is said to be a symmetry of (4). Then

$$F_t + [\![F, K]\!] = 0. \tag{7}$$

2.1 *Flows of the dKP equation*

In this section, we present the iso and non-isospectral flows of the dKP equation. To derive this, we follow Lax triad [20] approach, a convenient way to obtain the non-isospectral flows. The Lax triad system associated with isospectral flows of dKP is given as

$$\mathcal{L}_Y = \{\mathcal{B}_2, \mathcal{L}\}, \tag{8a}$$

$$\mathcal{L}_{T_m} = \{\mathcal{B}_m, \mathcal{L}\}, \tag{8b}$$

$$\mathcal{B}_{2,T_m} - \mathcal{B}_{m,Y} + \{\mathcal{B}_2, \mathcal{B}_m\} = 0, \quad m = 1, 2, \ldots, \tag{8c}$$

where $\mathcal{L}$ is the Laurent series in p, defined as $\mathcal{L} = p + \sum_{n=1}^{\infty} U_{n+1}(X, Y, T)p^{-n}$ and $\mathcal{B}_m = (\mathcal{L}^m)_{\geq 0}$, projection onto a polynomial in p, dropping negative powers. Here, T consists of infinite set of time variables, $T = (T_1, T_2, T_3, \ldots)$. The bracket $\{\cdot,\cdot\}$ in (8) is the Poisson bracket in 2D phase space (p, X), defined as

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial X} - \frac{\partial f}{\partial X}\frac{\partial g}{\partial p}. \tag{9}$$

From (8), we can generate the isospectral flows of the dKP equation and we list the first four as

$$U_{T_1} = K_1 = U_X, \tag{10a}$$

$$U_{T_2} = K_2 = U_Y, \tag{10b}$$

$$U_{T_3} = K_3 = 3UU_X + \frac{3}{4}\partial^{-1}U_{YY}, \tag{10c}$$

$$U_{T_4} = K_4 = 4UU_Y + 2U_X\partial^{-1}U_Y + \frac{1}{2}\partial^{-2}U_{YYY}. \tag{10d}$$

Here, one can observe that (10c) is the dKP equation. Now, we consider the Lax triad system for the non-isospectral flows of the dKP system as follows:

$$\mathcal{L}_Y = \{\mathcal{B}_2, \mathcal{L}\}, \tag{11a}$$

$$\mathcal{L}_{T_m} = \{\mathcal{A}_m, \mathcal{L}\} + \frac{1}{2}\mathcal{L}^{m-1}, \tag{11b}$$

$$\mathcal{B}_{2,T_m} - \mathcal{A}_{m,Y} + \{\mathcal{B}_2, \mathcal{A}_m\} = 0, \quad m = 1, 2, \ldots, \tag{11c}$$

where $\mathcal{A}_m$ is an m degree polynomial in p, given as $\mathcal{A}_m = \sum_{i=0}^{m} \hat{g}_i^{(m)} p^{m-i}$. The unknown functions $\hat{g}_i^{(m)}$ appearing in $\mathcal{A}_m$ can be explicitly found by using the following asymptotic condition:

$$\mathcal{A}_m|_{\mathbf{U}=0} = Yp^m + \frac{1}{2}Xp^{m-1}, \ \mathbf{U} = (U_2, U_3, \ldots). \quad (12)$$

From (11), we can deduce the non-isospectral flows of dKP and we present the first four as

$$U_{T_1} = \sigma_1 = YK_1, \quad (13a)$$

$$U_{T_2} = \sigma_2 = YK_2 + \frac{1}{2}XU_X + U, \quad (13b)$$

$$U_{T_3} = \sigma_3 = YK_3 + \frac{1}{2}XU_Y + \partial^{-1}U_Y, \quad (13c)$$

$$U_{T_4} = \sigma_4 = YK_4 + \frac{1}{2}XK_3 + 2U^2 + \frac{1}{2}U_X\partial^{-1}U + \frac{3}{4}\partial^{-2}U_{YY}. \quad (13d)$$

In the above flows, σ_3 becomes the master symmetry of the dKP equation.

2.2 *Flows of the mKP equation*

The non-isospectral flows of the mKP equation were investigated by Chen [21] using the Lax pair. Now, we extend this set-up in Lax triad approach for the mKP equation, introduced by Fu *et al* [20,22] for the KP, semidiscrete KP ($\partial\Delta KP$) and dKP equations. The interesting fact behind the construction of Lax triad is to overcome the difficulty in understanding the nature of variables y and t_2. In the isospectral case, t_2 is replaced by y whereas in the case of non-isospectral flows, t_2 can be thought of as completely independent of y. Now, the isospectral problem related to mKP equation is given as follows:

$$\tilde{L}\psi = \lambda\eta, \quad (14a)$$

$$\psi_y = \tilde{B}_2\psi, \quad \tilde{B}_2 = \partial^2 + 2v_0\partial, \quad (14b)$$

$$\psi_{t_m} = \tilde{B}_m\psi, \quad m = 1, 2, \ldots, \quad (14c)$$

where ψ is the eigenfunction and λ is the spectral parameter with $\lambda_{t_m} = 0$. Here, the Lax operator $\tilde{L}$ is given by

$$\tilde{L} = \partial + \sum_{j=0}^{\infty} v_j\partial^{-j}, \quad (15)$$

where $\partial = \partial_x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$ and $v_j = v_j(x, y, t)$ with the time variables $t = (t_1, t_2, \ldots)$ living in the Schwarz space $S(R^\infty)$, consisting of all rapidly decreasing functions. The operator $\tilde{B}_m$ is defined as $\tilde{B}_m = (\tilde{L}^m)_{\geq 1}$, strictly positive differential parts of $\tilde{L}^m$. Now, by the compatibility conditions of (14), we obtain the Lax triad equations associated with the isospectral flows of the mKP as follows:

$$\tilde{L}_y = [\tilde{B}_2, \tilde{L}], \quad (16a)$$

$$\tilde{L}_{t_m} = [\tilde{B}_m, L], \quad (16b)$$

$$\tilde{B}_{2,t_m} - \tilde{B}_{m,y} + [\tilde{B}_2, \tilde{B}_m] = 0, \quad m = 1, 2, \ldots, \quad (16c)$$

where $[, \cdot,]$ is defined as $[M, N] = MN - NM$. Next, we consider the spectral problem related to the non-isospectral flows of the mKP equation [21] with $\lambda_{t_m} = \frac{1}{2}\lambda^{m-1}$ as

$$L\psi = \lambda\psi, \quad (17a)$$

$$\psi_y = \tilde{B}_2\psi, \quad (17b)$$

$$\psi_{t_m} = \tilde{A}_m\psi, \quad m = 1, 2, \ldots, \quad (17c)$$

where the operator $\tilde{A}_m$ is of the form

$$\tilde{A}_m = \sum_{i=0}^{m-1} h_i^{(m)}\partial^{m-i}, \quad (18)$$

with the undetermined coefficients $h_i^{(m)}$ thought of functions in v_j and its derivatives. The compatibility conditions of (17) read as follows:

$$\tilde{L}_y = [\tilde{B}_2, \tilde{L}], \quad (19a)$$

$$\tilde{L}_{t_m} = [\tilde{A}_m, L] + \frac{1}{2}\tilde{L}^{m-1}, \quad (19b)$$

$$\tilde{B}_{2,t_m} - \tilde{A}_{m,y} + [\tilde{B}_2, \tilde{A}_m] = 0, \quad m = 1, 2, \ldots. \quad (19c)$$

The functions $h_i^{(m)}$ in $\tilde{A}_m$ can be found by substituting (18) in (19b) along with the following asymptotic condition [21]:

$$\tilde{A}_m|_{\mathbf{v}=0} = y\partial^m + \frac{1}{2}x\partial^{m-1} + \frac{m-2}{4}\partial^{m-2}, \ \mathbf{v} = (v_0, v_1, \ldots). \quad (20)$$

3. Flows of dispersionless mKP (dmKP) equation

3.1 *Isospectral flows dmKP*

It is understood that the dispersionless hierarchies are derived by the quasiclassical limit [3,4,10,11,23] of the dispersionful hierarchies. In [5,6], Takasaki and Takebe considered the derivation of dKP hierarchy through Sato's approach. Following this procedure, we take, $\epsilon x = X$, $\epsilon y = Y$, $\epsilon t = (\epsilon t_1, \epsilon t_2, \ldots) = (T_1, T_2, \ldots) = T$ and think of

$$v_j\left(\frac{X}{\epsilon}, \frac{Y}{\epsilon}, \frac{T}{\epsilon}\right) = V_j(X, Y, T) + \mathcal{O}(\epsilon).$$

When $\epsilon \to 0$ in the Lax operator (15), we get

$$\tilde{L}_\epsilon = \epsilon\partial + \sum_{j=1}^{\infty} v_j\left(\frac{X}{\epsilon}, \frac{Y}{\epsilon}, \frac{T}{\epsilon}\right)(\epsilon\partial)^{-j}$$
$$= \epsilon\partial + \sum_{j=1}^{\infty}\big(V_j\,(X, Y, T) + \mathcal{O}(\epsilon)\big)(\epsilon\partial)^{-j},\ \partial = \partial_X. \tag{21}$$

By considering the Wentzel–Kramers–Brillouin (WKB) asymptotic expansion of the wave function with action of S [6] as

$$\psi = \exp\left[\frac{1}{\epsilon}S(X, Y, T, \lambda)\right], \quad \epsilon \to 0, \tag{22}$$

and defining $p = \partial_X S$, called momentum function, we observe that $\epsilon^i\partial^i\psi \to p^i\psi$, $\epsilon \to 0$. Using this limit procedure, we arrive at

$$\tilde{\mathcal{L}} = p + \sum_{j=1}^{\infty} V_j(X, Y, T)p^{-j}. \tag{23}$$

Now, by taking the quasiclassical limit in (16), we get

$$\epsilon(\tilde{L}_\epsilon)_Y = [\tilde{B}_{2\epsilon}, \tilde{L}_\epsilon], \tag{24a}$$
$$\epsilon(\tilde{L}_\epsilon)_{T_m} = [\tilde{B}_{m\epsilon}, \tilde{L}_\epsilon], \tag{24b}$$
$$\epsilon\tilde{B}_{2,T_m} - \epsilon\tilde{B}_{m,Y} + [\tilde{B}_{2\epsilon}, \tilde{B}_{m\epsilon}] = 0, \quad m = 1, 2, \ldots. \tag{24c}$$

As ϵ approaches zero, we obtain the following system associated for the isospectral flows of dmKP:

$$\tilde{\mathcal{L}}_Y = \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{L}}\}, \tag{25a}$$
$$\tilde{\mathcal{L}}_{T_m} = \{\tilde{\mathcal{B}}_m, \tilde{\mathcal{L}}\}, \tag{25b}$$
$$\tilde{\mathcal{B}}_{2,T_m} - \tilde{\mathcal{B}}_{m,Y} + \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{B}}_m\} = 0, \tag{25c}$$

and $\tilde{\mathcal{B}}_m = (\tilde{\mathcal{L}}^m)_{\geq 1}$, projection onto a polynomial in p with strictly positive powers and the Poisson bracket is as defined in (9). Now, we list the first few $\tilde{\mathcal{B}}_m$'s:

$$\tilde{\mathcal{B}}_1 = p, \tag{26a}$$
$$\tilde{\mathcal{B}}_2 = p^2 + 2V_0p, \tag{26b}$$
$$\tilde{\mathcal{B}}_3 = p^3 + 3V_0p^2 + 3(V_0^2 + V_1)p, \tag{26c}$$
$$\tilde{\mathcal{B}}_4 = p^4 + 4V_0p^3 + 2(3V_0^2 + 2V_1)p^2 + 4(V_0^3 + 3V_1V_0 + V_2)p. \tag{26d}$$

From (25c), we consider the equation for the isospectral flows of the dmKP equation as follows:

$$V_{0,T_m} = \frac{1}{2p}(\tilde{\mathcal{B}}_{m,Y} - \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{B}}_m\}), \quad m = 1, 2, \ldots. \tag{27}$$

For various values of m, we get

$$V_{0,T_1} = V_{0,X}, \tag{28a}$$
$$V_{0,T_2} = V_{0,Y}, \tag{28b}$$
$$V_{0,T_3} = \frac{3}{2}(-2V_{1,X}V_0 + V_{1,Y} + 2V_{0,X}V_1 - 2V_{0,X}V_0^2 + 2V_{0,Y}V_0), \tag{28c}$$
$$V_{0,T_4} = 2(-2V_{2,X}V_0 + V_{2,Y} - 6V_{1,X}V_0^2 + 3V_{1,Y}V_0 + 2V_{0,X}V_2 - 4V_{0,X}V_0^3 + 3V_{0,Y}V_1 + 3V_{0,Y}V_0^2). \tag{28d}$$

From (25a), expressing $V_1, V_2, V_3, \ldots$ in terms of V_0 in the above expression, we get

$$V_{T_1} = \tilde{K}_1 = V_X, \tag{29a}$$
$$V_{T_2} = \tilde{K}_2 = V_Y, \tag{29b}$$
$$V_{T_3} = \tilde{K}_3 = \frac{3}{2}V_X\partial^{-1}V_Y - \frac{3}{2}V_XV^2 + \frac{3}{4}\partial^{-1}V_{YY}, \tag{29c}$$
$$V_{T_4} = \tilde{K}_4 = V_X\partial^{-2}V_{YY} - 2V_X\partial^{-1}(VV_Y) + 2V_Y\partial^{-1}V_Y - 2V_YV^2 + V\partial^{-1}V_{YY} - \partial^{-1}(V_Y^2) + \frac{1}{2}\partial^{-2}V_{YYY} - \partial^{-1}(VV_{YY}). \tag{29d}$$

Equation (29c) is the dmKP equation, which together with (29a), (29b), (29d), ... form the isospectral flows of the dmKP equation.

3.2 *Non-isospectral flows of the dmKP equation*

In (17), we considered the spectral parameter λ, which is a function of infinite set of variables corresponding to the non-isospectral problem of mKP equation. Using the averaging procedure in the variables, we get $\epsilon\lambda_{T_m} = (\epsilon/2)\lambda^{m-1}$. Along with this fact, we look into system (19) as follows:

$$\epsilon\tilde{L}_Y = [\tilde{B}_{2\epsilon}, \tilde{L}_\epsilon], \tag{30a}$$
$$\epsilon\tilde{L}_{T_m} = [\epsilon\tilde{A}_{m\epsilon}, L_\epsilon] + \frac{\epsilon}{2}\tilde{L}_\epsilon^{m-1}, \tag{30b}$$
$$\epsilon\tilde{B}_{2,T_m} - \epsilon^2\tilde{A}_{m,Y} + [\tilde{B}_{2\epsilon}, \epsilon\tilde{A}_{m\epsilon}] = 0, \quad m = 1, 2, \ldots \tag{30c}$$

and the undetermined operator (18) with boundary condition (20) becomes

$$\epsilon\tilde{A}_{m\epsilon} = \sum_{i=0}^{m-1} h_{i\epsilon}^{(m)}(\epsilon\partial)^{m-i}, \tag{31a}$$

$$\epsilon\tilde{A}_{m\epsilon}|_{\mathbf{V}=0} = \epsilon\left[\frac{Y}{\epsilon}(\epsilon\partial)^m + \frac{X}{2\epsilon}(\epsilon\partial)^{m-1} + \frac{m-2}{4}(\epsilon\partial^{m-2})\right], \tag{31b}$$

with $\mathbf{V} = (V_0, V_1, \ldots) + \mathcal{O}(\epsilon)$. Now, by taking the dispersionless limit, i.e. as $\epsilon \to 0$ in (30) and (31)

$$\tilde{\mathcal{L}}_Y = \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{L}}\}, \tag{32a}$$

$$\tilde{\mathcal{L}}_{T_m} = \{\tilde{\mathcal{A}}_m, \mathcal{L}\} + \frac{1}{2}\tilde{\mathcal{L}}^{m-1}, \tag{32b}$$

$$\tilde{\mathcal{B}}_{2,T_m} - \tilde{\mathcal{A}}_{m,Y} + \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_m\} = 0, \tag{32c}$$

where

$$\tilde{\mathcal{A}}_m = \sum_{i=0}^{m-1} \mathcal{H}_i^{(m)} p^{m-i}, \tag{33}$$

with the condition on $\tilde{\mathcal{A}}_m$ at $\mathbf{V} = 0$

$$\tilde{\mathcal{A}}_m|_{\mathbf{V}=0} = Yp^m + \frac{1}{2}Xp^{m-1}, \quad \mathbf{V} = (V_0, V_1, \ldots). \tag{34}$$

Here, we observe that the undetermined coefficient function $\mathcal{H}_i^{(m)}$ becomes a function of V_j and its derivative with respect to X and Y. Now, by substituting (33) in (32b) and using (34) for various values of m, we present the first few $\tilde{\mathcal{A}}_m$ as

$$\tilde{\mathcal{A}}_1 = Y\tilde{\mathcal{B}}_1, \tag{35a}$$

$$\tilde{\mathcal{A}}_2 = Y\tilde{\mathcal{B}}_2 + \frac{X}{2}\tilde{\mathcal{B}}_1, \tag{35b}$$

$$\tilde{\mathcal{A}}_3 = Y\tilde{\mathcal{B}}_3 + \frac{X}{2}\tilde{\mathcal{B}}_2, \tag{35c}$$

$$\tilde{\mathcal{A}}_4 = Y\tilde{\mathcal{B}}_4 + \frac{X}{2}\tilde{\mathcal{B}}_3 + \frac{1}{2}\partial^{-1}(V_1)p. \tag{35d}$$

One can observe that from (32c), we obtain the non-isospectral flow generating equation for dmKP as

$$V_{0,T_m} = \frac{1}{2p}(\tilde{\mathcal{A}}_{m,Y} - \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_m\}), \quad m = 1, 2, \ldots. \tag{36}$$

By considering $m = 1, 2, 3, 4$, we get

$$V_{0,T_1} = YV_{0,X} + \frac{1}{2}, \tag{37a}$$

$$V_{0,T_2} = YV_{0,Y} + \frac{X}{2}V_{0,X} + \frac{1}{2}V_0, \tag{37b}$$

$$V_{0,T_3} = \frac{3}{2}Y(-2V_{1,X}V_0 + V_{1,Y} + 2V_{0,X}V_1 - 2V_{0,X}V_0^2 + 2V_{0,Y}V_0) + \frac{1}{2}XV_{0,Y} + \frac{3}{2}V_1 + \frac{1}{2}V_0^2, \tag{37c}$$

$$\begin{aligned} V_{0,T_4} = {} & 2Y(-2V_{2,X}V_0 + V_{2,Y} - 6V_{1,X}V_0^2 + 3V_{1,Y}V_0 \\ & + 2V_{0,X}V_2 - 4V_{0,X}V_0^3 + 3V_{0,Y}V_1 + 3V_{0,Y}V_0^2) \\ & + \frac{3}{4}X(-2V_{1,X}V_0 + V_{1,Y} + 2V_{0,X}V_1 \\ & - 2V_{0,X}V_0^2 + 2V_{0,Y}V_0) + \frac{1}{4}\partial^{-1}V_{1,Y} \\ & + \frac{1}{2}V_{0,X}\partial^{-1}V_1 + 2V_2 + 4V_1V_0 + \frac{1}{2}V_0^3. \end{aligned} \tag{37d}$$

Replacing $V_1, V_2, V_3, \ldots$ in (37a), (37b), (37c) and (37d) with $V_0 = V$, we arrive at the first four non-isospectral flows of dmKP as follows:

$$V_{T_1} = \tilde{\sigma}_1 = Y\tilde{K}_1 + \frac{1}{2}, \tag{38a}$$

$$V_{T_2} = \tilde{\sigma}_2 = Y\tilde{K}_2 + \frac{X}{2}V_X + \frac{V}{2}, \tag{38b}$$

$$V_{T_3} = \tilde{\sigma}_3 = Y\tilde{K}_3 + \frac{X}{2}\tilde{K}_2 - \frac{1}{4}V^2 + \frac{3}{4}\partial^{-1}V_Y, \tag{38c}$$

$$\begin{aligned} V_{T_4} = \tilde{\sigma}_4 = {} & Y\tilde{K}_4 + \frac{X}{2}\tilde{K}_3 + \frac{1}{4}V_X\partial^{-2}V_Y - \frac{1}{4}V_X\partial^{-1}V^2 \\ & + V\partial^{-1}V_Y + \frac{5}{8}\partial^{-2}V_{YY} - \frac{5}{4}\partial^{-1}(VV_Y) - \frac{1}{2}V^3. \end{aligned} \tag{38d}$$

Equation (38c) is the non-isospectral dmKP equation, which together with (38a), (38b), (38d),... form the non-isospectral flows of the dmKP equation.

4. Algebraic structure of the dmKP equation

To study the Lie algebraic structure, we make use of the implicit flow representations on the iso and non-isospectral flows of the dmKP equation with $V_0 = V$ as follows:

Isospectral case:

$$\tilde{\mathcal{B}}_2'[\tilde{K}_n] = \tilde{\mathcal{B}}_{n,Y} - \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{B}}_n\}, \tag{39a}$$

$$\tilde{\mathcal{B}}_n|_{V=0} = p^n. \tag{39b}$$

Non-isospectral case:

$$\tilde{\mathcal{B}}_2'[\tilde{\sigma}_m] = \tilde{\mathcal{A}}_{m,Y} - \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_m\}, \quad (40a)$$

$$\tilde{\mathcal{A}}_m|_{V=0} = Yp^m + \frac{1}{2}Xp^{m-1}, \quad m \geq 2. \quad (40b)$$

Lemma 1. *If a function $\mathcal{R}$ takes the form $\sum_{i=0}^{m-1} a_i p^{m-i}$, $a_i \in \mathcal{M}$ and $Z = Z(V) \in \mathcal{M}$, then the equation*

$$\tilde{\mathcal{B}}_2'[Z] = \mathcal{R}_Y - \{\tilde{\mathcal{B}}_2, \mathcal{R}\}, \quad \mathcal{R}|_{V=0} = 0, \quad (41)$$

only admits zero solution $Z = 0$, $\mathcal{R} = 0$. Here, $\tilde{\mathcal{B}}_2 = p^2 + 2Vp$, where we have taken $V_0 = V$.

Proof. The proof is straightforward. □

Theorem 1. *Suppose that*

$$\langle \tilde{\mathcal{B}}_m, \tilde{\mathcal{B}}_n \rangle = \tilde{\mathcal{B}}_m'[\tilde{K}_n] - \tilde{\mathcal{B}}_n'[\tilde{K}_m] + \{\tilde{\mathcal{B}}_m, \tilde{\mathcal{B}}_n\}, \quad (42a)$$

$$\langle \tilde{\mathcal{B}}_m, \tilde{\mathcal{A}}_n \rangle = \tilde{\mathcal{B}}_m'[\tilde{\sigma}_n] - \tilde{\mathcal{A}}_n'[\tilde{K}_m] + \{\tilde{\mathcal{B}}_m, \tilde{\mathcal{A}}_n\}, \quad (42b)$$

$$\langle \tilde{\mathcal{A}}_m, \tilde{\mathcal{A}}_n \rangle = \tilde{\mathcal{A}}_m'[\tilde{\sigma}_n] - \tilde{\mathcal{A}}_n'[\tilde{\sigma}_m] + \{\tilde{\mathcal{A}}_m, \tilde{\mathcal{A}}_n\}. \quad (42c)$$

Then we have

$$\tilde{\mathcal{B}}_2'[[\![\tilde{K}_m, \tilde{K}_n]\!]] = \langle \tilde{\mathcal{B}}_m, \tilde{\mathcal{B}}_n \rangle_Y - \{\tilde{\mathcal{B}}_2, \langle \tilde{\mathcal{B}}_m, \tilde{\mathcal{B}}_n \rangle\}, \quad (43a)$$

$$\tilde{\mathcal{B}}_2'[[\![\tilde{K}_m, \tilde{\sigma}_n]\!]] = \langle \tilde{\mathcal{B}}_m, \tilde{\mathcal{A}}_n \rangle_Y - \{\tilde{\mathcal{B}}_2, \langle \tilde{\mathcal{B}}_m, \tilde{\mathcal{A}}_n \rangle\}, \quad (43b)$$

$$\tilde{\mathcal{B}}_2'[[\![\tilde{\sigma}_m, \tilde{\sigma}_n]\!]] = \langle \tilde{\mathcal{A}}_m, \tilde{\mathcal{A}}_n \rangle_Y - \{\tilde{\mathcal{B}}_2, \langle \tilde{\mathcal{A}}_m, \tilde{\mathcal{A}}_n \rangle\}, \quad (43c)$$

and

$$\langle \tilde{\mathcal{B}}_m, \tilde{\mathcal{B}}_n \rangle|_{V=0} = 0, \quad (44a)$$

$$\langle \tilde{\mathcal{B}}_m, \tilde{\mathcal{A}}_n \rangle|_{V=0} = \frac{1}{2}mp^{m+n-2}, \quad (44b)$$

$$\langle \tilde{\mathcal{A}}_m, \tilde{\mathcal{A}}_n \rangle|_{V=0} = \frac{1}{2}(m-n)\left(Yp^{m+n-2} + \frac{1}{2}Xp^{m+n-3}\right), \quad \forall\, m, n \geq 2. \quad (44c)$$

Proof. We only prove eqs (43c) and (44c), and the other equations can be proved in the same manner. Consider the LHS of (43c) and expand it as

$$\tilde{\mathcal{B}}_2'[[\![\tilde{\sigma}_m, \tilde{\sigma}_n]\!]] = (\tilde{\mathcal{B}}_2'[\sigma_m])'[\sigma_n] - (\tilde{\mathcal{B}}_2'[\sigma_n])'[\sigma_m]. \quad (45)$$

First, we take

$$\begin{aligned}(\tilde{\mathcal{B}}_2'[\sigma_m])'[\sigma_n] &= (\tilde{\mathcal{A}}_{m,Y} - \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_m\})'[\sigma_n] \\ &= \tilde{\mathcal{A}}_{m,Y}'[\sigma_n] - \{\tilde{\mathcal{B}}_2'[\sigma_n], \tilde{\mathcal{A}}_m\} - \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_m'[\sigma_n]\} \\ &= \tilde{\mathcal{A}}_{m,Y}'[\sigma_n] - \{\tilde{\mathcal{A}}_{n,Y}, \tilde{\mathcal{A}}_m\} + \{\{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_n\}, \tilde{\mathcal{A}}_m\} \\ &\quad - \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_m'[\sigma_n]\}. \end{aligned} \quad (46)$$

Similarly

$$\begin{aligned}(\tilde{\mathcal{B}}_2'[\sigma_n])'[\sigma_m] &= \tilde{\mathcal{A}}_{n,Y}'[\sigma_m] - \{\tilde{\mathcal{A}}_{m,Y}, \tilde{\mathcal{A}}_n\} \\ &\quad + \{\{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_m\}, \tilde{\mathcal{A}}_n\} - \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_n'[\sigma_m]\}. \end{aligned} \quad (47)$$

Subtracting (47) from (46) and using the Jacobi identity, we arrive

$$\tilde{\mathcal{B}}_2'[[\![\tilde{\sigma}_m, \tilde{\sigma}_n]\!]] = \langle \tilde{\mathcal{A}}_m, \tilde{\mathcal{A}}_n \rangle_Y - \{\tilde{\mathcal{B}}_2, \langle \tilde{\mathcal{A}}_m, \tilde{\mathcal{A}}_n \rangle\}.$$

Next, observe that $\tilde{\sigma}_n|_{V=0} = 0$, $\forall\, n \geq 2$. Using this in (42c) with (40b), we can easily prove (44c). Hence, the theorem is proved. □

Theorem 2. *The isospectral flows $\{\tilde{K}_i\}$ and non-isospectral flows $\{\tilde{\sigma}_i\}$ of dmKP equation constitute the following infinite-dimensional Lie algebra of Virasoro type*:

$$[\![\tilde{K}_n, \tilde{K}_m]\!] = 0, \quad (48a)$$

$$[\![\tilde{K}_m, \tilde{\sigma}_n]\!] = \frac{1}{2}m\tilde{K}_{n+m-2}, \quad (48b)$$

$$[\![\tilde{\sigma}_m, \tilde{\sigma}_n]\!] = \frac{1}{2}(m-n)\tilde{\sigma}_{n+m-2}, \quad \forall\, m, n \geq 2. \quad (48c)$$

Proof. We only prove (48c). In the same way (48a) and (48b) could be proved. Now, we take

$$2\omega_1 = 2[\![\tilde{\sigma}_m, \tilde{\sigma}_n]\!] - (m-n)\tilde{\sigma}_{m+n-2}, \quad (49a)$$

$$\omega_2 = \langle \tilde{\mathcal{A}}_m, \tilde{\mathcal{A}}_n \rangle - \frac{1}{2}(m-n)\tilde{\mathcal{A}}_{m+n-2}. \quad (49b)$$

Using (43c) and (44c) together with implicit flow representations (40a) and (40b), we have

$$2\omega_1 = \omega_{2,Y} - \{\tilde{\mathcal{B}}_2, \omega_2\}, \quad \omega_2|_{V=0} = 0. \quad (50)$$

From Lemma 1, it is immediate that $\omega_1 = 0$ and $\omega_2 = 0$. This shows that (48c) is true. Hence, the theorem is proved. □

5. Dispersionless Miura map

Let us consider the more generalised form of $\mathcal{L}$ as

$$\mathcal{L} = p^m + \sum_{i=1}^{m} a_{m-i}p^{m-i} + \sum_{j=1}^{\infty} a_{-j}p^{-j}, \quad (51)$$

where $a_{m-1}, a_{m-2}, \ldots, a_0, a_{-1}, \ldots$ are functions of X, Y and T. Now, take $\phi(X, Y, T)$ as any arbitrary function, independent of p and define [14]

$$\begin{aligned}\tilde{\mathcal{L}} &= \mathrm{e}^{-ad\phi}\mathcal{L} \\ &= \mathcal{L} - \{\phi, \mathcal{L}\} + \frac{1}{2!}\{\phi, \{\phi, \mathcal{L}\}\} \\ &\quad - \frac{1}{3!}\{\phi, \{\phi, \{\phi, \mathcal{L}\}\}\} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(\phi_X)^n \partial_p^n \mathcal{L}, \end{aligned}$$

$$\phi_X = \frac{\partial \phi}{\partial X}, \quad (52)$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined in (9). The map defined in (52) is called the dispersionless Miura map [14], which relates the isospectral flows between the dKP and dmKP equations. Now, we review some lemmas discussed in [14].

Lemma 2 [14]. *Let $\tilde{\mathcal{L}}$ be defined as above, then*

$$\tilde{\mathcal{L}}_{\geq 1} = \mathrm{e}^{-ad\phi}\mathcal{L}_{\geq 0} - \mathcal{L}_{\geq 0}|_{p=\phi_X}. \tag{53}$$

Lemma 3 [14]. *If $f(X,Y,T,p)$ and $g(X,Y,T,p)$ are any two functions, then*

$$\mathrm{e}^{-ad\phi}\{f,g\} = \{\mathrm{e}^{-ad\phi}f, \mathrm{e}^{-ad\phi}g\}.$$

Lemma 4 [14]. *If $\tilde{\mathcal{L}}$ is defined as above, then*

$$\tilde{\mathcal{L}}_{T_q} = \mathrm{e}^{-ad\phi}\mathcal{L}_{T_q} - \{\phi_{T_q}, \mathrm{e}^{-ad\phi}\mathcal{L}\}.$$

Lemma 5 [14]. *If $F(X,Y,T,p)$ and $G(X,Y,T,p)$ are any two functions, then*

$$\mathrm{e}^{-ad\phi}(FG) = (\mathrm{e}^{-ad\phi}F)(\mathrm{e}^{-ad\phi}G).$$

Using the above lemmas, we construct the following lemmas and theorems to prove the correspondence between the non-isospectral flows of the dKP and dmKP equations. From Lemma 5, we get $\tilde{\mathcal{L}}^k = \mathrm{e}^{-ad\phi}\mathcal{L}^k$. Now, we define $\tilde{\mathcal{B}}_k = (\tilde{\mathcal{L}}^k)_{\geq 1}$ and $\mathcal{B}_k = (\mathcal{L}^k)_{\geq 0}$.

Lemma 6. *If $\tilde{\mathcal{B}}_k$ and $\mathcal{B}_k$ are defined as above, then*

$$\tilde{\mathcal{B}}_{k,T_q} = \mathrm{e}^{-ad\phi}\mathcal{B}_{k,T_q} - \{\phi_{T_q}, \mathrm{e}^{-ad\phi}\mathcal{B}_k\} - (\mathcal{B}_k|_{p=\phi_X})_{T_q}.$$

Proof. Using Lemma 3, we obtain

$$\begin{aligned}\tilde{\mathcal{B}}_{k,T_q} &= (\mathrm{e}^{-ad\phi}\mathcal{B}_k - \mathcal{B}_k|_{p=\phi_X})_{T_q}\\ &= \mathrm{e}^{-ad\phi}\mathcal{B}_{k,T_q} + \phi_{X,T_q}\sum_{n=0}^{\infty}\frac{1}{n!}\phi_X^n\partial_p^{n+1}\mathcal{B}_k\\ &\quad - (\mathcal{B}_k|_{p=\phi_X})_{T_q}\\ &= \mathrm{e}^{-ad\phi}\mathcal{B}_{k,T_q} - \{\phi_{T_q}, \mathrm{e}^{-ad\phi}\mathcal{B}_k\} - (\mathcal{B}_k|_{p=\phi_X})_{T_q}.\end{aligned}$$

Hence, the lemma is proved. □

Lemma 7. *If $f(X,Y,T)$ and $G(X,Y,T,p)$ are any two functions, then*

$$\mathrm{e}^{-ad\phi}(fG) = f\mathrm{e}^{-ad\phi}G.$$

Proof. From (52), the proof is straightforward. □

Lemma 8. *Suppose,*

$$\tilde{\mathcal{A}}_k = Y\tilde{\mathcal{B}}_k + \frac{1}{2}X\tilde{\mathcal{B}}_{k-1} + \tilde{h}_k,$$

$$\mathcal{A}_k = Y\mathcal{B}_k + \frac{1}{2}X\mathcal{B}_{k-1} + h_k,$$

where $\tilde{h}_k(X,Y,T,p)$ and $h_k(X,Y,T,p)$ are any two functions with $\tilde{h}_k = \mathrm{e}^{-ad\phi}h_k - h_k|_{p=\phi_X}$, then

$$\tilde{\mathcal{A}}_k = \mathrm{e}^{-ad\phi}\mathcal{A}_k - \mathcal{A}_k|_{p=\phi_X}.$$

Proof. Using Lemma 7, we consider

$$\begin{aligned}\text{LHS} &= Y(\mathrm{e}^{-ad\phi}\mathcal{B}_k - \mathcal{B}_k|_{p=\phi_X})\\ &\quad + \frac{1}{2}X(\mathrm{e}^{-ad\phi}\mathcal{B}_{k-1} - \mathcal{B}_{k-1}|_{p=\phi_X}) + \tilde{h}_k\end{aligned}$$

and from Lemma 8, we get

$$\begin{aligned}\text{LHS} &= \mathrm{e}^{-ad\phi}\left(Y\mathcal{B}_k + \frac{1}{2}X\mathcal{B}_{k-1}\right)\\ &\quad - \left(Y\mathcal{B}_k|_{p=\phi_X} + \frac{1}{2}X\mathcal{B}_{k-1}|_{p=\phi_X}\right) + \tilde{h}_k\\ &= \mathrm{e}^{-ad\phi}\mathcal{A}_k - \mathrm{e}^{-ad\phi}h_k - \mathcal{A}_k|_{p=\phi_X} + h_k|_{p=\phi_X} + \tilde{h}_k\\ &= \mathrm{e}^{-ad\phi}\mathcal{A}_k - \mathcal{A}_k|_{p=\phi_X}.\end{aligned}$$

Hence, the lemma is proved. □

Lemma 9. *If $\tilde{\mathcal{A}}_k$ and $\mathcal{A}_k$ are defined as above, then*

$$\begin{aligned}\tilde{\mathcal{A}}_{k,T_q} &= \mathrm{e}^{-ad\phi}\mathcal{A}_{k,T_q} - \{\phi_{T_q}, e^{-ad\phi}\mathcal{A}_k\}\\ &\quad - (\mathcal{A}_k|_{p=\phi_X})_{T_q}.\end{aligned}$$

Proof. Using Lemma 9 and taking the proof similar with Lemma 7, it is easy to prove this lemma. □

Theorem 3. *If $\tilde{\mathcal{L}}$, $\tilde{\mathcal{A}}_k$ and $\mathcal{A}_k$ are defined as above, then*

$$\begin{aligned}&\tilde{\mathcal{L}}_{T_k} - \{\tilde{\mathcal{A}}_k, \tilde{\mathcal{L}}\} - \frac{1}{2}\tilde{\mathcal{L}}^{k-1}\\ &\quad = \mathrm{e}^{-ad\phi}\left(\mathcal{L}_{T_k} - \{\mathcal{A}_k, \mathcal{L}\} - \frac{1}{2}\mathcal{L}^{k-1}\right)\\ &\qquad + \{\mathcal{A}_k|_{p=\phi_X} - \phi_{T_k}, \mathrm{e}^{-ad\phi}\mathcal{L}\}.\end{aligned}$$

Proof. Using Lemmas 5 and 9,

$$\begin{aligned}\text{LHS} &= \mathrm{e}^{-ad\phi}\mathcal{L}_{T_k} - \{\phi_{T_k}, \mathrm{e}^{-ad\phi}\mathcal{L}\}\\ &\quad - \{\mathrm{e}^{-ad\phi}\mathcal{A}_k - \mathcal{A}_k|_{p=\phi_X}, \mathrm{e}^{-ad\phi}\mathcal{L}\}\\ &\quad - \frac{1}{2}\mathrm{e}^{-ad\phi}\mathcal{L}^{k-1}\\ &= \mathrm{e}^{-ad\phi}\left(\mathcal{L}_{T_k} - \{\mathcal{A}_k, \mathcal{L}\} - \frac{1}{2}\mathcal{L}^{k-1}\right)\\ &\quad + \{\mathcal{A}_k|_{p=\phi_X} - \phi_{T_k}, \mathrm{e}^{-ad\phi}\mathcal{L}\}\\ &\quad = \text{RHS}.\end{aligned}$$

This completes the theorem. □

Theorem 4. *If $\tilde{\mathcal{B}}_k$, $\mathcal{B}_k$, $\tilde{\mathcal{A}}_k$ and $\mathcal{A}_k$ are defined as above, then*

$$\begin{aligned}&\tilde{\mathcal{B}}_{2,T_k} - \tilde{\mathcal{A}}_{k,Y} + \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_k\}\\&\quad = \mathrm{e}^{-ad\phi}\left(\mathcal{B}_{2,T_k} - \mathcal{A}_{k,Y} + \{\mathcal{B}_2, \mathcal{A}_k\}\right)\\&\qquad + \{\phi_Y - \mathcal{B}_2|_{p=\phi_X}, \mathrm{e}^{-ad\phi}\mathcal{A}_k\}\\&\qquad - \{\phi_{T_k} - \mathcal{A}_k|_{p=\phi_X}, \mathrm{e}^{-ad\phi}\mathcal{B}_2\}\\&\qquad + (\mathcal{A}_k|_{p=\phi_X})_Y - (\mathcal{B}_2|_{p=\phi_X})_{T_k}.\end{aligned}$$

Proof. From Lemmas 7, 9 and 10, we consider

$$\begin{aligned}\text{LHS}=&\mathrm{e}^{-ad\phi}\mathcal{B}_{2,T_k} - \{\phi_{T_k}, \mathrm{e}^{-ad\phi}\mathcal{B}_2\} - (\mathcal{B}_2|_{p=\phi_X})_{T_k}\\&- \mathrm{e}^{-ad\phi}\mathcal{A}_{k,Y} + \{\phi_Y, \mathrm{e}^{-ad\phi}\mathcal{A}_k\} + (\mathcal{A}_k|_{p=\phi_X})_Y\\&+\{e^{-ad\phi}\mathcal{B}_2-\mathcal{B}_2|_{p=\phi_X}, \mathrm{e}^{-ad\phi}\mathcal{A}_k-\mathcal{A}_k|_{p=\phi_X}\}\\=&\mathrm{e}^{-ad\phi}\left(\mathcal{B}_{2,T_k}-\mathcal{A}_{k,Y}+\{\mathcal{B}_2, \mathcal{A}_k\}\right)+\{\phi_Y, \mathrm{e}^{-ad\phi}\mathcal{A}_k\}\\&- \{\phi_{T_k}, \mathrm{e}^{-ad\phi}\mathcal{B}_2\}+(\mathcal{A}_k|_{p=\phi_X})_Y-(\mathcal{B}_2|_{p=\phi_X})_{T_k}\\&- \{\mathrm{e}^{-ad\phi}\mathcal{B}_2, \mathcal{A}_k|_{p=\phi_X}\} - \{\mathcal{B}_2|_{p=\phi_X}, \mathrm{e}^{-ad\phi}\mathcal{A}_k\}\\&+ \{\mathcal{B}_2|_{p=\phi_X}, \mathcal{A}_k|_{p=\phi_X}\}.\end{aligned}$$

Observe that $\{\mathcal{B}_2|_{p=\phi_X}, \mathcal{A}_k|_{p=\phi_X}\} = 0$ since $\mathcal{B}_2|_{p=\phi_X}$ and $\mathcal{A}_k|_{p=\phi_X}$ are independent of p. Therefore,

$$\begin{aligned}\text{LHS} =& \mathrm{e}^{-ad\phi}\left(\mathcal{B}_{2,T_k} - \mathcal{A}_{k,Y} + \{\mathcal{B}_2, \mathcal{A}_k\}\right)\\&+ \{\phi_Y - \mathcal{B}_2|_{p=\phi_X}, \mathrm{e}^{-ad\phi}\mathcal{A}_k\}\\&- \{\phi_{T_k} - \mathcal{A}_k|_{p=\phi_X}, \mathrm{e}^{-ad\phi}\mathcal{B}_2\}\\&+ (\mathcal{A}_k|_{p=\phi_X})_Y - (\mathcal{B}_2|_{p=\phi_X})_{T_k}\\=& \text{R.H.S.}\end{aligned}$$

Hence, the theorem is proved. □

The given theorems and lemmas are discussed for the general form of $\mathcal{L}$ in (51). Now, we consider that the particular form of $\mathcal{L}$ defined for the dKP equation, becomes subcase of (51). It is easy to identify that the functions $\mathcal{B}_k$ and $\mathcal{A}_k$ for dmKP also become subcases of the general form of $\mathcal{B}_k$ and $\mathcal{A}_k$ which are discussed in the lemmas and theorems in this section.

Theorem 5. *If* $\mathcal{L} = p + \sum_{i=1}^{\infty} U_{i+1}p^{-i}$, *the associated* $\mathcal{B}_2 = p^2 + 2U_2$ *and* $\mathcal{A}_k$ *satisfy non-isospectral flows of dKP equation* (11) *with*

$$\phi_{T_m} = \mathcal{A}_m|_{p=\phi_X}, \quad \phi_Y = \mathcal{B}_2|_{p=\phi_X},$$

then $\tilde{\mathcal{L}} = \mathrm{e}^{-ad\phi}\mathcal{L}$ *satisfies the non-isospectral flows of dmKP equation* (32).

Proof. Equation (32a) is common to both iso and non-isospectral flows of the dKP equation and the proof of (32a) was already given in [14]. Hence, we omit the proof of (32a).

Now, using the given condition $\phi_{T_m} = \mathcal{A}_m|_{p=\phi_X}$ in Theorem 5, the proof of (32b) is obvious. Together with the given condition $\phi_Y = \mathcal{B}_2|_{p=\phi_X}$, consider Theorem 4, and we obtain

$$\tilde{\mathcal{B}}_{2,T_m} - \tilde{\mathcal{A}}_{m,Y} + \{\tilde{\mathcal{B}}_2, \tilde{\mathcal{A}}_m\} = (\mathcal{A}_m|_{p=\phi_X})_Y - (\mathcal{B}_2|_{p=\phi_X})_{T_m}.$$

Using the compatibility condition between ϕ_Y and ϕ_{T_m}, we conclude that (32c) also holds good. This completes the theorem. □

6. Conclusion

In this article, we have constructed Lax triad structures (14) for the mKP equation and subsequently implemented quasiclassical limit in (14) to obtain the associated Lax triad equations (25) for the dmKP equation. In [20,22], Fu *et al* clearly investigated that generating the non-isospectral flows for (2 + 1)-dimensional equations using triad formalisms overcome the difficulty in understanding the nature of time variable T_2 and space variable Y. Making use of this fact, from the triad equations, we derived the isospectral flows $\{K_l(U)\}$ and non-isospectral flows $\{\sigma_r(U)\}$ of the dmKP equation. By composing these flows, we have also obtained infinite-dimensional Lie algebras with centreless Kac–Moody–Virasoro structure. Finally, from Theorems 3–5 we demonstrated that the correspondence between the non-isospectral flows of the dKP and dmKP hierarchies through dispersionless Miura map is preserved as in the case of isospectral flows.

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References

[1] R Carroll, *J. Nonlinear Sci.* **4**, 519 (1994)
[2] R Carroll and Y Kodama, *J. Phys. A* **28**, 6373 (1995)
[3] D Lebedev and Y Manin, *Phys. Lett. A* **74**, 154 (1979)
[4] Y Kodama and J Gibbons, *Proceedings of the fourth workshop on nonlinear turbulant process in physics* (World Scientific, Singapore, 1990) p. 166
[5] T Takasaki and T Takebe, *Adv. Ser. Math. Phys.* **16**, 888 (1992)
[6] K Takasaki and T Takebe, *Rev. Math. Phys.* **7**, 743 (1995)
[7] S Aroyama and Y Kodama, *Commun. Math. Phys.* **182**, 185 (1996)
[8] B Dubrovin, *Nucl. Phys. B* **379**, 627 (1992)
[9] B Dubrovin, Integrable systems and quantum groups, *Lecture notes in mathematics* (Springer, 1996) Vol. 1620, pp. 120–348
[10] I Krichever, *Commun Math. Phys.* **143**, 415 (1992)
[11] I Krichever, *Commun. Math. Phys*. **47**, 437 (1994)

[12] Y Kodama, *Phys. Lett. A* **129**, 223 (1988)
[13] Y Kodama and J Gibbons, *Phys. Lett. A* **135**, 167 (1989)
[14] J H Chang and M H Tu, *J. Math. Phys.* **41**, 5391 (2000)
[15] T Takebe and L P Teo, *SIGMA* **2**, 72 (2006)
[16] T Takebe, *Lett. Math. Phys.* **59**, 157 (2002)
[17] T Xiao and Y Zeng, *Phys. Lett. A* **349**, 128 (2006)
[18] T Xiao and Y Zeng, *Inverse Problems* **22**, 869 (2006)
[19] H Wu and Y Zeng, *Commun. Nonlinear. Sci. Numer. Simul.* **17**, 2766 (2012)
[20] W Fu, R Ilangovane, K M Tamizhmani and D J Zhang, *J. Math. Phys.* **55**, 083504-17 (2014)
[21] D Y Chen, *Solitons, nonlinear evolution equations and inverse scattering* (Sciences Press, Beijing, 2006)
[22] W Fu, L Huang, K M Tamizhmani and D J Zhang, *Nonlinearity* **26**, 3197 (2013)
[23] V E Zakharov, Dispersionless limit of integrable systems in (2+1) dimensions, in *Singular limit of dispersive waves* edited by N M Erconali *et al* (Plenum, New York, 1994) pp. 165–174