Pramana – J. Phys. (2021) 95:192
https://doi.org/10.1007/s12043-021-02238-2

Discovering a celestial object using a non-parametric algorithm

JAYANTH P VYASANAKERE[1],*, SIDDHARTH BHATNAGAR[1,2] and JAYANT MURTHY[3]

[1]The School of Arts and Sciences, Azim Premji University, Bengaluru 562 125, India
[2]Department of Astronomy, University of Geneva, Chemin Pegasi 51, 1290 Versoix, Switzerland
[3]Indian Institute of Astrophysics, Bengaluru 560 034, India
*Corresponding author. E-mail: jayanth.vyasanakere@apu.edu.in

MS received 2 May 2021; revised 3 August 2021; accepted 3 August 2021

Abstract. We describe a method that does not use any orbital parameters, to arrive at the position and mass of a new celestial object, using high-precision orbital state vector data of the rest of the objects in the system. As an illustration of this approach, we rediscover Neptune with remarkable accuracy.

Keywords. Non-parametric algorithm; Neptune; iteration method.

PACS Nos 95.10.Ce; 45.50.Pk; 95.10.Eg; 96.30.Rm

1. Introduction

Mankind's fascination with space is led by the inspiration of discovery. Over the past two centuries, many solar system bodies have been discovered. The past three decades have witnessed the discovery of many Kuiper belt objects and exoplanets. Lately, there has also been renewed interest in finding a possible ninth planet in our solar system. All these started with the discovery of Uranus and Neptune, two centuries ago.

Neptune's discovery in Sept. 1846 CE, is mathematically interesting as it was discovered by analysing deviations of Uranus from its theoretical orbit [1–3]. Till date, it also serves as one of the greatest testimonies to the robustness of Newtonian Mechanics. The methods used by its discoverers are complex and there have been several developments to show how Neptune could have been discovered with simpler methods [4–9].

Here we develop a simple, but precise method to discover a perturber in a gravitational system. Our method works when the data of all objects in the system, except for the one to be discovered, is available. While the method is general, for the purpose of illustration, here we take the case of Neptune and 'rediscover' it.

The traditional methods characterise the orbit of the planet to be discovered with parameters describing a Keplerian orbit. Our method does not make use of any parameters and finds the position of the planet as a function of time. Another prime feature of our approach is that it not only looks at Neptune's effect on a single planet (say, Uranus), but on *two* planets (say, Uranus and Saturn), which is interesting in itself. This method can easily include a large number of objects in the system as the associated computation time increases only linearly.

This approach makes use of this age's high-precision orbital state vector data of solar system bodies. The data, given by [10], act as a proxy for the real data, to illustrate our method. We retrieve planetary data in Cartesian coordinates from Jan. 1800 CE to Jan. 1847 CE, at a time step of 2 h. The origin is the centre of the Sun, the z-axis is along the Sun's mean north pole at the reference epoch (J2000.0) and the x-axis is out along the ascending node of the Sun's mean equator on the reference plane (ICRF).

2. Model and equations

Our model consists of all significantly massive objects in the solar system until Neptune: The Sun, the 7 known planets (Mercury to Uranus), the 4 big asteroids (Ceres, Pallas, Vesta and Hygiea) and Neptune. Neptune, whose position and mass we deduce, is initially considered unknown for the purpose of demonstrating our method. All objects in the model include their moon(s) (if any); the position ($\boldsymbol{x}$) refers to that of the centre of mass of the respective planetary system and mass (M) refers to the total mass of that system.

Published online: 12 November 2021

Newton's law written for the ith object reads as

$$\frac{\mathrm{d}^2\boldsymbol{x}_i}{\mathrm{d}t^2} = -\sum_{j\neq i} GM_j \frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^3} \qquad (i = 1 \text{ to } 13), \qquad (1)$$

where G is the universal gravitational constant. We denote the Sun by $\odot$ and Neptune by N. Subtracting eq. (1) written for the Sun from the same equation written for the object i ($i \neq \odot, N$),

$$GM_N\left(\frac{\boldsymbol{r}_N - \boldsymbol{r}_i}{|\boldsymbol{r}_N - \boldsymbol{r}_i|^3} - \frac{\boldsymbol{r}_N}{|\boldsymbol{r}_N|^3}\right) = \boldsymbol{V}_i(t), \qquad (2)$$

where

$$\boldsymbol{V}_i(t) = \frac{\mathrm{d}^2\boldsymbol{r}_i}{\mathrm{d}t^2} - \sum_{j\neq i,N} GM_j \frac{\boldsymbol{r}_j - \boldsymbol{r}_i}{|\boldsymbol{r}_j - \boldsymbol{r}_i|^3} + \sum_{j\neq\odot,N} GM_j \frac{\boldsymbol{r}_j}{|\boldsymbol{r}_j|^3} \qquad (3)$$

and $\boldsymbol{r}_i = \boldsymbol{x}_i - \boldsymbol{x}_\odot$, which are relative coordinates with respect to the Sun, whose data can be obtained from [10]. Hence, $\boldsymbol{V}_i(t)$ can be obtained from eq. (3) without an *a priori* knowledge of Neptune. Thus, in eq. (2), the unknowns are $\boldsymbol{r}_N(t)$ and M_N. At this stage, one might be tempted to use orbital parameters to describe $\boldsymbol{r}_N(t)$ and obtain these values through curve fitting. This task is accomplished in ref. [9], using a geometric method without using curve fitting. In contrast to all these methods, the current method is completely different where we obtain $\boldsymbol{r}_N(t)$ at 'every' given instant of time 'without' using any orbital parameter.

3. The method

We adopt the iteration method to solve for $\boldsymbol{r}_N(t)$. Equation (2) can be rewritten in the following two ways:

$$\boldsymbol{r}_N = \boldsymbol{r}_i + |\boldsymbol{r}_N - \boldsymbol{r}_i|^3\left(\frac{\boldsymbol{V}_i}{GM_N} + \frac{\boldsymbol{r}_N}{|\boldsymbol{r}_N|^3}\right) \qquad (4)$$

$$\boldsymbol{r}_N = |\boldsymbol{r}_N|^3\left(\frac{\boldsymbol{r}_N - \boldsymbol{r}_i}{|\boldsymbol{r}_N - \boldsymbol{r}_i|^3} - \frac{\boldsymbol{V}_i}{GM_N}\right). \qquad (5)$$

With $V_i(t) = |\boldsymbol{V}_i(t)|$, the mass can be written as

$$GM_N = V_i \Big/ \left|\frac{\boldsymbol{r}_N - \boldsymbol{r}_i}{|\boldsymbol{r}_N - \boldsymbol{r}_i|^3} - \frac{\boldsymbol{r}_N}{|\boldsymbol{r}_N|^3}\right|. \qquad (6)$$

When eq. (6) is substituted in eqs (4) and (5), those two equations, at any given time, can be thought of as representing the position of Neptune as a *fixed point* of two separate three-dimensional nonlinear maps A and B:

$$\boldsymbol{r}''_N = A(\boldsymbol{r}'_N) \quad \text{and} \quad \boldsymbol{r}''_N = B(\boldsymbol{r}'_N). \qquad (7)$$

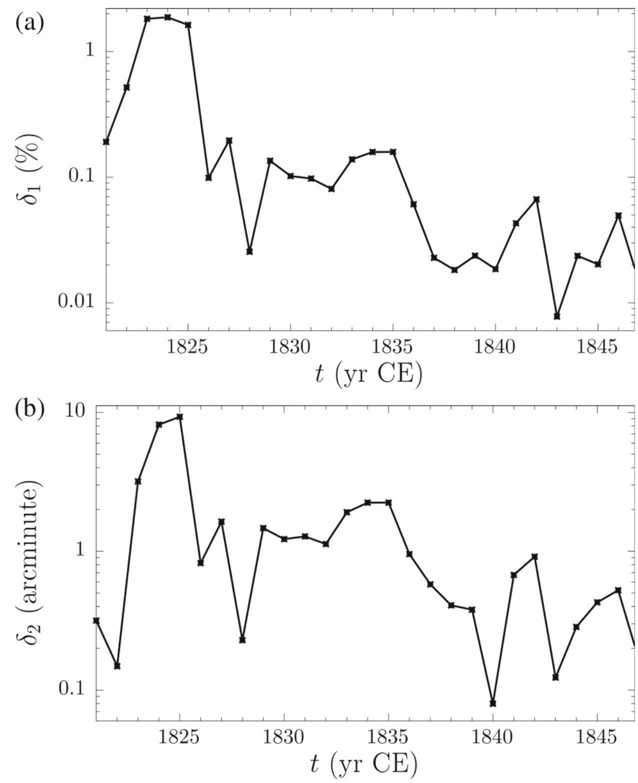

Figure 1. (**a**) The percentage deviation of the predicted position of Neptune from its actual position from Sun and (**b**) the angle between the predicted and the actual positions of Neptune as viewed from Earth.

$\boldsymbol{r}_i$ and $\hat{\boldsymbol{V}}_i$ (unit vector along $\boldsymbol{V}_i$) are constants in the maps. The details of this map are given in Appendix A.

Further, for any given time, there is no single fixed point, but 'a line of fixed points'! This means that $\boldsymbol{r}_N$, which solves eq. (2) for a given $\boldsymbol{r}_i$ and $\hat{\boldsymbol{V}}_i$, is not unique. These different solutions will have different values for the mass M_N.

Hence, by running the above algorithm separately for i = Uranus and i = Saturn, we generate 'two' lines of fixed points at any time, one for each planet. These two lines must theoretically intersect right at the true position of Neptune at the time considered. So, we identify the two points on these lines that correspond to the closest approach of the said lines. We then define the position of Neptune as the mid-point of the line segment joining these two points.

4. Results and discussions

Carrying out this algorithm for different times, the trajectory of Neptune ($\boldsymbol{r}_N(t)$) is calculated. The deviation of this calculated position from the true position of Neptune (obtained from the data given in [10]) is within δ_1%

of the Sun–Neptune distance, where δ_1 is shown in figure 1a. $\boldsymbol{r}_N(t)$ obtained, can further be used to determine Neptune's direction in the sky, as viewed from Earth. The angle between this direction and the actual direction of Neptune is called δ_2 and is shown in figure 1b. Angle δ_2 is what matters for an Earth-based astronomer to precisely locate Neptune in the sky. This angle is particularly small because the line of fixed points makes only a small angle with the line of sight (γ does not deviate much from α in figure 3) and hence, most of the error contributing to δ_1, will be along the line of sight.

This is indeed a very accurate prediction of Neptune's position. While this is better than the other methods in the literature to (re)discover Neptune [1,2,5,7,8] by at least an order of magnitude, a comparison is not fair, because the nature of data that this method relies on, is more sophisticated than those methods. However, the method used in ref. [9] uses the same kind of data [10] and also revolves around eqs (2) and (3). So we compare these two methods. Bhatnagar *et al* [9] analysed the deviation in Uranus' orbit for around two-and-a-half centuries and took a geometrical approach to obtain Neptune's orbital parameters, thereby, its position within around a degree. On the other hand, the current method uses the deviations of both Uranus and Saturn for less than half a century, and with an iteration procedure, obtains Neptune's position within around an arc minute, bypassing the calculation of orbital parameters.

The accuracy of this method is determined by the accuracy of evaluation of $\boldsymbol{V}_i$. In turn, this is limited by the following two considerations: On the one hand, the error in the evaluation of $\boldsymbol{V}_i$ (eq. (3)) is decided by whether all objects in the system (except the one to be discovered) which have significant mass have been included in the model, and how accurate their mass values are. On the other hand, this error has to be negligible compared to the magnitude of the actual $\boldsymbol{V}_i$, which using eq. (2) is determined by the mass of the perturber (in our case, Neptune), its location with respect to the Sun and the planet being analysed (in our case, Uranus/Saturn). Thus, when the respective V_is are significant for both Uranus and Saturn (around 1846, see figure 4), the error can be expected to be small (see figure 1).

Neptune's trajectory obtained here, can be plugged into eq. (6), to obtain its mass, which comes out to be $(1.025 \pm 0.039) \times 10^{26}$ kg. The actual mass is within this range [10].

No differential equations need to be solved in this method. The computation time of the problem increases only linearly with the number of objects in the system and therefore, this approach can easily incorporate a large number of objects. Moreover, as no orbital parameters are to be found, this method is most suitable for bodies that significantly deviate from their Keplerian orbits due to perturbations by other bodies. In principle, this method can be extended to find the ever-elusive Planet Nine, given sufficiently precise observed data of the known solar system.

Acknowledgements

The authors thank Rajaram Nityananda, Azim Premji University for illuminating discussions.

Appendix A. Details of the iteration method

Referring to eq. (7), let $\boldsymbol{r}'_N$ be defined by the radial distance r', the polar angle θ' with $\boldsymbol{r}_i$ along the polar axis and the azimuthal angle ϕ' (the spherical polar coordinates with Sun at the centre). Let $\boldsymbol{r}''_N$ be similarly defined by r'', θ'' and ϕ''. We carry out a linear stability analysis by writing the Jacobian matrix at $\boldsymbol{r}_N$ and calculating its eigenvalues for both the maps.

Now we define two dimensionless quantities α and β. α is the angle between $\boldsymbol{r}_N$ and $\boldsymbol{r}_i$, which ranges from 0 to π radian. β is the ratio $|\boldsymbol{r}_N|/|\boldsymbol{r}_i|$ which can, in principle, range from 0 to ∞. Here are a few interesting results about maps A and B:

1. By symmetry, the eigenvalues mentioned above (which also determine stability) for either map can only depend on α and β.
2. Note that $\boldsymbol{V}_i$ is in the $i \odot N$ plane (plane defined by object i, Sun and Neptune). Hence, again by symmetry, $\partial r''/\partial\phi'$, $\partial\theta''/\partial\phi'$, $\partial\phi''/\partial r'$ and $\partial\phi''/\partial\theta'$ are zero for both the maps.
3. This means that $\partial\phi''/\partial\phi'$ is one of the three eigenvalues (called λ_3) of each of the maps, with the corresponding eigenvector along $\hat{\boldsymbol{\phi}}$ (perpendicular to the $i \odot N$ plane). This evaluates to $(1+\beta^2 - 2\beta\cos\alpha)^{3/2}/\beta^3$ for map A and $\beta^3/(1+\beta^2 - 2\beta\cos\alpha)^{3/2}$ for map B. Since their product is 1, at least one of the two maps will be unstable for any α and β.
4. It is straightforward, although laborious (hence not shown), to obtain the remaining 2×2 submatrices of the Jacobians analytically. The determinants (det) and the trace (Tr) for both the maps satisfy det $+ 1 =$ Tr. This means that another eigenvalue (called λ_1) is 1 for both the maps, for any α and β. This is consistent with the non-uniqueness of the solution of eq. (2) and the existence of a line of fixed points.
5. We call the remaining eigenvalue λ_2. $|\lambda_2|$ and $|\lambda_3|$ are either both less than 1, both greater than 1 or

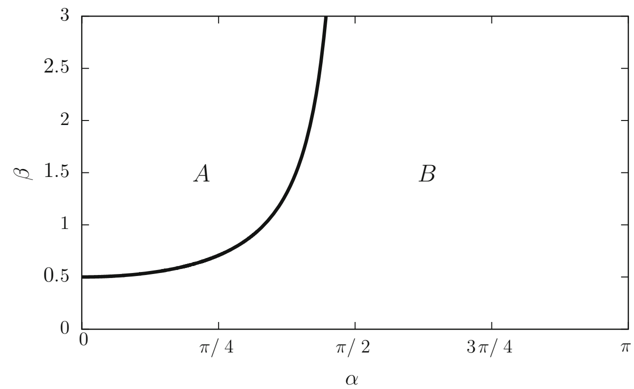

Figure 2. In the regimes marked A and B, maps A and B respectively are neutral. α is in radian.

both equal to 1. This means that for generic α and β values, exactly one of the maps will be neutral ($\lambda_1 = 1, |\lambda_2| < 1, |\lambda_3| < 1$) and the other map will be unstable ($\lambda_1 = 1, |\lambda_2| > 1, |\lambda_3| > 1$). Figure 2 gives the regimes of α and β, where maps A and B respectively, are neutral and the other map is unstable. The boundary between the two regimes is given by $\beta = 1/(2\cos\alpha)$. Along this curve, all the eigenvalues of both the maps are unity. As $\beta \to \infty$, the regimes are decided by whether the angle α is acute or obtuse.

6. Both the maps share the same set of eigenvectors. Above, it was noted that $\hat{\boldsymbol{\phi}}$ is one of the eigenvectors. The other two will be perpendicular to $\hat{\boldsymbol{\phi}}$ (i.e., in the $i \odot N$ plane). Let $\boldsymbol{e}_1$ be the eigenvector of the neutral map corresponding to λ_1. The coordinate along $\boldsymbol{e}_1$ does not change on iteration. Let γ be the angle made by $\boldsymbol{e}_1$ with $\boldsymbol{r}_i$ in the $i \odot N$ plane. γ is shown in figure 3 for β corresponding to the mean orbital radii of Uranus and Saturn. Note that γ does not deviate much from α for a large enough β. In the limit $\beta \to \infty$, $\gamma = \alpha$, i.e., $\boldsymbol{e}_1$ will be along $\boldsymbol{r}_N$.

Appendix B. Making the initial guess

We need some preliminary analysis to employ this method to locate Neptune. Initially, we model the orbit of Neptune as a circle around the Sun, coplanar with that of Uranus and Saturn. The parameters characterising Neptune's orbit will then be its radius R and the position on this circular orbit at a given time.

It is easy to discern that the peaks in figure 4 correspond to conjunctions of Neptune with Uranus and Saturn. Pursuing this, we get an estimate for the time of conjunction with Uranus (T_c) as around 1822 CE and

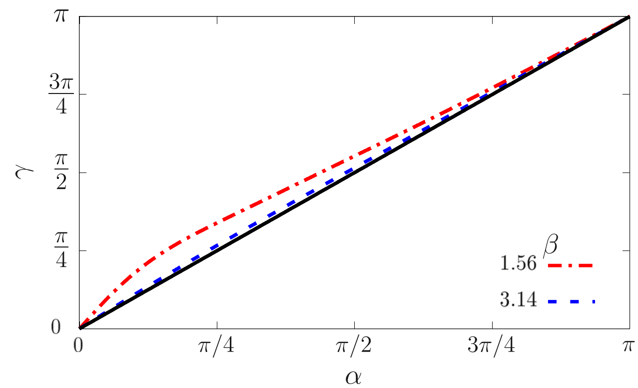

Figure 3. α is shown as a black solid line. β correspond to Uranus (1.56) and Saturn (3.14). The consequent γ are shown as a red dot–dashed line and a blue dashed line respectively. α and γ are in radian.

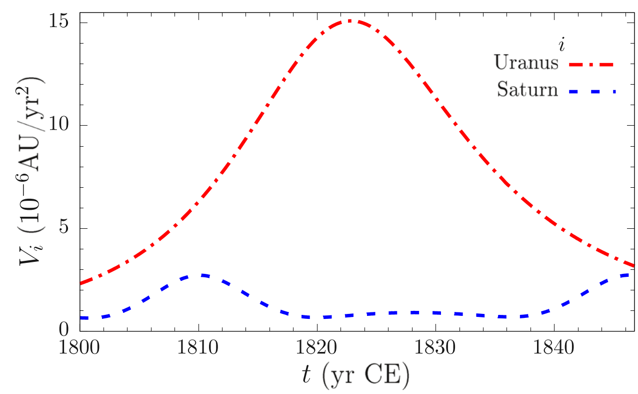

Figure 4. $V_i(t) = |\boldsymbol{V}_i(t)|$ obtained from5 eq. (3) is shown as a function of time for i = Uranus and i = Saturn.

the duration between the two conjunctions with Saturn as around 36 years. This gives us 160 years, as an estimate for Neptune's orbital period, which by Kepler's third law provides $R \approx 30$ AU.

At any given time, an initial guess for the position of Neptune will result in a unique point on the line of fixed points after convergence. For the initial guess, we use a particular choice of angular coordinates and a range of values around R for the radial coordinate. We start the analysis around T_c. For the first time step, we use the same angular coordinates as that of Uranus at conjunction. For the subsequent time steps, the position of Neptune obtained after convergence, can be used to make the initial guess for the angular coordinates.

References

[1] U J Le Verrier, *Astron. Nachr.* **25**, 53 (1846)
[2] J C Adams, *Mon. Not. R. Astron. Soc.* **7**, 149 (1846)
[3] J G Galle, *Mon. Not. R. Astron. Soc.* **7**, 153 (1846)
[4] E W Brown, *Mon. Not. R. Astron. Soc.* **92**, 80 (1931)
[5] R A Lyttleton, *Vistas Astron.* **3**, 25 (1960)

[6] C J Brookes, *Mon. Not. R. Astron. Soc*. **158**, 79 (1972)
[7] H M Lai, C Lam and K Young, *Am. J. Phys.* **58**, 946 (1990)
[8] G Eriksson and K Garcia Martin, Discovery of Neptune, TRITA-SCI-GRU No. 2018-210 (KTH, School of Engineering Sciences (SCI), 2018)
[9] S Bhatnagar, J P Vyasanakere and J Murthy, *Am. J. Phys.* **89**(**5**), 454 (2021)
[10] J D Giorgini, JPL Solar System Dynamics Group, Data retrieved 2020-04-04, NASA/JPL Horizons On-Line Ephemeris System, https://ssd.jpl.nasa.gov/horizons.cgi