Pramana – J. Phys. (2021) 95:190
https://doi.org/10.1007/s12043-021-02190-1

Critical thickness problem for tetra-anisotropic scattering in the reflected reactor system

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MS received 16 November 2020; revised 6 May 2021; accepted 7 May 2021

Abstract. Critical thicknesses are calculated in reflected systems for high-order anisotropic scattering by using neutron transport theory. The anisotropic systems are taken into account from isotropic to tetra-anisotropic scattering terms one by one. Neutron transport equation is solved by using the Legendre polynomial P_N method and then Chebyshev polynomial T_N method. The eigenfunctions and eigenvalues are calculated for different numbers of secondary neutrons (c) up to the ninth-order term in the iteration of the two methods. The Marshak boundary condition is applied to find critical thickness for the reflected reactor system. Thus, a wide-range critical thickness spectrum has been generated, depending on the number of secondary neutrons, anisotropic scattering coefficients and different range of reflection coefficients. Finally, the calculated critical thickness values are compared with those in the literature and it is observed that our results are in agreement with them.

Keywords. Neutron transport equation; critical thickness; anisotropic scattering.

PACS Nos 28.20.Gd; 28.90.+i; 02.90.+p

1. Introduction

In the nuclear fuel criticality calculations, it is very important to use anisotropic scattering in higher orders for the collision treatment in neutron transport to calculate the effective multiplication constant. The anisotropy effect is very important in studying particle transport problems. There have been various researches about criticality problems with anisotropic scattering [1–8]. Güleçyüz *et al* solved the critical slab problem for the linearly anisotropic scattering of reflected boundary condition using the H_N method [9]. Atalay presented the reflected slab problem for linearly anisotropic scattering with Case's singular eigenfunction method [10]. The Chebyshev polynomials of the first and second kind are used effectively in the series expansion of the neutron angular flux and accurate results are obtained for the critical thickness and also for the reflected slabs by Öztürk and Anlı [11,12]. Türeci presented a study about İnönü linear quadratic anisotropic scattering with the F_N method and criticality problem with reflected boundary condition for triplet anisotropic scattering [7,13]. In this work, the critical thickness problem is taken into account for the reflected slab reactor from isotropic to tetraanisotropic scattering cases by applying the T_N and P_N methods.

The aim of this study is to show the dependence of critical thickness for the bare and reflected reactor systems on the anisotropic scattering effect. Inspired by the work done so far, this study examines the effects on critical thickness by changing the anisotropic scattering types. In order to understand the effectiveness of methods in tetra and other scattering types, the iteration is performed up to the 9th order and the convergence in the results are observed up to three digits. The variations of the critical thickness according to the number of secondary neutrons and reflector thickness are shown in tables. We also verify that the pure tetra-anisotropic scattering coefficient is equal to the isotropic scattering when $f_4 = 0$ by interpolating from the maximum value to the minimum value. As the neutron transport equation explains the distribution and conservation of neutrons in the reactor core, the general expression of the equation is related to the position, velocity, time etc. Thus, the neutron transport equation has seven unknown parameters. For this reason, some assumptions are required to solve the neutron transport equation. The steady-state, one speed and plane geometrical neutron transport equation is written as

Published online: 08 November 2021

[14]

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{c}{2}\int_{-1}^{1} f(\mu,\mu')\psi(x,\mu')\,\mathrm{d}\mu', \tag{1}$$

where μ' is the direction of scattering, $f(\mu,\mu')$ is the scattering function which defines the scattering probability of neutrons, $\psi(x,\mu)$ is the number of neutrons at x-position and μ is the direction with distance measured in units of mean free path (mfp), c is the number of secondary neutrons related with the material cross-sections by the equation $c\sigma_t = \upsilon\sigma_f + \sigma_s$. Here σ_f is the fission cross-section and σ_s is the scattering cross-section, υ the number of neutrons per fission. In a fissionable medium, the value of c is bigger than 1, and in a non-fissionable medium, the value of c is smaller than 1. So, c bigger than one is considered to calculate the critical thickness values. The scattering function in eq. (1) is expanded with the Legendre polynomials as [15]

$$f(\mu,\mu') = \sum_{n=0}^{N} (2n+1) f_n P_n(\mu) P_n(\mu'), \tag{2}$$

where $P_n(\mu)$ and $P_n(\mu')$ are Legendre polynomials and f_n is the scattering coefficient. It can also be written as $f(\Omega,\Omega') = f(\cos\theta_0)$ where $\cos\theta_0 = \Omega\cdot\Omega'$. If it is the anisotropy scattering function, it can be developed into a Legendre polynomial series of the argument $\mu_0 = \Omega\cdot\Omega'$ where Ω and Ω' are the directions of neutrons before and after the collision; μ_0 is the cosine of the difference of scattering angles. The scattering function defines the probability of each scattering, and the range of f_n coefficients must be determined for every scattering case. The value of the scattering coefficients changes with the case of the scattering function and cosine angle. It is known that the scattering function can take numerical values between zero and one, and the cosine angles' range is between minus one and plus one. Furthermore, f_n is restricted to the range $|f_n| \le \frac{1}{(2n+1)}$ to ensure positivity of the distribution function for all scattering angles [16]. The scattering types are named as follows:

$$f(\mu_0) = \frac{1}{4\pi}\left(f_0 P_0(\mu_0)\right),$$
isotropic scattering

$$f(\mu_0) = \frac{1}{4\pi}\left(f_0 P_0(\mu_0) + 3 f_1 P_1(\mu_0)\right),$$
linear anisotropic scattering

$$f(\mu_0) = \frac{1}{4\pi}\left(f_0 P_0(\mu_0) + 3 f_1 P_1(\mu_0) + 5 f_2 P_2(\mu_0)\right),$$
quadratic anisotropic scattering

$$f(\mu_0) = \frac{1}{4\pi}\left(f_0 P_0(\mu_0) + 5 f_2 P_2(\mu_0)\right),$$
pure quadratic anisotropic scattering

$$f(\mu_0) = \frac{1}{4\pi}\left(f_0 P_0(\mu_0) + 3 f_1 P_1(\mu_0) + 5 f_2 P_2(\mu_0) + 7 f_3 P_3(\mu_0)\right),$$
triplet anisotropic scattering

$$f(\mu_0) = \frac{1}{4\pi}\left(f_0 P_0(\mu_0) + 7 f_3 P_3(\mu_0)\right),$$
pure triplet anisotropic scattering

$$f(\mu_0) = \frac{1}{4\pi}\left(f_0 P_0(\mu_0) + 3 f_1 P_1(\mu_0) + 5 f_2 P_2(\mu_0) + 7 f_3 P_3(\mu_0) + 9 f_4 P_4(\mu_0)\right),$$
tetra-anisotropic scattering

$$f(\mu_0) = \frac{1}{4\pi}\left(f_0 P_0(\mu_0) + 9 f_4 P_4(\mu_0)\right),$$
pure tetra-anisotropic scattering.

1.1 *P_N Solution of the neutron transport equation for tetra-anisotropic scattering*

The P_N method is an analytical technique based on the integro-differential form of the transport equation. It is often used for criticality problems [5]. The angular flux for the Legendre polynomials is defined in [17] as

$$\psi(x,\mu) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \phi_n(x) P_n(\mu). \tag{3}$$

Legendre moments of the flux are given by

$$\phi_n(x) = \int_{-1}^{1} P_n(\mu')\psi(x,\mu')\mathrm{d}\mu'. \tag{4}$$

The scattering function in eq. (2) is used for each scattering type one by one. The tetra-anisotropic scattering function can be written as

$$\begin{aligned} f(\mu,\mu') = {} & f_0 P_0(\mu) P_0(\mu') \\ & + 3 f_1 P_1(\mu) P_1(\mu') + 5 f_2 P_2(\mu) P_2(\mu') \\ & + 7 f_3 P_3(\mu) P_3(\mu') + 9 f_4 P_4(\mu) P_4(\mu'). \end{aligned} \tag{5}$$

If the scattering function in eq. (5) and Legendre moments in eq. (4) are substituted into eq. (1), one gets

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\psi(x,\mu)=\frac{c}{2}\left[\begin{array}{l}P_0(\mu)f_0\phi_0(x)+3P_1(\mu)f_1\phi_1(x)+5P_2(\mu)f_2\phi_2(x)\\+7P_3(\mu)f_3\phi_3(x)+9P_4(\mu)f_4\phi_4(x)\end{array}\right]. \quad (6)$$

One can insert eq. (3) into eq. (6) and then integrate with $P_m(\mu)$ over $\mu\in(-1,1)$. The recursion relation

$$\mu P_n(\mu)=\frac{1}{2n+1}\left[(n+1)\,P_{n+1}(\mu)+nP_{n-1}(\mu)\right] \quad (7)$$

and the orthogonality of the Legendre polynomials

$$\int_{-1}^{1}P_m(x)P_n(x)\mathrm{d}x=\begin{cases}0 & m\neq n\\ \frac{2}{2n+1} & m=n\end{cases} \quad (8)$$

are applied to eq. (6) [18]. After some algebra, one can obtain

$$\begin{aligned}&(n+1)\,\frac{\mathrm{d}\phi_{n+1}(x)}{\mathrm{d}x}+n\frac{\mathrm{d}\phi_{n-1}(x)}{\mathrm{d}x}\\&\quad+(2n+1)(1-cf_n\delta_{n0}+cf_n\delta_{n1}+cf_n\delta_{n2}\\&\quad+cf_n\delta_{n3}+cf_n\delta_{n4})\phi_n(x)=0,\qquad n=0,1,2,\dots,N\end{aligned} \quad (9)$$

where the Kronecker delta is defined as

$$\delta_{nm}=\begin{cases}1, & n=m\\ 0, & n\neq m\end{cases}.$$

A well-known ansatz for the solutions of eq. (9) is employed as [19]

$$\phi_n(x)=G_n(v)e^{-x/v}, \quad (10)$$

where G_n is the eigenfunction and v is the corresponding eigenvalue. G_n has been studied by İnönü for general anisotropic scattering [20]. Replacing eq. (10) in eq. (9), a system of equations is obtained for the analytic expressions of $G_n(v)$ where $G_0(v)=1$. The general form of eq. (9) may be given by

$$\begin{aligned}&(n+1)G_{n+1}(v)+nG_{n-1}(v)+(2n+1)\\&\quad\times[1-\{cf_0\delta_{n,0}+cf_1\delta_{n,1}\\&\quad+cf_2\delta_{n,2}+cf_3\delta_{n,3}+cf_4\delta_{n,4}\}]vG_n(v)=0,\\&\quad n=0,1,2,\dots,N.\end{aligned} \quad (12)$$

As shown in eq. (12), the analytical solution of $G_n(v)$ gives the discrete eigenvalues v_k by solving $G_{n+1}(v)=0$, for any c. Here, it has $(N+1)/2$ eigenvalues v_k, $k=1,2,3,\dots,N+1$ roots are used to find the flux moment. In the iteration of P_1 the eigenvalue is

$$v=\pm\frac{1}{\sqrt{3-3cf_0-3cf_1+3c^2f_0f_1}}.$$

It is seen that the eigenvalue depends on the scattering coefficients and the number of secondary neutrons c. By using numerical values such as $c=1.1$, $f_0=1$ and $f_1=0.3$, they are found to be $v=2.2305$i and $v=-2.2305$i. If one wants to calculate G_9, then five pairs are found by solving eq. (12) with the numerical values such as $c=1.1$, $f_0=1.0$, $f_1=0.3$, $f_2=0.2$, $f_3=0.14$ and $f_4=0.11$. Thus, the eigenvalues are $v_1=\pm 2.12202$i, $v_2=\pm 0.19189$, $v_3=\pm 0.54312$, $v_4=\pm 0.83110$ and $v_5=\pm 1.10193$. Five pairs of eigenvalues are found for using in the P_9 iteration. The positive eigenvalues are used in the calculation of critical thickness. The range of results depends on the value of secondary neutrons c. If c is bigger than one, it means that the medium is fissionable. After determining the discrete eigenvalues of v_k, the roots of the Legendre polynomials are found by $P_{n+1}(\mu_k)=0$, where

$$\mu_k=\cos\left(\frac{2k-1}{2n}\pi\right),\qquad \frac{N+1}{2}<k\leq N+1.$$

The general solution of the flux moments in eq. (12) can be written for odd numbers of N:

$$\begin{aligned}\phi_n(x)&=\sum_{k=1}^{\frac{N+1}{2}}\beta_kG_n(v_k)\\&\times\left[\mathrm{e}^{x/v_k}+(-1)^n\mathrm{e}^{-x/v_k}\right],\quad (N+1)/2<k\leq(N+1),\end{aligned} \quad (13)$$

where β_k is a constant which can be determined from the physical boundary conditions of the system. The parity relation is defined as

$$G_n(-v)=(-1)^nG_n(v).$$

Therefore, the general solution of eq. (1) for the neutron angular flux can be found by substituting eq. (13) into eq. (3),

$$\begin{aligned}\psi(x,\mu_k)&=\sum_{n=0}^{\infty}\sum_{k=1}^{\frac{N+1}{2}}\frac{2n+1}{2}\beta_kG_n(v_k)\\&\quad\times\left[\left(1+(-1)^n\right)\cosh\left(\frac{x}{v_k}\right)\right.\\&\quad\left.+\left(1-(-1)^n\right)\sinh\left(\frac{x}{v_k}\right)\right]P_n(\mu_k).\end{aligned} \quad (14)$$

1.2 T_N *Solution of the neutron transport equation for tetra-anisotropic scattering*

The neutron angular flux in the neutron transport equation can be expanded in terms of Chebyshev polynomials of the first type [21]

$$\psi(x,\mu)=\frac{\phi_0(x)}{\pi\sqrt{1-\mu^2}}T_0(\mu)+\frac{2}{\pi\sqrt{1-\mu^2}}\sum_{n=1}^{N}\phi_n(x)T_n(\mu), \tag{15}$$

where $\phi_n(x)$ is the flux moment and $T_n(\mu)$ is the term of the Chebyshev polynomial of the first type. Equation (15) is substituted in eq. (1),

$$\mu\frac{\partial}{\partial x}\left(\frac{\phi_0(x)}{\pi\sqrt{1-\mu^2}}T_0(\mu)+\frac{2}{\pi\sqrt{1-\mu^2}}\sum_{n=1}^{N}\phi_n(x)T_n(\mu)\right)+\frac{\phi_0(x)}{\pi\sqrt{1-\mu^2}}T_0(\mu)+\frac{2}{\pi\sqrt{1-\mu^2}}\sum_{n=1}^{N}\phi_n(x)T_n(\mu)=\frac{c}{2}\int_{-1}^{1}f(\mu,\mu')\left(\frac{\phi_0(x)}{\pi\sqrt{1-\mu^2}}T_0(\mu')+\frac{2}{\pi\sqrt{1-\mu^2}}\sum_{n=1}^{N}\phi_n(x)T_n(\mu')\right)\mathrm{d}\mu'. \tag{16}$$

Here the scattering function is used for tetra-anisotropic scattering and the resultant equation is found when eq. (5) is substituted in eq. (16);

$$\mu\frac{\partial}{\partial x}\left(\frac{\phi_0(x)}{\pi\sqrt{1-\mu^2}}T_0(\mu)+\frac{2}{\pi\sqrt{1-\mu^2}}\sum_{n=1}^{N}\phi_n(x)T_n(\mu)\right)+\frac{\phi_0(x)}{\pi\sqrt{1-\mu^2}}T_0(\mu)+\frac{2}{\pi\sqrt{1-\mu^2}}\sum_{n=1}^{N}\phi_n(x)T_n(\mu)=\frac{c}{2}\int_{-1}^{1}f_0P_0(\mu)P_0(\mu')+3f_1P_1(\mu)P_1(\mu')+5f_2P_2(\mu)P_2(\mu')+7f_3P_3(\mu)P_3(\mu')+9f_4P_4(\mu)P_4(\mu')\times\left(\frac{\phi_0(x)}{\pi\sqrt{1-\mu^2}}T_0(\mu')+\frac{2}{\pi\sqrt{1-\mu^2}}\sum_{n=1}^{N}\phi_n(x)T_n(\mu')\right)\mathrm{d}\mu'. \tag{17}$$

Some definite integrals are used and the recursion relation for the first type Chebyshev polynomial is given by

$$T_{n+1}(\mu)-2\mu T_n(\mu)+T_{n-1}(\mu)=0 \tag{18}$$

and the orthogonal relation is

$$\int_{-1}^{1}T_m(\mu)T_n(\mu)(1-\mu^2)^{-1/2}\mathrm{d}\mu=\begin{cases}0, & m\neq n,\\ \pi/2, & m=n\neq 0,\\ \pi, & m=n=0.\end{cases} \tag{19}$$

The recursion relation (eq. (18)) and the orthogonal relation (eq. (19)) are replaced by eq. (17) which is multiplied with $T_m(\mu)$and then integrated over $\mu\in(-1,1)$. Thus, one obtains

$$\int_{-1}^{1}\frac{1}{\pi\sqrt{1-\mu^2}}\left\{2\sum_{n=1}^{N}T_n(\mu)\phi_n(x)+\sum_{n=1}^{N}T_{n-1}(\mu)\phi_n'(x)+\sum_{n=1}^{N}T_{n+1}(\mu)\phi_n'(x)+\phi_0(x)+\mu\phi_0'(x)\right\}T_m(\mu)\mathrm{d}\mu-\frac{c}{2}\int_{-1}^{1}\int_{-1}^{1}f(\mu,\mu')\frac{1}{\pi\sqrt{1-\mu^2}}\left(\phi_0(x)T_0(\mu')+2\sum_{n=1}^{N}\phi_n(x)T_n(\mu')\right)\mathrm{d}\,\mu\,\mathrm{d}\mu'=0 \tag{20}$$

from which a set of differential equations can be found for varying m values that are related to the $\phi_n(x)$ flux moments. A general expression is proposed to solve the recursion equations as

$$\phi_n(x)=G_n(v)\exp(x/v). \tag{21}$$

The series expression of $G_n(v)$ is found by substituting into eq. (21) the flux moments that are obtained from eq. (20). The eigenvalues of the T_N method is found by calculating $G_{n+1}(v)=0$, for any c and $(N+1)/2$ eigenvalues v_k, $k=1,2,3,\ldots,N+1$ is obtained. As an example, eigenvalues for T_2 are obtained as

$$v=\pm\frac{1}{\sqrt{2-2cf_0-2cf_1+2c^2f_0f_1}}.$$

It is shown that the eigenvalues depend on the secondary neutron number c and the scattering coefficients. The numerical values are substituted ($c=1.1$, $f_0=1$, $f_1=0.3$, $f_2=0.2$, $f_3=0.14$ and $f_4=0.11$) and the result becomes $v=\pm 2.73179\mathrm{i}$. The roots of the second-type Chebyshev polynomial are calculated by $T_{n+1}(\mu_k)=0$, where

$$\mu_k=\cos\left(\frac{(2k-1)\pi}{2(N+1)}\right),\qquad \frac{N+1}{2}<k\leq N+1.$$

The general expression can be written as, for odd N values,

Table 1. The critical half-thickness for tetra-anisotropic scattering with reflected slab by the P_N method ($f_1 = 0.3$, $f_2 = 0.20$, $f_3 = 0.142$, $f_4 = 0.11$).

c		P_1	P_3	P_5	P_7	P_9
1.01	$R = 0.00$	9.91238	9.80926	9.80427	9.80283	9.80221
	$R = 0.25$	9.29606	9.18618	9.17927	9.17716	9.17623
	$R = 0.50$	8.14290	8.03640	8.02800	8.02533	8.02413
	$R = 0.75$	5.54325	5.47384	5.46723	5.46505	5.46405
	$R = 0.99$	0.25115	0.25015	0.25010	0.25009	0.25008
1.05	$R = 0.00$	3.95698	3.81493	3.80872	3.80714	3.80648
	$R = 0.25$	3.40456	3.27738	3.26984	3.26780	3.26691
	$R = 0.50$	2.55344	2.46264	2.45558	2.45365	2.45279
	$R = 0.75$	1.33966	1.30358	1.29994	1.29909	1.29872
	$R = 0.99$	0.05025	0.04931	0.04926	0.04925	0.04924
1.20	$R = 0.00$	1.60986	1.44908	1.43309	1.43061	1.42974
	$R = 0.25$	1.20851	1.09792	1.08392	1.08134	1.08047
	$R = 0.50$	0.76924	0.70856	0.70012	0.69817	0.69753
	$R = 0.75$	0.35148	0.32742	0.32462	0.32392	0.32365
	$R = 0.99$	0.01256	0.01175	0.01167	0.01165	0.01164
1.40	$R = 0.00$	0.96655	0.82296	0.79621	0.79070	0.78919
	$R = 0.25$	0.67017	0.58154	0.56427	0.55985	0.55840
	$R = 0.50$	0.40100	0.35338	0.34486	0.34249	0.34159
	$R = 0.75$	0.17727	0.15751	0.15441	0.15362	0.15333
	$R = 0.99$	0.00628	0.00559	0.00549	0.00547	0.00546
1.60	$R = 0.00$	0.70134	0.57667	0.54600	0.53757	0.53483
	$R = 0.25$	0.46570	0.39150	0.37398	0.36858	0.36651
	$R = 0.50$	0.27137	0.23136	0.22282	0.22021	0.21916
	$R = 0.75$	0.11853	0.10171	0.09840	0.09746	0.09710
	$R = 0.99$	0.00419	0.00360	0.00349	0.00346	0.00344
1.80	$R = 0.00$	0.55302	0.44393	0.41285	0.40269	0.39881
	$R = 0.25$	0.35722	0.29336	0.27641	0.27060	0.26814
	$R = 0.50$	0.20501	0.17057	0.16218	0.15940	0.15822
	$R = 0.75$	0.08903	0.07444	0.07108	0.07003	0.06960
	$R = 0.99$	0.00314	0.00263	0.00251	0.00248	0.00246
2.00	$R = 0.00$	0.45743	0.36081	0.33074	0.31980	0.31513
	$R = 0.25$	0.28985	0.23382	0.21770	0.21176	0.20908
	$R = 0.50$	0.16485	0.13452	0.12641	0.12353	0.12226
	$R = 0.75$	0.07128	0.05844	0.05514	0.05401	0.05353
	$R = 0.99$	0.00251	0.00206	0.00195	0.00191	0.00189

$$\phi_n(x) = \sum_{k=1}^{\frac{N+1}{2}} \beta_k G_n(v_k) \times \left[\exp(x/v_k) + (-1)^n \exp(-x/v_k)\right], \quad n = 1, \ldots, N. \tag{22}$$

Here the parity rule is $G_n(-v_k) = (-1)^n G_n(v_k)$ and β_k can be determined from the boundary condition of the system. One finally obtains the angular flux for the T_N method as follows:

$$\psi(x, \mu_k) = \frac{T_0(\mu)}{\pi\sqrt{1-\mu^2}} \sum_{k=1}^{N+1/2} \beta_k G_0(v_k) \left[(2)\cosh\left(\frac{x}{v_k}\right)\right] + \frac{2}{\pi\sqrt{1-\mu^2}} \sum_{n=1}^{N} \sum_{k=1}^{N+1/2} \beta_k G_n(v_k) \left[\left(1+(-1)^n\right)\cosh\left(\frac{x}{v_k}\right) + \left(1-(-1)^n\right)\sinh\left(\frac{x}{v_k}\right)\right] T_n(\mu_k). \tag{23}$$

Table 2. Critical half-thickness for different reflection coefficients and secondary neutron numbers c for the P_N method obtained in P_9 order.

	c	$R=0$	$R=0.25$	$R=0.50$	$R=0.75$	$R=0.99$
Isotropic	1.01	8.33032	7.88905	7.05989	5.08259	0.25023
	1.20	1.29038	1.01696	0.68308	0.32602	0.01183
	1.60	0.51363	0.36647	0.22496	0.10097	0.00356
	2.00	0.31418	0.21570	0.12866	0.05685	0.00201
Linear anisotropic	1.01	9.81302	9.18686	8.03379	5.47017	0.25028
	1.20	1.45549	1.09846	0.70847	0.32866	0.01183
	1.60	0.56081	0.38300	0.22868	0.10130	0.00359
	2.00	0.33729	0.22285	0.13014	0.05698	0.00201
Pure quadratic	1.01	8.32003	7.87817	7.04897	5.07447	0.25004
	1.20	1.26539	0.99844	0.67249	0.32110	0.01165
	1.60	0.49008	0.35112	0.21617	0.09715	0.00346
	2.00	0.29507	0.20354	0.12172	0.05384	0.00191
Quadratic	1.01	9.80016	9.17317	8.02018	5.46079	0.25008
	1.20	1.42359	1.07656	0.69621	0.32365	0.01165
	1.60	0.53107	0.36572	0.21950	0.09745	0.00346
	2.00	0.31375	0.20948	0.12297	0.05394	0.00191
Pure triplet	1.01	8.33184	7.89127	7.06278	5.08520	0.25024
	1.20	1.29496	1.02034	0.68535	0.32625	0.01183
	1.60	0.51815	0.36848	0.22550	0.10102	0.00359
	2.00	0.31770	0.21698	0.12896	0.05688	0.00201
Triplet	1.01	9.80234	9.17643	8.02439	5.46427	0.25009
	1.20	1.43082	1.08145	0.69818	0.32392	0.01165
	1.60	0.53806	0.36841	0.22015	0.09751	0.00346
	2.00	0.31878	0.21109	012331	0.05397	0.00191
Pure tetra	1.01	8.33024	7.88891	7.05971	5.08241	0.25023
	1.20	1.28971	1.01631	0.68331	0.32579	0.01182
	1.60	0.51168	0.36515	0.22420	0.10064	0.00358
	2.00	0.31184	0.21423	0.12784	0.05650	0.00200
Tetra	1.01	9.80221	9.17623	8.02413	5.46405	0.25008
	1.20	1.42974	1.08047	0.69753	0.32365	0.01164
	1.60	0.53483	0.36651	0.21916	0.09710	0.00344
	2.00	0.31513	0.20908	0.12226	0.05353	0.00189

2. Criticality conditions

The critical thickness is calculated for a reflected slab reactor system. The core is surrounded by a reflecting material from all sides of the core for the high-order anisotropic scattering case. The reflector condition is represented as [12]

$$\psi(a,-\mu) = R\psi(a,\mu), \;\; \mu > 0. \tag{24}$$

Here, the critical half-thickness is demonstrated by the symbol a. The critical half-thickness is found for the P_N and T_N methods with the Marshak boundary condition as follows:

$$\int_0^1 (\psi(a,-\mu_k) - R\psi(a,\mu_k))P_m(-\mu_k)\mathrm{d}\mu = 0, \quad m = 1,3,\ldots,N$$

$$\int_0^1 (\psi(a,-\mu_k) - R\psi(a,\mu_k))T_m(-\mu_k)\mathrm{d}\mu = 0, \quad m = 1,3,\ldots,N. \tag{25}$$

The critical half-thickness for high-order anisotropic scattering is obtained by substituting eq. (25) into eq. (14) for the P_N method and into eq. (23) for the T_N method.

3. Numerical results

The solution of eq. (25) can be obtained by using any computer code. The criticality thickness equation depends on the secondary neutron number c, reflection coefficient R and scattering coefficients f_n. The critical thicknesses by varying c and R values are given in tables 1–9. The scattering coefficients used are $f_1 = 0.30$,

Table 3. The critical half-thickness for tetra-anisotropic scattering with reflected slab by the T_N method ($f_1 = 0.3$, $f_2 = 0.20$, $f_3 = 0.142$, $f_4 = 0.11$).

c		T_1	T_3	T_5	T_7	T_9
1.01	$R = 0.00$	12.18390	9.79805	9.80962	9.80659	9.80617
	$R = 0.25$	11.45600	9.16998	9.18723	9.18227	9.18182
	$R = 0.50$	10.08780	8.01215	8.03983	8.03229	8.03210
	$R = 0.75$	6.95328	5.44139	5.48237	5.47318	5.47308
	$R = 0.99$	0.31976	0.24774	0.25111	0.25058	0.25054
1.05	$R = 0.00$	4.88781	3.80591	3.81512	3.81180	3.81132
	$R = 0.25$	4.22976	3.26668	3.27778	3.27309	3.27261
	$R = 0.50$	3.20122	2.45078	2.64408	2.45884	2.45863
	$R = 0.75$	1.69762	1.29421	1.30575	1.30230	1.30258
	$R = 0.99$	0.06398	0.04885	0.04948	0.04936	0.04939
1.20	$R = 0.00$	2.00635	1.45229	1.44153	1.43657	1.43588
	$R = 0.25$	1.51843	1.09872	1.09151	1.08625	1.08566
	$R = 0.50$	0.97376	0.70706	0.70548	0.70139	0.70104
	$R = 0.75$	0.44695	0.32583	0.32706	0.32533	0.32527
	$R = 0.99$	0.01600	0.01168	0.01175	0.01170	0.01170
1.40	$R = 0.00$	1.21223	0.83259	0.80630	0.79728	0.79580
	$R = 0.25$	0.84664	0.58597	0.57154	0.56440	0.56305
	$R = 0.50$	0.50908	0.35473	0.34909	0.34508	0.34433
	$R = 0.75$	0.22557	0.15777	0.15619	0.15469	0.15451
	$R = 0.99$	0.00800	0.00560	0.00555	0.00550	0.00550
1.60	$R = 0.00$	0.88308	0.58779	0.55651	0.54430	0.54143
	$R = 0.25$	0.58987	0.39692	0.38069	0.37282	0.37077
	$R = 0.50$	0.34488	0.23371	0.22649	0.22250	0.22154
	$R = 0.75$	0.15086	0.10256	0.09993	0.09840	0.09811
	$R = 0.99$	0.00533	0.00363	0.00354	0.00349	0.00348
1.80	$R = 0.00$	0.69824	0.45506	0.42307	0.40927	0.40517
	$R = 0.25$	0.45316	0.29898	0.28254	0.27453	0.27205
	$R = 0.50$	0.26082	0.17325	0.16547	0.16149	0.16036
	$R = 0.75$	0.11332	0.07550	0.07246	0.07089	0.07051
	$R = 0.99$	0.00400	0.00267	0.00256	0.00251	0.00249
2.00	$R = 0.00$	0.57872	0.37147	0.34039	0.32607	0.32114
	$R = 0.25$	0.36807	0.23934	0.22332	0.21539	0.21267
	$R = 0.50$	0.20971	0.13726	0.12940	0.12545	0.12421
	$R = 0.75$	0.09075	0.05955	0.05640	0.05481	0.05435
	$R = 0.99$	0.00320	0.00210	0.00199	0.00194	0.00192

$f_2 = 0.20$, $f_3 = 0.142$, $f_4 = 0.11$ for each scattering type.

In tables 1 and 3, the critical half-thickness calculations are done for tetra-anisotropic scattering using the P_N and T_N methods. These calculations constitute the basis of our study.

In tables 2 and 4, the secondary number of neutrons is fixed and the scattering types are listed from isotropic to tetra-anisotropic scattering cases. The same scattering coefficients are applied for $c = 1.01$, 1.2, 1.6 and 2.0, and the reflection coefficients are changed from the bare system ($R = 0$) to the maximum reflection coefficient ($R = 0.99$) value.

In tables 5 and 6, the critical thickness values are obtained for pure tetra-anisotropic scattering by changing the value of the scattering coefficient f_4 from 0.11 to zero. It is well known that it drops to isotropic scattering if the scattering coefficient is zero. Our results in table 7 are compared with those of the isotropic scattering [12] for T_9 and P_9. Also in table 8, our P_N and T_N solutions on linear anisotropic scattering solutions are compared with Atalay's [10] results for the singular eigenfunction method. The reflector coefficient is examined for 0.25, 0.50, 0.75 and 0.99 and also for $R = 0$ which is known as the bare system. The secondary number of neutrons c is examined for $c = 1.01$, 1.05, 1.20, 1.40, 1.60, 1.80 and 2.00. This range for the secondary number of neutrons represents the criticality in the reactor system.

Because the critical thickness is examined in different scattering types for bare and reflected reactor systems by varying secondary number of neutrons, a wide spectrum of critical thickness is presented in table 9 in which a

Table 4. Critical half-thickness values for different reflector coefficients and secondary neutron numbers for the T_N method obtained in T_9 order.

	c	$R = 0$	$R = 0.25$	$R = 0.50$	$R = 0.75$	$R = 0.99$
İsotropic	1.01	8.33048	7.88931	7.06032	5.08320	0.25028
	1.20	1.29057	1.01719	0.68401	0.32613	0.01183
	1.60	0.51403	0.36683	0.22518	0.10106	0.00360
	2.00	0.31483	0.21614	0.12890	0.05694	0.00202
Linear anisotropic	1.01	9.81324	9.18723	8.03438	5.47092	0.25032
	1.20	1.45573	1.09872	0.70868	0.32877	0.01183
	1.60	0.56119	0.38335	0.22890	0.10139	0.00360
	2.00	0.33793	0.22330	0.13038	0.05707	0.00202
Pure quadratic	1.01	8.32017	7.87842	7.04940	5.07508	0.25008
	1.20	1.26559	0.99867	0.67270	0.32122	0.01167
	1.60	0.49057	0.35155	0.21642	0.09725	0.00346
	2.00	0.29588	0.20407	0.12199	0.05394	0.00191
Quadratic	1.01	9.80036	9.17352	8.02075	5.46153	0.25013
	1.20	1.42382	1.07682	0.69642	0.32377	0.01166
	1.60	0.53153	0.36613	0.21974	0.09755	0.00346
	2.00	0.31456	0.21001	0.12324	0.05405	0.00191
Pure triplet	1.01	8.33197	7.89151	7.06318	5.08578	0.25028
	1.20	1.29510	1.02053	0.68553	0.32636	0.01183
	1.60	0.51849	0.36881	0.22571	0.10111	0.00360
	2.00	0.31831	0.21741	0.12919	0.05697	0.00202
Triplet	1.01	9.80251	9.17674	8.02491	5.46497	0.25013
	1.20	1.43098	1.08165	0.69836	0.32403	0.01166
	1.60	0.53842	0.36878	0.22038	0.09761	0.00346
	2.00	0.31953	0.21160	0.12358	0.05408	0.00191
Pure tetra	1.01	8.33332	7.89320	7.06585	5.09071	0.25088
	1.20	1.29402	1.02025	0.68625	0.32724	0.01187
	1.60	0.51624	0.36843	0.22617	0.10150	0.00361
	2.00	0.31616	0.21708	0.12947	0.05720	0.00203
Tetra	1.01	9.80617	9.18182	8.03210	5.47398	0.25074
	1.20	1.43588	1.08566	0.70104	0.32527	0.01170
	1.60	0.54143	0.37077	0.22154	0.09811	0.00348
	2.00	0.32114	0.21267	0.12421	0.05435	0.00192

comparison of our results by the P_N and T_N methods with the exact ones by Kaper *et al* [22] and P_N results of Lee and Dias [23] (in which the Marshak boundary and 9th iteration has been used) is presented. It is seen that our results are in good agreement with those of Lee and Dias [23] and are in agreement with those of Kaper *et al* [22] for c values around one.

4. Conclusion

The solution of the critical thickness problem for the reflected boundary condition is done by many researchers. The T_N and P_N methods are applied to the linear anisotropic scattering, and the high-order anisotropic scattering calculations are performed by the F_N and H_N methods. Different from other studies, we presented a study by including the results of eight different scattering types in a single paper. Additionally, we present our results for a system without and with a reflector. The calculations were done using two independent methods. In tables 1 and 3, it is seen that the critical thickness decreases gradually as the reflector coefficient is increased. The values of critical half-thickness are decreased as expected by increasing the reflector coefficient, and it is also observed that the critical values are decreased when the second number of neutrons c is increased. Therefore, it can be concluded that the methods and the scattering types give reasonable results about the system under consideration. In each row of tables 3 and 5, it is seen that the decrease in critical thickness is obvious in the presence of reflector for each type of scattering, for different values of secondary number of neutrons. Since the P_N method is not affected by the

Table 5. Critical half-thickness for pure tetra-anisotropic scattering for P_9 in the P_N method with varying values of scattering coefficient f_4 ($f_1 = f_2 = f_3 = 0$, $f_4 = 0.11 - 0.0$).

c	f_4	$R = 0.00$	$R = 0.25$	$R = 0.50$	$R = 0.75$	$R = 0.99$
1.01	0.11	8.33024	7.88891	7.05971	5.08241	0.25023
	0.09	8.33025	7.88894	7.05974	5.08245	0.25023
	0.07	8.33027	7.88896	7.05978	5.08248	0.25023
	0.05	8.33029	7.88899	7.05981	5.08251	0.25023
	0.03	8.33030	7.88901	7.05984	5.08254	0.25023
	0.00	8.33032	7.88904	7.05989	5.08259	0.25023
1.2	0.11	1.28971	1.01631	0.68331	0.32579	0.01182
	0.09	1.28985	1.01644	0.68341	0.32584	0.01182
	0.07	1.28997	1.01656	0.68351	0.32588	0.01182
	0.05	1.29009	1.01668	0.68360	0.32592	0.01182
	0.03	1.29021	1.01680	0.68368	0.32596	0.01183
	0.00	1.29038	1.01696	0.68381	0.32602	0.01183
1.6	0.11	0.51168	0.36515	0.22420	0.10064	0.00358
	0.09	0.51207	0.36541	0.22435	0.10071	0.00358
	0.07	0.51244	0.36566	0.22450	0.10077	0.00359
	0.05	0.51280	0.36591	0.22464	0.10083	0.00359
	0.03	0.51314	0.36614	0.22477	0.10089	0.00359
	0.00	0.51363	0.36647	0.22496	0.10097	0.00359
2.0	0.11	0.31184	0.21433	0.12784	0.05650	0.00200
	0.09	0.31230	0.21452	0.12801	0.05657	0.00200
	0.07	0.31274	0.21480	0.12816	0.05663	0.00201
	0.05	0.31317	0.21507	0.12831	0.05670	0.00201
	0.03	0.31358	0.21533	0.12846	0.05676	0.00201
	0.00	0.31418	0.21570	0.12866	0.05685	0.00201

Table 6. Critical half-thickness for pure tetra-anisotropic scattering for T_9 in T_N method with varying values of scattering coefficient f_4 ($f_1 = f_2 = f_3 = 0$, $f_4 = 0.11 - 0.0$).

c	f_4	$R = 0.00$	$R = 0.25$	$R = 0.50$	$R = 0.75$	$R = 0.99$
1.01	0.11	8.33332	7.89320	7.06585	5.09071	0.25088
	0.09	8.33277	7.89245	7.06478	5.08925	0.25076
	0.07	8.33224	7.89172	7.06374	5.08784	0.25065
	0.05	8.33172	7.89101	7.06273	5.08646	0.25057
	0.03	8.33122	7.89031	7.06174	5.08513	0.25043
	0.00	8.33048	7.88931	7.06032	5.08320	0.25028
1.2	0.11	1.29402	1.02025	0.68625	0.32724	0.01187
	0.09	1.29336	1.01966	0.68581	0.32702	0.01186
	0.07	1.29271	1.01908	0.68539	0.32682	0.01186
	0.05	1.29208	1.01853	0.68499	0.32661	0.01185
	0.03	1.29147	1.01798	0.68459	0.32642	0.01184
	0.00	1.29057	1.01719	0.68401	0.32613	0.01183
1.6	0.11	0.51624	0.36843	0.22617	0.10150	0.00361
	0.09	0.51581	0.36812	0.22598	0.10142	0.00361
	0.07	0.51540	0.36782	0.22580	0.10133	0.00361
	0.05	0.51500	0.36753	0.22561	0.10125	0.00360
	0.03	0.51460	0.36725	0.22544	0.10117	0.00360
	0.00	0.51403	0.36683	0.22518	0.10106	0.00360
2.0	0.11	0.31616	0.21708	0.12947	0.05720	0.00203
	0.09	0.31590	0.21690	0.12936	0.05715	0.00202
	0.07	0.31565	0.21672	0.12925	0.05710	0.00202
	0.05	0.31541	0.21655	0.12915	0.05705	0.00202
	0.03	0.31517	0.21639	0.12905	0.05701	0.00202
	0.00	0.31483	0.21614	0.12890	0.05694	0.00202

Table 7. Comparison of our results with that of the isotropic scattering by Anli *et al* [12].

R	*c*	T_9		P_9	
		Isotropic [12]	Present work	Isotropic [12]	Present work
0.00	1.01	8.3308	8.3305	8.3305	8.3303
	1.10	2.1144	2.1144	2.1142	2.1142
	1.60	0.5140	0.5140	0.5136	0.5136
	1.80	0.3917	0.3917	0.3911	0.3911
	2.00	0.3148	0.3148	0.3142	0.3142
0.25	1.01	7.8894	7.8893	7.8891	7.8891
	1.10	1.7717	1.7717	1.7715	1.7715
	1.60	0.3668	0.3668	0.3665	0.3665
	1.80	0.2733	0.2733	0.2729	0.2729
	2.00	0.2161	0.2161	0.2157	0.2157
0.50	1.01	7.0604	7.0603	7.0599	7.0599
	1.10	1.2845	1.2845	1.2843	1.2843
	1.60	0.2252	0.2252	0.2250	0.2250
	1.80	0.1649	0.1649	0.1647	0.1647
	2.00	0.1289	0.1289	0.1287	0.1287
0.75	1.01	5.0832	5.0832	5.0826	5.0859
	1.10	0.6543	0.6543	0.6541	0.6541
	1.60	0.1011	0.1011	0.1010	0.1010
	1.80	0.0733	0.0733	0.0732	0.0732
	2.00	0.0569	0.0569	0.0569	0.0569
0.99	1.01	0.2503	0.2503	0.2502	0.2502
	1.10	0.0243	0.0243	0.0243	0.0243
	1.60	0.0047	0.0036	0.0047	0.0036
	1.80	0.0030	0.0026	–	0.0026
	2.00	0.0021	0.0020	–	0.0020

Table 8. Comparison of our critical slab thickness on a linearly anisotropic scattering with Atalay [10].

R	*c*	Case singular eigenfunction method	P_N Method	T_N Method
0.00	1.01	19.62374	19.62604	19.62648
	1.10	4.84124	4.84364	4.84409
	1.60	1.13657	1.12162	1.12238
	1.80	0.87360	0.84987	0.84702
	2.00	0.71692	0.67458	0.67586
0.25	1.01	18.32167	18.37372	18.37446
	1.10	3.87306	3.91570	3.91631
	1.60	0.90838	0.76600	0.76670
	1.80	0.57121	0.56649	0.56731
	2.00	0.46329	0.44570	0.44660
0.50	1.01	15.90535	16.06758	16.06876
	1.10	2.63239	2.71308	2.71366
	1.60	0.44286	0.45736	0.45780
	1.80	0.33005	0.33380	0.33426
	2.00	0.26606	0.26028	0.26076
0.75	1.01	10.60644	10.94034	10.94184
	1.10	1.25773	1.32740	1.32771
	1.60	0.19364	0.20260	0.20278
	1.80	0.14364	0.14681	0.14699
	2.00	0.11550	0.11396	0.11414
0.99	1.01	0.46963	0.50056	0.50064
	1.10	0.04555	0.04868	0.04870
	1.60	0.00684	0.00718	0.00720
	1.80	0.00507	0.00520	0.00520
	2.00	0.00407	0.00402	0.00404

Table 9. Comparison of our critical slab thickness (by the P_N and T_N methods) on isotropic scattering with the exact results by Kaper [22] and Lee and Dias [23].

c	Exact [22]	Lee and Dias [23]	P_9	T_9
1.02	5.665505456	5.66632540	5.666325399	5.666482775 (13.31s)
1.05	3.300263772	3.30111271	3.301112713	3.301277022 (12.95s)
1.10	2.113309666	2.11420844	2.114208435	2.114380010 (14.56s)
1.20	1.289379285	1.29037912	1.290379124	1.290572302 (13.31s)
1.30	0.937725560	0.93882295	0.938822945	0.939043947 (13.26s)
1.40	0.736603550	0.73782620	0.737862004	0.738092111 (13.39s)
1.60	0.511962980	0.51363113	0.513631132	0.514030650 (13.15s)
2.00	0.311025980	0.31417523	0.314175228	0.314829239 (13.72s)

small differences of coefficients (see eq. (14)), the critical thickness values for $R = 0.99$ are almost similar for the same number of secondary neutron number, as seen in table 5. Additionally, it is seen that the critical half-thickness decreases as the reflector coefficient increases for each value of f_4 for any value of c. The scattering coefficient f_4 is decreased from 0.11 to 0 to compare our results to those of isotropic scattering. Finally, we compare our results with the isotropic scattering done by Anli *et al* [12], and with the results of linear anisotropic scattering with reflection coefficients by Atalay [10]. Not only our results in tables 7 and 8 are in good agreement with those of Anli *et al* [12] and Atalay [10], but they are also in agreement with those of Kaper *et al* [22] and Lee and Dias [23] in table 9. In table 8, the results of T_N expansion seem to be more accurate than those of P_N method as the reflection coefficient R increases. The comparison of all results shows that the calculations are logical and coherent with each other and with literature. Comparison with the results of Lee and Dias [23] in table 9 shows that the P_N method is better than the T_N expansion. As Lee and Dias [23] have already presented their results up to P_{15} by using Marshak boundary condition in their study to compare with those of Kaper *et al* [22], we then compare our results for P_9 by considering those of Lee and Dias [23] and Kaper *et al* [22] to observe the consistency. Therefore, one can conclude that the results of the P_N method are better than those of the T_N expansion. The accuracy of our results in the frame of Marshak boundary condition must be validated with that of Mark boundary condition. It may be the content of another study. Furthermore, our results yield a wide range of spectrum of the criticality data for bare and reflected reactor on eight different scattering types. Critical thickness values with respect to many parameters have been tabulated in tables throughout the study, and it is believed that any researcher can compare her/his own results with the data presented in this study.

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments on the article.

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