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# Off-shell T-matrix for the Manning–Rosen potential

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**Abstract.** New analytical expressions for the off-shell wave functions and T-matrix with the Manning–Rosen potential are constructed in terms of generalised hypergeometric functions. The off-shell T-matrices are computed for the neutron–proton and neutron–deuteron systems. The limiting behaviours of our expression for the off-shell T-matrix are verified and found correct.

**Keywords.** Manning–Rosen potential; off-shell wave function; off-shell T-matrix; limiting behaviours.

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## 1. Introduction

The two-particle T-matrix plays an important role in the quantum theoretical description of scattering/reaction between two or more than two particles. Reactions between neutral particles, whose pairwise interaction has a short range, are effectively described with the help of the integral equations of Faddeev [1,2] and similar equations that have been developed later. The two-particle T-matrices are the basic ingredients of these integral equations. It is well known that in two-particle scattering theory, virtually all the relevant scattering quantities can be obtained directly from the T-matrices [3–13]. The closed form analytical expressions for the Coulomb off-shell T-matrix elements in all angular momentum states can be seen in refs [11,13,14]. However, for some exponential type of potentials like Hulthén, Morse etc. the same analytical expressions [15–17] are restricted to the lowest angular momentum state ($\ell = 0$). The Manning–Rosen potential [18–21] is generally used in the atomic and molecular domains. In the recent past, we have applied it successfully in the realm of nuclear physics [22–25] in the context of elastic and semielastic scattering. In ref. [24] we have derived off-shell Jost function and half-shell T-matrix without calculating off-shell Jost and physical solutions of the Schrödinger equation. The main difficulties involved in deriving the off-shell solutions are the evaluations of some indefinite integrals involving Manning–Rosen Green's function. The present text deals with the evaluation of these indefinite integrals and thereby construction of off-shell solutions. It is of importance to have the off-shell Jost and physical solutions as well as the T-matrices in the literature associated with the scattering by the Manning–Rosen potential which is frequently used in atomic, molecular and nuclear physics. Exploiting the off-shell solutions we derive a closed form expression of the s-wave off-shell T-matrix for the Manning–Rosen potential. To the authors' knowledge, the results for the off-shell solutions and T-matrix associated with the potential under consideration are new. In §2 we construct analytical expressions for the off-shell Jost and physical solutions. In §3 we calculate off-shell T-matrix by using an expression which does not involve the potential explicitly. Section 4 is related to the results and discussion. We conclude in §5.

## 2. Off-shell Jost and physical solutions

### 2.1 *Jost solution*

In the framework of our approach we derive off-shell solutions by directly integrating the particular integrals of the inhomogeneous Schrödinger equation. The inhomogeneous Schrödinger equation for the off-shell Jost solution reads as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-\frac{1}{b^2}\left\{\frac{\alpha\,(\alpha-1)\,\mathrm{e}^{-2r/b}}{\left(1-\mathrm{e}^{-r/b}\right)^2}-\frac{A\mathrm{e}^{-r/b}}{\left(1-\mathrm{e}^{-r/b}\right)}\right\}\right]\times f_m\,(k,q,r)=\left(k^2-q^2\right)\,\mathrm{e}^{iqr}. \quad (1)$$

Here $A$, $b$ and $\alpha$ are three adjustable parameters of the Manning–Rosen potential. The particular integral

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of eq. (1) represents the off-shell Jost solution [26–28]

$$f_m(k,q,r) = \left(k^2-q^2\right)\int_r^\infty \mathrm{d}r' \mathrm{e}^{iqr'} G_m^{(I)}\left(r,r'\right)$$
$$= \left(k^2-q^2\right)\int_0^\infty \mathrm{d}r' \mathrm{e}^{iqr'} G_m^{(I)}\left(r,r'\right)$$
$$+\left(k^2-q^2\right)\int_0^r \mathrm{d}r' \mathrm{e}^{iqr'} G_m^{(R)}\left(r,r'\right), \quad (2)$$

where the Manning–Rosen Green's functions are related to the irregular and regular boundary conditions

$$G_m^{(I)}\left(r,r'\right) = \frac{1}{f_m(k)}\left[\varphi_m\left(k,r'\right) f_m(k,r) - \varphi_m(k,r) f_m\left(k,r'\right)\right] \quad (3)$$

and

$$G_m^{(R)}\left(r,r'\right) = \frac{1}{f_m(k)}\left[\varphi_m(k,r) f_m\left(k,r'\right) - \varphi_m\left(k,r'\right) f_m(k,r)\right]. \quad (4)$$

The functions $\varphi_m(k,r)$ and $f_m(k,r)$ represent the on-shell regular and irregular solutions of the potential under consideration and $f_m(k)$, the on-shell Jost function, is written as [22–24]

$$\varphi_m(k,r) = b^{\alpha+1}\left(1-\mathrm{e}^{-r/b}\right)^{\alpha+1}\mathrm{e}^{ikr} \times {}_2F_1\left(A',B';C';1-\mathrm{e}^{-r/b}\right), \quad (5)$$

$$f_m(k,r) = \left(1-\mathrm{e}^{-r/b}\right)^{-\alpha}\mathrm{e}^{ikr} \times {}_2F_1\Big(A'-2\alpha-1,B'-2\alpha-1; 1-2ikb;\mathrm{e}^{-r/b}\Big) \quad (6)$$

and

$$f_m(k) = b^\alpha \frac{\Gamma(1-2ikb)\,\Gamma(2\alpha+2)}{\Gamma(A')\,\Gamma(B')} \quad (7)$$

with

$$A' = 1+\alpha-ikb+\left(\alpha^2+\alpha-k^2b^2+A\right)^{1/2}, \quad (8a)$$
$$B' = 1+\alpha-ikb-\left(\alpha^2+\alpha-k^2b^2+A\right)^{1/2}, \quad (8b)$$
$$C' = 2\alpha+2. \quad (8c)$$

To evaluate the definite and indefinite integrals involved in eq. (2) we proceed as follows: Combination of eqs (3), (5) and (6), along with the use of the standard integrals for the Gaussian hypergeometric functions [29–32] the definite integral yields

$$\left(k^2-q^2\right)\int_0^\infty \mathrm{d}r' \mathrm{e}^{iqr'} G_m^{(I)}\left(r,r'\right)$$
$$= -\frac{\left(k^2-q^2\right)}{f_m(k)} b^{\alpha+2}\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/b}\right)^{\alpha+1} \times {}_2F_1\left(A',B';C';\ 1-\mathrm{e}^{-r/b}\right)$$
$$\times \frac{\Gamma(-ib(k+q))\,\Gamma(1-\alpha)}{\Gamma(1-\alpha-ib(k+q))} \times {}_3F_2\big(A'-2\alpha-1,\ B'-2\alpha-1,\ -ib(k+q);\ 1-2ikb,\ 1-\alpha-ib(k+q);\ 1\big)$$
$$+\frac{f_m(k,q)}{f_m(k)}\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/b}\right)^{-\alpha} \times {}_2F_1\left(A'-2\alpha-1,\ B'-2\alpha-1;\ 1-2ikb;\ \mathrm{e}^{-r/b}\right), \quad (9)$$

where the off-shell Jost function [24]

$$f_m(k,q) = \left(k^2-q^2\right) b^{\alpha+2}\frac{\Gamma(\alpha+2)\,\Gamma(-ib(q+k))}{\Gamma(\alpha+2-ib(q+k))} \times {}_3F_2\Big(A',B',\ \alpha+2;\ 2\alpha+2,\alpha+2 -ib(q+k);1\Big). \quad (10)$$

In deriving the above expression, we have used [29–32]

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\,{}_2F_1(\alpha,\beta;\gamma;x)\,\mathrm{d}x = \frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\,{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;1). \quad (11)$$

For the indefinite integral in eq. (2) we first rewrite the regular Green's function as

$$G_m^{(R)}\left(r,r'\right) = \frac{b}{(1+2\alpha)}\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/b}\right)^{\alpha+1}$$
$$\times\Bigg[{}_2F_1\left(A',B';C';\ 1-\mathrm{e}^{-r/b}\right)\mathrm{e}^{ikr'}\left(1-\mathrm{e}^{-r'/b}\right)^{-\alpha} \times {}_2F_1\left(A'-1-2\alpha,\ B'-1-2\alpha;\ -2\alpha;\ 1-\mathrm{e}^{-r'/b}\right)$$
$$-(1-\mathrm{e}^{-r/b})^{-2\alpha-1} \times {}_2F_1\left(A'-1-2\alpha,\ B'-1-2\alpha;\ -2\alpha;\ 1-\mathrm{e}^{-r/b}\right)$$
$$\times\ \mathrm{e}^{ikr'}(1-\mathrm{e}^{-r'/b})^{\alpha+1}\,{}_2F_1\left(A',B';C';\ 1-\mathrm{e}^{-r'/b}\right)\Bigg]. \quad (12)$$

Substituting $z' = (1-\mathrm{e}^{-r'/b})$ and utilising the standard integral [33]

$$f_\sigma(a,b;c;z) = \frac{1}{c-1}\Bigg[{}_2F_1(a,b;c;z) \times \int_0^z s^{\sigma-1}(1-s)^{a+b-c}$$

$$\times{}_2F_1\left(a-c+1, b-c+1; 2-c; s\right)\, \mathrm{d}s$$
$$-z^{1-c}{}_2F_1\left(a-c+1, b-c+1; 2-c; z\right)$$
$$\left.\times\int_0^z s^{\sigma+c-2}(1-s)^{a+b-c}{}_2F_1\left(a, b; c; s\right)\, \mathrm{d}s\right] \quad (13)$$

in the indefinite integral of eq. (2) one gets

$$\left(k^2-q^2\right)\int_0^r \mathrm{d}r'\mathrm{e}^{iqr'}G_m^{(R)}\left(r, r'\right)$$
$$= b^2\left(k^2-q^2\right)\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/b}\right)^{\alpha+1}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(1-ib\left(k-q\right)+n\right)}{\Gamma\left(1-ib\left(k-q\right)\right)n!}$$
$$\times f_{1-\alpha+n}\left(A', B'; C'; 1-\mathrm{e}^{-r/b}\right). \quad (14)$$

Therefore, eqs (2), (9) and (14) reproduce the desired expression for the off-shell Jost solution as

$$f_m\left(k, q, r\right) = -\frac{\left(k^2-q^2\right)}{f_m\left(k\right)}b^{\alpha+2}\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/b}\right)^{\alpha+1}$$
$$\times{}_2F_1\left(A', B'; C'; 1-\mathrm{e}^{-r/b}\right)$$
$$\times\frac{\Gamma\left(-ib\left(k+q\right)\right)\Gamma\left(1-\alpha\right)}{\Gamma\left(1-\alpha-ib\left(k+q\right)\right)}$$
$$\times{}_3F_2\left(A'-2\alpha-1, B'-2\alpha-1, -ib\left(k+q\right); 1\right.$$
$$\left.-2ikb, 1-\alpha-ib\left(k+q\right);\ 1\right)$$
$$+\frac{f_m\left(k, q\right)}{f_m\left(k\right)}\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/b}\right)^{-\alpha}$$
$$\times{}_2F_1\left(A'-2\alpha-1, B'-2\alpha-1; 1-2ikb; \mathrm{e}^{-r/b}\right)$$
$$+b^2\left(k^2-q^2\right)\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/b}\right)^{\alpha+1}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(1-ib\left(k-q\right)+n\right)}{\Gamma\left(1-ib\left(k-q\right)\right)n!}$$
$$\times f_{1-\alpha+n}\left(A', B'; C'; 1-\mathrm{e}^{-r/b}\right). \quad (15)$$

It is well known that when $r\to 0$, the off-shell Jost solution goes over to Jost function. Thus, using the relation [23]

$$f_m\left(k, q\right) = \left(2\alpha+1\right)b^{\alpha}\lim_{r\to 0}\left(1-\mathrm{e}^{-r/b}\right)^{\alpha}f_m\left(k, q, r\right) \quad (16)$$

it can easily be verified that eq. (15) produces eq. (10).

## 2.2 *Physical solution*

The off-shell physical solution satisfies the inhomogeneous differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-\frac{1}{b^2}\left\{\frac{\alpha\left(\alpha-1\right)\mathrm{e}^{-2r/b}}{\left(1-\mathrm{e}^{-r/b}\right)^2}\right.\right.$$
$$\left.\left.-\frac{A\mathrm{e}^{-r/b}}{\left(1-\mathrm{e}^{-r/b}\right)}\right\}\right]\psi_m^{(+)}\left(k, q, r\right) = \left(k^2-q^2\right)\sin qr. \quad (17)$$

The particular solution to eq. (17) is

$$\psi_m^{(+)}\left(k, q, r\right) = \left(k^2-q^2\right)\int_0^{\infty}\mathrm{d}r'\sin qr'G_m^{(+)}\left(r, r'\right)$$
$$= \frac{\left(k^2-q^2\right)}{2i}\left[\bar{G}_m^{(+)}\left(r, q\right)-\bar{G}_m^{(+)}\left(r, -q\right)\right], \quad (18)$$

where

$$\bar{G}_m^{(+)}\left(r, q\right) = \int_0^{\infty}\mathrm{d}r'\, G_m^{(+)}\left(r, r'\right)\mathrm{e}^{iqr'} \quad (19)$$

and

$$\bar{G}_m^{(+)}\left(r, -q\right) = \bar{G}_m^{(+)}\left(r, q\right)\Big|_{q\to -q}.$$

The physical Green's function $G_m^{(+)}\left(r, r'\right)$ is expressed as

$$G_m^{(+)}\left(r, r'\right) = -\frac{1}{k}\psi_m^{(+)}\left(k, r_<\right)f_m\left(k, r_>\right). \quad (20)$$

The quantity $\psi_m^{(+)}\left(k, r\right)$ stands for the on-shell physical solution of the Manning–Rosen potential given by

$$\psi_m^{(+)}\left(k, r\right) = \frac{k\varphi_m\left(k, r\right)}{f_m\left(k\right)}. \quad (21)$$

Combination of eqs (19)–(21) yields

$$\bar{G}_m^{(+)}\left(r, q\right) = -k\left[\int_0^r G_m^{(R)}\left(r, r'\right)\mathrm{e}^{iqr'}\mathrm{d}r'\right.$$
$$\left.-\frac{\varphi_m\left(k, r\right)}{f_m\left(k\right)}\int_0^{\infty}\mathrm{d}r' f_m\left(k, r'\right)\mathrm{e}^{iqr'}\right]. \quad (22)$$

Using eq. (11) and the standard result in eq. (14), eq. (22) leads to

$$\bar{G}_m^{(+)}\left(r, q\right) = b^2\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/b}\right)^{\alpha+1}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(n+1-ib\left(k-q\right)\right)}{\Gamma\left(1-ib\left(k-q\right)\right)n!}$$
$$\times f_{1-\alpha+n}\left(A', B'; C'; 1-\mathrm{e}^{-r/b}\right)$$

$$-\frac{b^{\alpha+2}}{f_m(k)}\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/b}\right)^{\alpha+1}$$
$$\times\frac{\Gamma(-ib(k+q))\Gamma(1-\alpha)}{\Gamma(1-\alpha-ib(k+q))}$$
$$\times{}_2F_1\left(A',B';C';1-\mathrm{e}^{-r/b}\right)$$
$$\times{}_3F_2\left(A'-2\alpha-1,B'-2\alpha-1,-ib(k+q);\right.$$
$$\left.1-2ikb,1-\alpha-ib(k+q);1\right). \tag{23}$$

Substituting $q$ by $-q$ in eq. (23) one gets an expression for $\bar{G}_m^{(+)}(r,-q)$, thereby the complete solution $\psi_m^{(+)}(k,q,r)$ from eq. (18). The off-shell T-matrix has two representations: one with explicit dependence of the potential and the other without [13,14]. In the following section we shall calculate the off-shell T-matrix by using the expression which does not involve the potential explicitly.

## 3. Off-shell T-matrix

The relation between off-shell physical solution and T-matrix without explicit dependence on the potential reads as [13,14,34,35]

$$T\left(p,q,k^2\right)=\frac{2\left(k^2-p^2\right)}{\pi pq}\int_0^\infty \mathrm{d}r\sin(pr)\ \psi^{(+)}(k,q,r)$$
$$=\frac{\left(k^2-p^2\right)\left(q^2-k^2\right)}{2\pi pq}\left\{T_1\left(p,q,k^2\right)\right.$$
$$-T_2\left(p,-q,k^2\right)-T_3\left(-p,q,k^2\right)$$
$$\left.+T_4\left(-p,-q,k^2\right)\right\} \tag{24}$$

with

$$T_1\left(p,q,k^2\right)=\int_0^\infty \mathrm{d}r\mathrm{e}^{ipr}\bar{G}_m^{(+)}(r,q), \tag{25a}$$

$$T_2\left(p,-q,k^2\right)=\int_0^\infty \mathrm{d}r\mathrm{e}^{ipr}\bar{G}_m^{(+)}(r,-q), \tag{25b}$$

$$T_3\left(-p,q,k^2\right)=\int_0^\infty \mathrm{d}r\mathrm{e}^{-ipr}\bar{G}_m^{(+)}(r,q) \tag{25c}$$

and

$$T_4\left(-p,-q,k^2\right)=\int_0^\infty \mathrm{d}r\mathrm{e}^{-ipr}\bar{G}_m^{(+)}(r,-q). \tag{25d}$$

To have an explicit expression for $T\left(p,q,k^2\right)$ one needs to calculate one of the four integrals in eq. (24) and the other integrals follow automatically either by substituting $q\to -q$ or $p\to -p$. For the first integral in eq. (24) we put the independent variable $1-\mathrm{e}^{-r/b}=z$, utilise the following relation [33]:

$$\int_0^s z^{\rho-1}(s-z)^{\nu-1} f_\sigma\left(a,b;c;p'z\right)\mathrm{d}z$$
$$=\frac{\Gamma(\rho+\sigma)\Gamma(\nu)}{\sigma(\sigma+c-1)\Gamma(\rho+\sigma+\nu)}p'^{\sigma}s^{\rho+\sigma+\nu-1}$$
$$\times{}_4F_3\left(1,\sigma+a,\sigma+b,\rho+\sigma;\right.$$
$$\left.\sigma+1,\sigma+c,\rho+\sigma+\nu;p's\right) \tag{26}$$

and make some algebraic manipulation to get

$$T_1\left(p,q,k^2\right)$$
$$=\frac{b^{\alpha+1}}{f_m(k)}\frac{\Gamma(1-ib(k+q))\Gamma(1-\alpha)}{\Gamma(1-\alpha-ib(k+q))\ (k+p)(k+q)}$$
$$\times{}_3F_2\left(A'-1-2\alpha,B'-1-2\alpha,-ib(k+q);\right.$$
$$\left.1-2ikb,1-\alpha-ib(k+q);1\right)$$
$$\times\frac{\Gamma(\alpha+2)\Gamma(1-ib(k+p))}{\Gamma(\alpha+2-ib(k+p))}$$
$$\times{}_3F_2\left(A',B',\alpha+2;C',\alpha+2-ib(k+p);1\right)$$
$$+\frac{ib^2}{(k+p)}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-ib(k-q))}{\Gamma(1-ib(k-q))\ n!}$$
$$\times\frac{\Gamma(3+n)\Gamma(1-ib(k+p))}{(n+2+\alpha)(n+1-\alpha)\Gamma(3+n-ib(k+p))}$$
$$\times{}_4F_3\left(1,1-\alpha+n+A',1-\alpha+n+B',3+n;\right.$$
$$\left.2-\alpha+n,\alpha+n+3,3+n-ib(k+p);1\right). \tag{27}$$

Thus, judicious combination of eqs (24), (25) and (27) produces the desired expression for the off-shell T-matrix for motion in the Manning–Rosen potential. We shall make some useful checks on our expression for the off-shell T-matrix with respect to its half-shell and on-shell limits. Also in the limit $\alpha\to 0$, the Manning–Rosen potential becomes the Hulthén one. In Appendix A we shall show that the Manning–Rosen off-shell T-matrix reproduces the correct Hulthén limit.

*Half-off shell limit*

From eqs (24) and (27) it is noticed that for $q=k$ the first and third terms in eq. (24) vanish and the remaining second and fourth terms read as

$$\left.\frac{\left(k^2-p^2\right)\left(q^2-k^2\right)}{2\pi pq}T_2\left(p,q,k^2\right)\right|_{q=k}$$
$$=\frac{(k-p)}{\pi p}\left\{-\frac{b^{\alpha+1}}{f_m(k)}\frac{\Gamma(\alpha+2)\Gamma(1-ib(k+p))}{\Gamma(\alpha+2-ib(k+p))}\right.$$
$$\left.\times{}_3F_2(A',B',\alpha+2;C',\alpha+2-ib(k+p);1)\right\} \tag{28}$$

and

$$\left.\frac{\left(k^2-p^2\right)\left(q^2-k^2\right)}{2\pi pq}T_4\left(p,q,k^2\right)\right|_{q=k}$$
$$=\frac{(k+p)}{\pi p}\left\{-\frac{b^{\alpha+1}}{f_m(k)}\frac{\Gamma(\alpha+2)\Gamma(1-ib(k-p))}{\Gamma(\alpha+2-ib(k-p))}\right.$$
$$\left.\times{}_3F_2(A',B',\alpha+2;C',\alpha+2-ib(k-p);1)\right\}. \tag{29}$$

Thus, eq. (24) in conjunction with eqs (7), (28) and (29) becomes

$$T\left(p,k,k^2\right)=\frac{f_m(k,p)-f_m(k,-p)}{i\pi pf_m(k)}. \tag{30}$$

Our expression for the off-shell T-matrix produces the correct half-shell limit.

*On-shell limit*

For $p=q=k$ one gets

$$\left.\frac{\left(k^2-p^2\right)\left(q^2-k^2\right)}{2\pi pq}T_4\left(p,q,k^2\right)\right|_{p=q=k}$$
$$=-\frac{2}{\pi}\frac{b^{\alpha+1}}{f_m(k)}{}_2F_1(A',B';C';1). \tag{31}$$

Using eq. (7) and the following relation [29–31]

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{32}$$

eq. (31) yields

$$T(k,k,k^2)=-\frac{\Gamma(1+2ikb)}{ik\pi\,\Gamma(1-2ikb))}. \tag{33}$$

## 4. Results and discussion

To compute the off-shell T-matrices for (n–p) and (n–d) systems we use the parameters of the $^1S_0$, $^3S_1$ and $^2S_{1/2}$ states as given in refs [22,23]. These parameters are $A=0.952$, $b=1.152$ fm, $\alpha=-0.0043$ for $^1S_0$, $A=1.57$, $b=1.2135$ fm, $\alpha=0.005$ for $^3S_1$ and $A=2.1054$, $b=1.1069$ fm, $\alpha=0.005$ for $^2S_{1/2}$ states. In computing the off-shell T-matrices we use $\hbar^2/m_p=41.47$ MeV fm$^2$. Our results for the T-matrices for $^1S_0$ and $^3S_1$ (n–p) states are presented in tables 1 and 2 with $E_{\text{Lab}}=10$ and 20 MeV and the same for the $^2S_{1/2}$ (n–d) state in table 3 with $E_{\text{Lab}}=6$ and 10.5 MeV as a function of $p$. The values of the other off-shell momenta are considered to be $q=0.25$ and 0.55 fm$^{-1}$.

From table 1, it is observed that for $E_{\text{lab}}=10$ MeV and $q=0.25$ fm$^{-1}$ the real and imaginary parts of the T-matrices decrease continuously over the whole range of the off-shell momentum $p$ which is in agreement with those of ref. [14], computed based on Yamaguchi non-local potential. Although the T-matrices for the Manning–Rosen and Yamaguchi potentials follow the same trend, they differ numerically. For the same laboratory energy but with $q=0.55$ fm$^{-1}$, Re $T(p,q,k^2)$ initially decreases up to $p=0.5$ fm$^{-1}$ and beyond that point increases smoothly to zero, while Im $T(p,q,k^2)$ increases with $p$ in accordance with ref. [14]. For $E_{\text{lab}}=20$ MeV and $q=0.25$ fm$^{-1}$, Re $T(p,q,k^2)$ increases monotonically with $p$ while Im $T(p,q,k^2)$ first decreases up to $p=0.7$ fm$^{-1}$ and then increases continuously up to $p=6.0$ fm$^{-1}$. On the other hand, with $q=0.55$ fm$^{-1}$, Re $T(p,q,k^2)$ decreases up to $p=0.3$ fm$^{-1}$ at the beginning and then increases whereas Im $T(p,q,k^2)$ increases with off-shell momentum $p$. This different behaviour of the T-matrices for Manning–Rosen and Yamaguchi potentials is quite obvious because there are several nuclear potentials which produce the same on-shell effects but lead to different off-shell characteristics [36]. For (n–p) $^3S_1$ state, Re $T(p,q,k^2)$s, presented in table 2, initially increase up to $p=0.5$ and 1.0 fm$^{-1}$ with $E_{\text{Lab}}=10$ and 20 MeV respectively for $q=0.25$ fm$^{-1}$ and then decrease gradually as $p$ increases, whereas Im $T(p,q,k^2)$s for $E_{\text{Lab}}=10$ and 20 MeV initially decrease up to $p=0.5$ and 0.7 fm$^{-1}$ respectively and then increase smoothly. For $q=0.55$ fm$^{-1}$, Re $T(p,q,k^2)$ at $E_{\text{Lab}}=10$ MeV reaches its lower peak at $p=0.6$ fm$^{-1}$ and for $E_{\text{Lab}}=20$ MeV it shows small oscillations between $p=0.4$ and 0.9 fm$^{-1}$. Imaginary parts are smooth increasing functions of $p$.

In table 3, it is noticed that T-matrices for the (n–d) system follow the same trend like the $^3S_1$ (n–p) state for both $E_{\text{Lab}}=6$ and 10.5 MeV with $q=0.25$ fm$^{-1}$. For $q=0.55$ fm$^{-1}$ both Re $T(p,q,k^2)$ and Im $T(p,q,k^2)$ show opposite trends for the laboratory energies under consideration. $T(p,q,k^2)$ and Im $T(p,q,k^2)$ parts at $E_{\text{Lab}}=6$ MeV move towards zero from their positive and negative values and for $E_{\text{Lab}}=10.5$ MeV both of them attain their positive and negative peaks at $p=0.8$ fm$^{-1}$. The transition matrices approach zero with the increment of the off-shell momentum $p$ which is clearly seen in our data tables 1–3. Also we have verified that our off-shell T-matrices reproduce the correct half-shell limits and proper phase shifts at respective laboratory energies. Our phase shift values, computed from half-shell T-matrix, are presented in table 4 along with those of Arndt

**Table 1.** Off-shell T-matrix $T(p, q, k^2)$ in (fm) for (n–p) $^1S_0$ state.

| $p$ (fm$^{-1}$) | $E_{\text{Lab}} = 10$ MeV | | | | | | | | $E_{\text{Lab}} = 20$ MeV | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $q = 0.25$ fm$^{-1}$ | | | | $q = 0.55$ fm$^{-1}$ | | | | $q = 0.25$ fm$^{-1}$ | | $q = 0.55$ fm$^{-1}$ | |
| | Re $T$ | Im $T$ | Re $T$ (Ref. [14]) | Im $T$ Ref. [14] | Re $T$ | Im $T$ | Re $T$ Ref. [14] | Im $T$ Ref. [14] | Re $T$ | Im $T$ | Re $T$ | Im $T$ |
| 0.1 | −0.987 | −1.063 | −0.979 | −1.411 | −0.539 | −2.516 | −0.824 | −1.187 | −0.650 | 0.413 | −0.691 | −1.290 |
| 0.2 | −0.918 | −1.103 | −0.956 | −1.377 | −0.689 | −2.276 | −0.804 | −1.159 | −0.582 | 0.198 | −0.709 | −1.207 |
| 0.3 | −0.826 | −1.149 | −0.919 | −1.324 | −0.860 | −1.941 | −0.773 | −1.114 | −0.505 | −0.090 | −0.722 | −1.090 |
| 0.4 | −0.734 | −1.179 | −0.872 | −1.257 | −0.987 | −1.584 | −0.734 | −1.057 | −0.447 | −0.372 | −0.720 | −0.965 |
| 0.5 | −0.654 | −1.175 | −0.818 | −1.179 | −1.036 | −1.273 | −0.688 | −0.992 | −0.415 | −0.581 | −0.699 | −0.852 |
| 0.6 | −0.590 | −1.130 | −0.761 | −1.097 | −1.010 | −1.045 | −0.640 | −0.923 | −0.396 | −0.685 | −0.663 | −0.763 |
| 0.7 | −0.536 | −1.053 | −0.703 | −1.013 | −0.937 | −0.901 | −0.591 | −0.852 | −0.373 | −0.694 | −0.620 | −0.697 |
| 0.8 | −0.488 | −0.956 | −0.646 | −0.931 | −0.847 | −0.816 | −0.543 | −0.783 | −0.335 | −0.643 | −0.578 | −0.645 |
| 0.9 | −0.442 | −0.856 | −0.591 | −0.852 | −0.760 | −0.758 | −0.498 | −0.717 | −0.287 | −0.571 | −0.537 | −0.598 |
| 1.0 | −0.398 | −0.763 | −0.541 | −0.779 | −0.681 | −0.705 | −0.455 | −0.655 | −0.239 | −0.504 | −0.496 | −0.549 |
| 1.5 | −0.240 | −0.458 | – | – | −0.392 | −0.402 | – | – | −0.147 | −0.335 | −0.301 | −0.324 |
| 2.0 | −0.156 | −0.292 | −0.229 | −0.330 | −0.235 | −0.247 | −0.192 | −0.278 | −0.107 | −0.223 | −0.186 | −0.202 |
| 2.5 | −0.108 | −0.199 | – | – | −0.154 | −0.168 | – | – | −0.079 | −0.153 | −0.125 | −0.137 |
| 3.0 | −0.079 | −0.143 | −0.117 | −0.169 | −0.108 | −0.121 | −0.098 | −0.142 | −0.061 | −0.110 | −0.089 | −0.099 |
| 4.0 | −0.048 | −0.086 | −0.069 | −0.100 | −0.062 | −0.072 | −0.058 | −0.084 | −0.038 | −0.066 | −0.052 | −0.059 |
| 5.0 | −0.032 | −0.057 | −0.046 | −0.066 | −0.040 | −0.048 | −0.038 | −0.055 | −0.026 | −0.043 | −0.034 | −0.039 |
| 6.0 | −0.20 | −0.035 | −0.032 | −0.046 | −0.028 | −0.034 | −0.027 | −0.039 | −0.019 | −0.031 | −0.024 | −0.028 |

**Table 2.** Off-shell T-matrix $T(p, q, k^2)$ in (fm) for (n–p) $^3S_1$ state.

| $p$ (fm$^{-1}$) | $E_{\text{Lab}} = 10$ MeV | | | | $E_{\text{Lab}} = 0$ MeV | | | |
|---|---|---|---|---|---|---|---|---|
| | $q = 0.25$ fm$^{-1}$ | | $q = 0.55$ fm$^{-1}$ | | $q = 0.25$ fm$^{-1}$ | | $q = 0.55$ fm$^{-1}$ | |
| | Re $T$ | Im $T$ | Re $T$ | Im $T$ | Re $T$ | Im $T$ | Re $T$ | Im $T$ |
| 0.1 | 0.363 | −1.352 | .426 | −2.923 | 0.138 | 0.047 | −0.122 | −1.651 |
| 0.2 | 0.415 | −1.434 | .267 | −2.705 | 0.181 | −0.193 | −0.140 | −1.591 |
| 0.3 | 0.476 | −1.537 | .082 | −2.409 | 0.220 | −0.510 | −0.155 | −1.512 |
| 0.4 | 0.520 | −1.623 | −0.066 | −2.105 | 0.237 | −0.813 | −0.162 | −1.434 |
| 0.5 | 0.538 | −1.664 | −0.149 | −1.853 | 0.233 | −1.029 | −0.161 | −1.370 |
| 0.6 | 0.531 | −1.648 | −0.176 | −1.676 | 0.224 | −1.131 | −0.157 | −1.321 |
| 0.7 | 0.508 | −1.584 | −0.173 | −1.562 | 0.227 | −1.137 | −0.156 | −1.279 |
| 0.8 | 0.478 | −1.487 | −0.162 | −1.479 | 0.243 | −1.086 | −0.159 | −1.230 |
| 0.9 | 0.445 | −1.379 | −0.151 | −1.400 | 0.261 | −1.017 | −0.161 | −1.169 |
| 1.0 | 0.412 | −1.270 | −0.142 | −1.308 | 0.270 | −0.949 | −0.160 | −1.096 |
| 1.5 | 0.260 | −0.831 | −0.072 | −0.814 | 0.163 | −0.675 | −0.101 | −0.713 |
| 2.0 | 0.167 | −0.549 | −0.034 | −0.531 | 0.094 | −0.456 | −0.062 | −0.469 |
| 2.5 | 0.113 | −0.381 | −0.016 | −0.367 | 0.058 | −0.317 | −0.040 | −0.325 |
| 3.0 | 0.081 | −0.276 | −0.008 | −0.267 | 0.038 | −0.230 | −0.028 | −0.236 |
| 4.0 | 0.048 | −0.164 | −0.003 | −0.158 | 0.021 | −0.136 | −0.016 | −0.139 |
| 5.0 | 0.031 | −0.107 | −0.001 | −0.103 | 0.013 | −0.089 | −0.010 | −0.088 |
| 6.0 | 0.022 | −0.076 | −0.001 | −0.073 | 0.009 | −0.063 | −0.007 | −0.064 |

*et al* [37] and Hüber *et al* [38]. The results obtained for the $s$-wave (n–p) and (n–d) phase parameters are in good agreement with the benchmark calculations of refs [37,38]. The differences found may be attributed, in large part, to the rather simple potential model used. It is clear that our phase shifts reproduced by the half-shell T-matrices are correct.

## 5. Conclusion

In practice, the two-nucleon scattering data are used to determine the on-shell nucleon–nucleon interaction. The off-shell properties of nuclear potentials and many-body force contributions must be tested in systems with

**Table 3.** Off-shell T-matrix $T(p, q, k^2)$ in (fm) for (n-d) $^2S_{1/2}$ state.

| $p$ (fm$^{-1}$) | $E_{\text{Lab}} = 6$ MeV | | | | $E_{\text{Lab}} = 10.5$ MeV | | | |
|---|---|---|---|---|---|---|---|---|
| | $q = 0.25$ fm$^{-1}$ | | $q = 0.55$ fm$^{-1}$ | | $q = 0.25$ fm$^{-1}$ | | $q = 0.55$ fm$^{-1}$ | |
| | Re $T$ | Im $T$ | Re $T$ | Im $T$ | Re $T$ | Im $T$ | Re $T$ | Im $T$ |
| 0.1 | 0.755 | −0.521 | 0.848 | −1.686 | 0.506 | 0.034 | 0.463 | −1.197 |
| 0.2 | 0.827 | −0.599 | 0.780 | −1.613 | 0.575 | −0.107 | 0.459 | −1.179 |
| 0.3 | 0.923 | −0.708 | 0.700 | −1.514 | 0.664 | −0.305 | 0.461 | −1.159 |
| 0.4 | 1.020 | −0.825 | 0.634 | −1.412 | 0.752 | −0.517 | 0.471 | −1.146 |
| 0.5 | 1.099 | −0.925 | 0.594 | −1.329 | 0.822 | −0.700 | 0.488 | −1.143 |
| 0.6 | 1.150 | −0.994 | 0.579 | −1.275 | 0.871 | −0.828 | 0.506 | −1.152 |
| 0.7 | 1.171 | −1.027 | 0.577 | −1.248 | 0.901 | −0.898 | 0.519 | −1.164 |
| 0.8 | 1.168 | −1.028 | 0.575 | −1.233 | 0.918 | −0.918 | 0.521 | −1.171 |
| 0.9 | 1.145 | −1.006 | 0.567 | −1.216 | 0.922 | −0.908 | 0.513 | −1.165 |
| 1.0 | 1.108 | −0.970 | 0.551 | −1.187 | 0.912 | −0.883 | 0.496 | −1.142 |
| 1.5 | 0.837 | −0.742 | 0.432 | −0.889 | 0.710 | −0.712 | 0.388 | −0.880 |
| 2.0 | 0.596 | −0.537 | 0.324 | −0.629 | 0.499 | −0.525 | 0.287 | −0.631 |
| 2.5 | 0.431 | −0.390 | 0.242 | −0.457 | 0.357 | −0.384 | 0.212 | −0.459 |
| 3.0 | 0.320 | −0.292 | 0.185 | −0.342 | 0.263 | −0.287 | 0.160 | −0.343 |
| 4.0 | 0.195 | −0.178 | 0.115 | −0.208 | 0.159 | −0.175 | 0.099 | −0.209 |
| 5.0 | 0.129 | −0.118 | 0.077 | −0.138 | 0.105 | −0.116 | 0.066 | −0.139 |
| 6.0 | 0.092 | −0.084 | 0.055 | −0.098 | 0.074 | −0.082 | 0.047 | −0.099 |
| 7.0 | 0.068 | −0.062 | 0.041 | −0.073 | 0.055 | −0.061 | 0.035 | −0.074 |
| 10.0 | 0.034 | −0.031 | 0.021 | −0.037 | 0.027 | −0.031 | 0.018 | −0.037 |

**Table 4.** Phase shifts for (n–p) and (n–d) systems obtained from half-shell T-matrices.

| $E_{\text{Lab}}$ (MeV) | $\delta_{0s}$(deg) | $\delta_{0s}$(deg) [37] |
|---|---|---|
| 10 | 57.66 | 59.35 |
| 20 | 51.15 | 52.9 |
| $^3S_1$ (n–p) system | | |
| $E_{\text{Lab}}$ (MeV) | $\delta_{0t}$(deg) | $\delta_{0t}$(deg) [37] |
| 10 | 101.53 | 102.76 |
| 20 | 87.03 | 86.92 |
| $^2S_{1/2}$ (n–d) system | | |
| $E_{\text{Lab}}$ (MeV) | $\delta_{1/2+}$(deg) | $\delta_{1/2+}$(deg) [38] |
| 6 | −48.02 | −45.80 |
| 10.5 | −59.70 | −60.80 |

three or more nucleons. As a consequence, Faddeev formalism [1,2] is important for understanding the three-nucleon bound and scattering states. The Faddeev theory has been extensively applied to this problem and, in particular, to the study of the n–d scattering under and above the deuteron breakup threshold. Generally, the scattering amplitude, in traditional potential scattering theory, is obtained from the on-shell (elastic) version of the off-shell T-matrix and we have shown that our off-shell T-matrix reproduced it properly. As the Schrödinger equation for the Manning–Rosen potential admits exact analytical solution, it may be used appropriately with the Faddeev equation related to the three-body systems [1,2] and in nuclear matter calculation [39,40]. In the Faddeev equations, the basic inputs are the channel two-body off-shell transition matrices of each two-body subsystem in which only two of the particles are believed to interact while the third one plays the role of a spectator. Consequently, two-body T-matrices, in such a situation, appear off-the-energy-shell. The T-matrix has immense importance in scattering theory as it is closely related to the experiment. In the three-body problem, the off-shell T-matrix acts as a bridge between experimental two-nucleon data and three-nucleon observables. Therefore, the T-matrix formalism presents a rational and constructive approach for the parametrisation of the unknown parts of the involved forces. It is well-known that traditional two-body forces are not sufficiently strong to reproduce the observed $^3$H and $^3$He binding energies, and the role of three-body forces in resolving this discrepancy has been studied extensively [41–43]. The present text deals with the (n–p) and (n–d) systems with a simple two-nucleon model potential involving three adjustable parameters while the standard models have several parameters to be fitted [37]. The (n–d) system is also treated within the two-body model of interaction instead of the three-body model. Therefore, the observed small differences in the phase shift parameters should be regarded as ‘fine tuning’ of the nuclear forces in three-nucleon systems. In particular, the simple potential models are good enough to reproduce low-energy scattering parameters for two-nucleon systems and do not fit high-energy data. However, the small differences observed in phase parameters do not point towards the existence of any serious insufficiency in the calculations. We conclude by noting that the overall quality of the conformity between our approach and more sophisticated theoretical calculations, in the low-energy region, is noteworthy.

## Appendix A

The Manning–Rosen potential

$$V_{\mathrm{MR}}(r) = \frac{1}{b^2}\left\{\frac{\alpha(\alpha-1)\,\mathrm{e}^{-2r/b}}{\left(1-\mathrm{e}^{-r/b}\right)^2} - \frac{A\mathrm{e}^{-r/b}}{\left(1-\mathrm{e}^{-r/b}\right)}\right\} \tag{A1}$$

converts into the Hulthén potential when $\alpha = 0$ and reads as

$$V_{\mathrm{H}}(r) = V_0\frac{\mathrm{e}^{-r/b}}{\left(1-\mathrm{e}^{-r/b}\right)}; \qquad V_0 = -A/b^2. \tag{A2}$$

Under the limit eq. (27) leads to

$$\begin{aligned} T_1(q,p,k^2) = {} & b^3\sum_{n=0}^{\infty}\frac{\Gamma(n+1-i(k-p)b)\Gamma(-i(k+q)b)\Gamma(n+2)}{\Gamma(1-i(k-p)b)\Gamma(n+2-i(k+q)b)\Gamma(n+1)} \\ & \times {}_3F_2\,(1,1+n+X,1+n+Y;\ 2+n,\ 3+n-ib(k+p);1) \\ & -\frac{b^2\Gamma(-i(k+q)b)\Gamma(i(k-q)b)}{f_{\mathrm{H}}(k)i(k+p)\Gamma(1+i(k-q)b+X)\Gamma(1+i(k-q)b+Y)} \\ & \times {}_3F_2\,(X,\ Y,\ -i(k+p)b;\ 1-2ikb,\ 1-i(k+p)b;\ 1) \end{aligned} \tag{A3}$$

with

$$X = 1-ikb+ib\left(k^2+V_0\right)^{1/2}$$

and

$$Y = 1-ikb-ib\left(k^2+V_0\right)^{1/2}. \tag{A4}$$

Using the following relations of $f_n(a,b;c;z)$ and ${}_3F_2(*)$ functions [33,44]

$$f_n(a,b;c;z) = \frac{z^n}{n(n+c-1)} \times {}_3F_2\,(1,\ n+a,\ n+b; n+1,\ n+c;\ z), \tag{A5}$$

$$\begin{aligned} f_m(a,b;c;z) = {} & \frac{\Gamma(m)\Gamma(m+c-1)\Gamma(a)\Gamma(b)}{\Gamma(m+a)\Gamma(m+b)\Gamma(c)} \\ & \times\sum_{i=0}^{\infty}\frac{\Gamma(m+i+a)\Gamma(m+a+b)\Gamma(c)\,z^{m+i}}{\Gamma(a)\Gamma(b)\Gamma(m+i+c)\Gamma(m+i+1)}; \\ & m\to\text{positive integer}, \end{aligned} \tag{A6}$$

$$\begin{aligned} & {}_3F_2\,(\alpha,\ \beta,\ \gamma;\ \delta,\ \varepsilon;\ 1) \\ & = \frac{\Gamma(\varepsilon)\Gamma(\delta+\varepsilon-\alpha-\beta-\gamma)}{\Gamma(\varepsilon-\gamma)\Gamma(\delta+\varepsilon-\alpha-\beta)} \\ & \quad\times{}_3F_2\,(\delta-\alpha,\ -\beta,\ \gamma;\ \delta,\ \varepsilon+\delta-\alpha-\beta;\ 1), \end{aligned} \tag{A7}$$

$$\begin{aligned} & {}_3F_2\,(a,\ b,\ c;\ e,\ f;\ 1) \\ & = \frac{\Gamma(s)\Gamma(f)}{\Gamma(f-a)\Gamma(s+a)} \\ & \quad\times{}_3F_2\,(a,\ e-b,\ e-c;\ e,\ s+a;\ 1); \\ & s = e+f-a-b-c \end{aligned} \tag{A8}$$

and

$$\sum_{L=0}^{n}\frac{(a)_L\,(b)_L}{(c)_L\,\Gamma(L+1)} = \frac{\Gamma(a+n+1)\Gamma(b+n+1)}{\Gamma(a+b+n+1)\Gamma(n+1)} \times{}_3F_2\,(a,\ b,\ c+n;\ c,\ a+b+n+1;\ 1) \tag{A9}$$

in eq. (A3) one gets

$$\begin{aligned} T_1(q,p,k^2) = {} & -b^3\sum_{n=0}^{\infty}\frac{\Gamma(Y+1)\Gamma(-i(k+q)b)}{(X+1)\Gamma(1-i(k-p)b)\Gamma(2-i(k+q)b)} \\ & \times\sum_{n=0}^{\infty}\frac{(n+1)\Gamma(n+1-i(k-p)b)}{\Gamma(n+Y+2)}{}_3F_2\,(-n,\ X+1,\ 1-i(k+q)b-Y;\ X+2,\ 2-i(k+q)b;\ 1). \end{aligned} \tag{A10}$$

Expanding the sum, rearranging the terms along with eq. (32) we finally obtain

$$
\begin{aligned}
T_1(q, p, k^2) &= -b^3 \\
&\times \frac{Y\,\Gamma(Y + i(k-p)b - 1)\Gamma(-i\,(k+q)b)}{(X+1)\Gamma(Y+1+i\,(k-p)b)\Gamma(2-i\,(k+q)b)} \\
&\times {}_4F_3\,(2,\ X+1,\ 1-i\,(k-p)b,\ 1-i\,(k+q)b-Y; \\
&\quad X+2,\ 2-i\,(k+q)b,\ 2-Y \\
&\quad -i\,(k-p)b;\ 1)\,. \qquad \text{(A11)}
\end{aligned}
$$

Therefore, eq. (24) together with eq. (A11) produce the Hulthén off-shell T-matrix [17].

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