Pramana – J. Phys. (2021) 95:175
https://doi.org/10.1007/s12043-021-02209-7

An improved q-deformed logistic map and its implications

DIVYA GUPTA* and V V M S CHANDRAMOULI

Department of Mathematics, Indian Institute of Technology Jodhpur, Karwar 342 037, India

*Corresponding author. E-mail: gupta.8@iitj.ac.in

MS received 6 May 2021; accepted 21 June 2021

Abstract. In this paper, we show that the q-deformation scheme applied on both sides of the difference equation of the logistic map is topologically conjugate to the canonical logistic map and therefore there is no dynamical changes by this q-deformation. We propose a correction on this q-deformation scheme and apply it on the logistic map to describe the dynamical changes. We illustrate the Parrondo's paradox by assuming chaotic region as the gain. Further, we compute the topological entropy in the parameter plane and show the existence of Li-Yorke chaos. Finally, we show that in the neighbourhood of a particular parameter value, q-logistic map has stochastically stable chaos.

Keywords. q-Deformed logistic map; Heine deformation on nonlinear map; topological entropy; stochastically stable chaos.

PACS Nos 12.60.Jv; 12.10.Dm; 98.80.Cq; 11.30.Hv

1. Introduction

A q-deformed physical system in quantum group structure is an exploration of the possible deformation in the well-known physical phenomena models. The expectation is that deviation from the exact system would be detected as a result of the deformed system. This deviation may help to observe the changes in physical behaviour of the system due to deformation. Deformation of a function introduces an additional parameter q into the function's definition in such a way that the original function can be recovered under the limit $q \to 1$. There exists multiple deformations for the same function. The deformed parameter can be considered as a constant that must be modified to fit the observational data.

Cryptography is used to protect the information in digital form to provide secure communication. Several encryption methods are used in cryptography sciences. We are always looking for an encryption method which improves the security of the system. The encryption and chaotic system are highly correlated due to the randomness and aperiodic behaviour of the chaotic system. A system which shows chaos is highly recommended for encryption. The encrypted system shows complex behaviour which is useful to keep the data more secure.

The logistic map is taken as a model of nonlinear system which is the most common chaotic generator to encrypt the data in cryptology. It is suggested that nonlinear equation can be deformed to increase the chaotic region, which helps in secure communication. By applying the deformation on logistic map, we can increase the chaotic part of the map. There are so many aspects by which we can determine the chaotic region in nonlinear maps. These aspects include Lyapunov exponent, topological entropy and by calculating the boundary where transition takes place from simple to chaotic dynamics. The population growth model of the logistic map is given by the non-linear difference equation:

$$x_{n+1} = ax_n(1 - x_n), \tag{1}$$

where the parameter $a \in [0, 4]$ and x_n is the population at nth generation.

The function $f_a : [0, 1] \to [0, 1]$ corresponding to eq. (1) is given by

$$f_a(x) = ax(1 - x). \tag{2}$$

The period n-cycle of the logistic map $f_a(x)$ is denoted by P_n and can be calculated by solving x such that $f_a^n(x) - x = 0$.

Published online: 16 October 2021

Deformation scheme on the nonlinear map is first introduced by Jaganathan and Sinha [1]. They used the following deformation:

$$x_{n+1} = a[x_n](1 - [x_n]), \tag{3}$$

where

$$[x_n] = \frac{x_n}{1 + \epsilon(1 - x_n)}$$

and $\epsilon \in (-1, \infty)$. They observed that the deformed logistic map using the above scheme shows the coexistence of the attractor which is a very rare property in one-dimensional dynamics. With this inspiration, Banerjee and Parthasarathy [2] proposed a new deformation scheme on the logistic map by applying deformation on both sides, which is given by

$$[x_{n+1}] = a[x_n](1 - [x_n]) \tag{4}$$

and the deformation number is given by

$$[x] = \frac{1 - q^x}{1 - q}, \tag{5}$$

where $x \in \mathbb{R}$ and $q \in \mathbb{R}$. As $q \to 1$ the deformed number $[x] \to x$. We refer the deformation eq. (5) along with eq. (4) as Type-A deformation scheme. They observed that using Type-A deformation scheme on the logistic map, the dynamics upto period 2 orbit remained unchanged and became chaotic thereafter (this can be seen in figure 6 of [2]). They examined the Parrondo's paradox for some values of the deformed parameter q by showing concavity in the q-deformed logistic map which implies rapid growth and slower decay.

In §2, we discuss the dynamics of q-logistic map of Type-A obtained by applying the deformation eq. (5) using Type-A deformation scheme. We redetermine the results of Banerjee and Parthasarathy [2] by discussing the dynamics related to the 2^n-periodic attractors, Lyapunov exponent and topological entropy. We explain that there is no dynamical changes in the deformed logistic map of Type-A compared to the canonical logistic map eq. (1).

In §3, we discuss the dynamics of q-logistic map which is found by applying the deformation eq. (5) using eq. (3). Then we have

$$x_{n+1} = a[x_n](1 - [x_n]).$$

We call this the Type-B deformation scheme. We prove that there are remarkable dynamical changes in the deformed logistic map of Type-B. First, we calculate the periodic points of period 2^n and the topological entropy in the parameter plane. Further, we show that for a given deformed parameter q, there exists an open interval around a particular parameter such that q-logistic map of Type-B has positive Lyapunov exponent everywhere which ensures stochastically stable chaos.

2. q-Logistic map of Type-A

Consider the deformation $[x]$ given by eq. (5) on x_n of logistic map eq. (1) in the way of eq. (4). From eqs (4) and (5), the q-logistic map of Type-A is given by

$$x_{n+1} = \frac{\log\left(1 - \frac{a(1-q^{x_n})(q^{x_n}-q)}{1-q}\right)}{\log(q)}, \tag{6}$$

where $a \in [0, 4]$ and $q \in (0, \infty)\backslash\{1\}$. The q-logistic map of Type-A (eq. (6)) becomes the canonical logistic map $f_a(x)$ as the limit $q \to 1$. We consider $q \in (0, 2)$ in our discussion.

The function corresponding to the difference equation (eq. (6)) is given by

$$F_{a,q}(x) = \frac{\log\left(1 - \frac{a(1-q^{x})(q^{x}-q)}{1-q}\right)}{\log q}. \tag{7}$$

Lemma 1. *The function $F_{a,q}(x)$ is topological conjugate to the canonical logistic map $f_a(x)$.*

Proof. Let $h(x) = (1 - q^x)/(1 - q)$,

$$f_a \circ h(x) = f_a\left(\frac{1 - q^x}{1 - q}\right) = \frac{a(1 - q^x)(q^x - q)}{(1 - q)^2}.$$

From eq. (7) we can write

$$\frac{1 - q^{F_{a,q}(x)}}{1 - q} = \frac{a(1 - q^x)(q^x - q)}{(1 - q)^2},$$

which implies

$$h \circ F_{a,q}(x) = f_a \circ h(x).$$

Hence, $F_{a,q}(x)$ is a topological conjugate to the logistic map $f_a(x)$ through the homeomorphism $h(x)$. Therefore, the dynamics of two topologically conjugate maps $F_{a,q}(x)$ and $f_a(x)$ are similar. □

2.1 *Period 2^n-cycle*

To calculate the fixed points of q-logistic map of Type-A $F_{a,q}(x)$, we solve $[x_{n+1}] = [x_n]$ then by eq. (4), we get the following equation:

$$a[x_n](1 - [x_n]) - [x_n] = 0$$
$$f_a([x_n]) - [x_n] = 0. \tag{8}$$

Let $[x_n] = X_n$. From eq. (8) we get $f_a(X_n) - X_n = 0$ which is the same as to solve the fixed point of logistic map. Fixed points of the logistic map is given by $X_n = 0$ and $X_n = 1 - 1/a$. Therefore, the fixed points of the q-logistic map of Type-A can be calculated as $[x_n] = 0$ and $[x_n] = 1 - 1/a$ which are

$$x_n = 0 \quad \text{and} \quad x_n = \frac{\log(q + \frac{1-q}{a})}{\log q}.$$

The fixed point $x_n = 0$ is stable for $0 < a < 1$ and the non-trivial fixed point is stable for $1 < a < 3$. To calculate the period-2 cycle, we solve $[x_{n+2}] = [x_n]$ which implies

$$f_a^2([x_n]) - [x_n] = 0$$
$$f_a^2(X_n) - X_n = 0. \tag{9}$$

The solution of eq. (9) is the periodic point of period-2 of logistic map $f_a(X_n)$ which is given by

$$X_n = \frac{1}{2a}(a + 1 \pm \sqrt{(a-3)(a+1)}).$$

The period-2 cycles of eq. (6) are

$$[x_n] = \frac{1}{2a}(a + 1 \pm \sqrt{(a-3)(a+1)})$$

and

$$x_n = \frac{\log(\frac{a(1+q)-1+q\pm(1-q)\sqrt{a^2-2a-3}}{2a})}{\log q}.$$

which are real for $a \geq 3$.

To calculate the 2^n periodic cycle of q-logistic map of Type-A, we need to compute $[x_{n+2^n}] = [x_n]$ which is similar to

$$f_a^{2^n}([x_n]) - [x_n] = 0,$$
$$f_a^{2^n}(X_n) - X_n = 0.$$

Therefore, the 2^n-cycle of eq. (6) is $X_n = P_{2^n}$ which is given by

$$x_n = \frac{\log(1 - (1-q)P_{2^n})}{\log q},$$

where P_{2^n} = periodic point of period 2^n of the logistic map f_a.

For the real values of 2^n periodic points of eq. (6), the term inside 'log' which is $1 - (1-q)P_{2^n}$ should be non-negative which implies

$$P_{2^n} < \frac{1}{1-q},$$

which is true for all q. The stability of the 2^n periodic point of the q-logistic map of Type-A eq. (6) depends on the real values of 2^n periodic point of the logistic map. Hence, the birth and decay of 2^n-cycle in the q-logistic map of Type-A is the same as 2^n-cycle in the logistic map. Therefore, for any value of q, the birth of 2^n periodic points will not deviate and will be the same as in the logistic map. The bifurcation diagram of the q-logistic map of Type-A and the canonical logistic map are plotted in red and blue respectively (see figure 1). The vertical lines show the period doubling bifurcations in both the maps. It is clear that the period doubling bifurcation occurs at the same parameter a for both the maps and it is independent of the deformed parameter q.

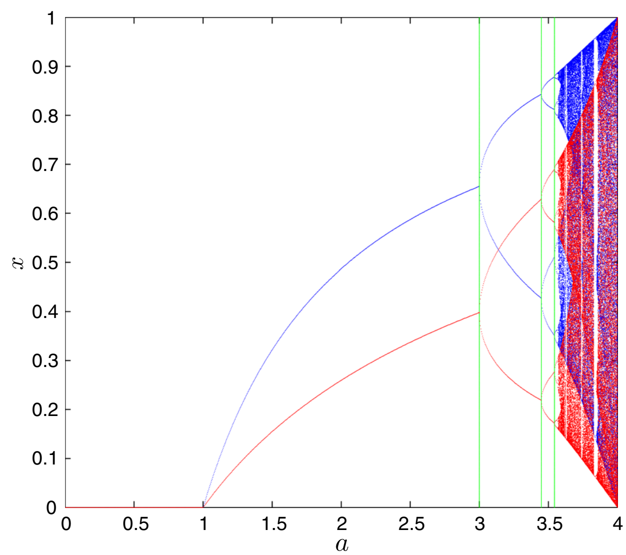

Figure 1. Bifurcation diagram of the q-logistic map of Type-A (eq. (6)) in which red colour is at $q = 0.1$, blue colour is at $q = 0.9$ and green vertical lines are for the period doubling bifurcation points.

The calculated 2^n-periodic points of the q-logistic map of Type-A and the bifurcation diagram (figure 1) reveal that there is no dynamical changes upto 2^n-periodic orbits in the q-logistic map of Type-A with respect to the canonical logistic map.

2.2 *Lyapunov exponent and topological entropy*

The Lyapunov exponent of $F_{a,q}(x)$ is given by

$$\text{LE} = \lim_{N\to\infty} \left\{ \frac{1}{N} \sum_{i=0}^{n-1} \ln |F'_{a,q}(x_i)| \right\}, \tag{10}$$

where $x_{i+1} = F_{a,q}(x_i)$. To compute this, we differentiate $F_{a,q}(x)$ with respect to x, and then we obtain

$$F'_{a,q}(x) = \frac{aq^x(1 + q - 2q^x)}{a(1-q^x)(q^x - q) + q - 1}. \tag{11}$$

Note that the critical point of $F_{a,q}(x)$ is obtained at

$$x = \frac{\log \frac{1+q}{2}}{\log q}.$$

We use eq. (10) to calculate the Lyapunov exponent of $F_{a,q}(x)$. In figure 2, Lyapunov exponent of q-logistic map of Type-A and canonical logistic map are shown by red and blue colour respectively. We observe that Lyapunov exponent of $F_{a,q}(x)$ for any q is similar to the logistic map which implies that the path from the periodic to the chaotic region is the same for both maps.

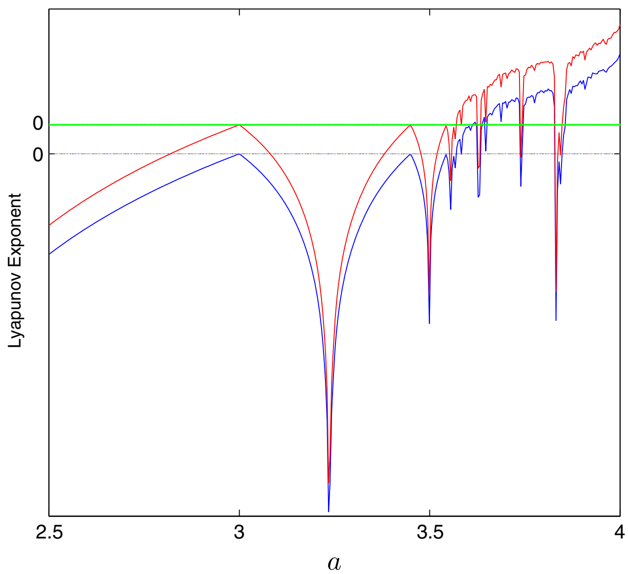

Figure 2. Red indicates Lyapunov exponent of q-logistic map of Type-A (eq. (6)) at $q = 0.15$ and for $a \in (2, 4)$ and blue shows Lyapunov exponent of the logistic map.

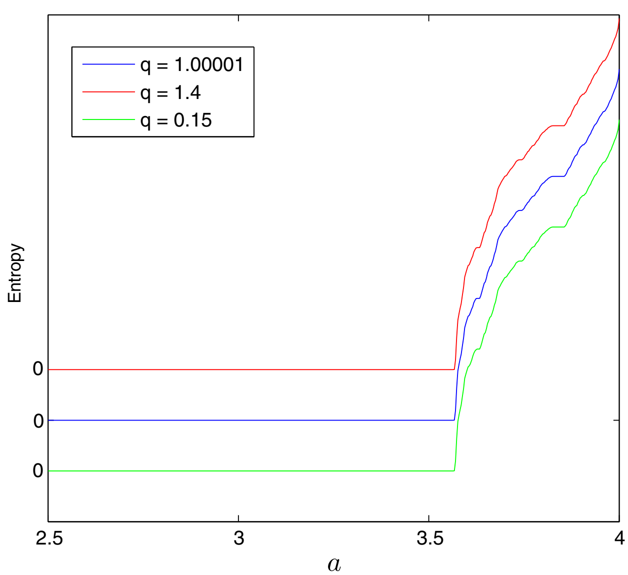

Figure 3. Topological entropy of the q-logistic map of Type-A for different values of q and $a \in (2.5, 4)$.

Further, both maps become chaotic with positive Lyapunov exponent at the same parameter a and which is independent of the deformed parameter q.

Topological entropy [3]: Assume that f is piecewise strictly monotone mapping of an interval. Let c_n be the number of points at which f^n has extrema. Then, according to [3], the topological entropy is defined as

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(c_n)$$

$$h(f) \leq \frac{1}{n} \log(c_n): \text{ for any } n.$$

Topological entropy can be positive or zero and it is invariant through topological conjugacy. The topological entropy is a measure of the chaotic behaviour and chaos can be recognised with positive topological entropy. Dynamical systems are topologically simple in the sense that their topological entropy is equal to zero.

Let f be a continuous interval map on I to itself. A set $U \in I$ is said to be a scrambled set if $\delta > 0$ such that every $x, y \in U$ with $x \neq y$ satisfies the following conditions:

$$\liminf_{n\to\infty} | f^n(x) - f^n(y) | = 0$$

and

$$\limsup_{n\to\infty} | f^n(x) - f^n(y) | \geq \delta.$$

According to Li and Yorke [4], f is called chaotic if there is an uncountable scrambled set $U \subset I$. Relationship between the topological entropy and Li–Yorke chaos for the continuous interval map is proved in [5], according to which positive topological entropy is a sufficient condition for the map to be chaotic in the sense of Li and Yorke. Converse of the statement is not true in general.

To obtain the region of chaos, we compute the topological entropy using the algorithm given in [6]. The topological entropy of $F_{a,q}(x)$ is shown in figure 3 for different values of the parameter q. As the parameter $q \to 1$, the $F_{a,q}(x)$ coincides with the canonical logistic map. It can be noted that the topological entropy changes from 0 to positive value simultaneously for different values of q which implies that the region of Li–Yorke chaos for $F_{a,q}(x)$ and $f_a(x)$ is similar.

The above discussion concludes that dynamical or topological changes do not occur with the deformation scheme (eq. (4)).

3. *q*-Logistic map of Type-B

We apply the deformation given in eq. (5) on the logistic map using eq. (3) and obtain the q-logistic map of Type-B which is given by

$$G_{a,q}(x) = \frac{a(1-q^x)(q^x-q)}{(1-q)^2}. \tag{12}$$

As deformed parameter $q \to 1$, then we obtain the logistic map $f_a(x)$. The above equation is a transcendental equation and analytical solution of the fixed points is not possible. We use Newton–Raphson Method to calculate the fixed point of $G_{a,q}(x)$.

For fixed a and varying values of the deformed parameter q, $G_{a,q}(x)$ has three fixed points $\{0, x_+^*, x_-^*\}$ in the order $0 < x_-^* < x_+^*$ whose existence depends on the value of a and q. The fixed point $x^* = 0$ exists for all values of a and q. Further, it is stable for $a < (q-1)/(\log(q))$. The fixed point x_+^* undergoes reverse periodic doubling bifurcation route to chaos. The attractors on the period doubling route of x_+^* coexists with $x^* = 0$ for higher values of q. We notice that whenever two attractors coexist, the fixed point x_-^* exists as a repeller between these attractors.

The fixed points of $G_{a,q}(x)$ are shown in figure 4 for different values of a. The black line represents the stable fixed point, grey line represents the unstable fixed point and x-axis shows the deformed parameter value q. In figure 4a, x_+^* is a stable fixed point (black colour) and becomes unstable fixed point (grey colour). The two black lines together show the coexistence of two stable fixed points and x_-^* is the unstable fixed point between them. In figure 4b, the fixed point x_+^* is stable for values of q beyond 15. The black line of $x^* = 0$ and unstable fixed point x_-^* show that attractors en route of x_+^* coexist with $x^* = 0$.

3.1 *Superstable periodic cycle of period* 2^n

The superstable periodic cycles of period 2^i of the function $G_{a,q}(x)$ for fixed values of q are given by

$$a_{2^i} = \left\{a \ : \ G_{a,q}^{2^i}(c) = c \text{ for } i \geq 0\right\},$$

where c is the critical point of $G_{a,q}(x)$.

The superstable fixed point of $G_{a,q}(x)$ which is the solution of $G_{a,q}(c) - c = 0$ is given by

$$a_1 = \frac{4\log\left(\frac{q+1}{2}\right)}{\log q}.$$

We calculate the values a_{2^i} by solving the equation $\mathcal{I}_{2^i}(a) = G_{a,q}^{2^i}(c) - c$ for $i \geq 1$ for fixed values of q, for which we need two initial guess $a_{2^i}^0$ and $a_{2^i}^1$. We estimate $a_{2^i}^0$ with the value of previous superstable point $a_{2^{i-1}}$ and approximate the value $a_{2^i}^1$ in such a way that the sequence $a_{2^{i-2}}$, $a_{2^{i-1}}$ and $a_{2^i}^1$ converges to the universal Feigenbaum constant $\delta = 4.669201\ldots$. The relation found using Feigenbaum theory is given by

$$a_{2^i}^1 = a_{2^{i-1}} - \frac{a_{2^{i-1}} - a_{2^{i-2}}}{\delta}.$$

To calculate the superstable points, we need to find the zeros of the function $\mathcal{I}_{2^i}(a)$, which are provided by the secant method

$$a_{2^i} = a_{2^{i-1}} - \frac{a_{2^i}^0 - a_{2^i}^1}{F_{2^i}(a_{2^i}^1) - F_{2^i}(a_{2^i}^0)} F_{2^i}(a_{2^i}^1).$$

We use the software MATLAB to calculate the superstable periodic points a_{2^i} of period 2^i. The calculated values are given in tables 1 and 2 for different values of q.

We observe that when $q \in (0.01, 0.982)$, the a_∞ value of q-logistic map $G_{a,q}(x)$ increases and hit the maximum at $a_\infty = 3.5700\ldots$. As $q \in (0.982, 5.0569)$, the a_∞ value decreases and as $q \to 1$, the value $a_\infty \to 3.5699\ldots$ which is the value of canonical logistic map $f_a(x)$ and ultimately a_∞ approaches $3.556\ldots$ for $q = 5.0569$.

In table 1, for $q = 0.01$ the minimum of a_∞ value is $3.095495\ldots$ which is very much less than the a_∞ value

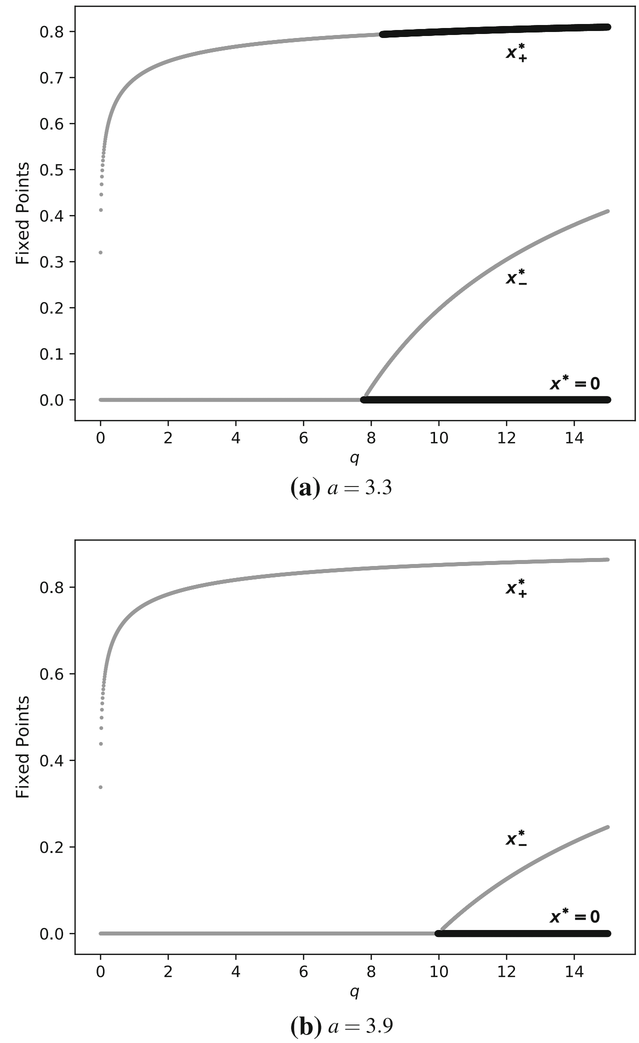

Figure 4. Fixed points of $Ga, q(x)$ for the deformed parameter $q \in (0.0001, 15)$.

Table 1. Periodic points and the corresponding Feigenbaum ratio of $G_{a,q}(x)$ for different values of q.

Per	$q = 0.01$	δ for $q = 0.01$
2^0	0.593417243762	–
2^1	2.155456474286	–
2^2	2.879435954753	–
2^3	3.048791037611	4.27492029320
2^4	3.085469745532	4.61725868919
2^5	3.093347621967	4.65591307811
2^6	3.095035678955	4.66683085537
2^7	3.095397252764	4.66863733834
2^8	3.095474692682	4.66908824694
2^9	3.095491278030	4.66917641043
2^{10}	3.095494830108	4.66919618653
2^{11}	3.095495590855	4.66920070785
2^{12}	3.095495753783	4.66920071311
Per	$q = 0.15$	δ for $q = 0.15$
2^0	1.166790171614	–
2^1	2.978015520938	–
2^2	3.399813047742	–
2^3	3.487720711575	4.79818833091
2^4	3.506542761766	4.67046166282
2^5	3.510572364181	4.67094473702
2^6	3.511435348436	4.66938115410
2^7	3.511620170878	4.66926117617
2^8	3.511659754100	4.66921172884
2^9	3.511668231609	4.66920411917
2^{10}	3.511670047232	4.66920204996
2^{11}	3.511670436082	4.66920150540
2^{12}	3.511670519362	4.66920151615

Table 2. Periodic points and the corresponding Feigenbaum ratio of $G_{a,q}(x)$ for different values of q.

Per	$q = 1.05$	δ for $q = 1.05$
2^0	2.024392662801	–
2^1	3.240205330936	–
2^2	3.499235788929	–
2^3	3.554635537699	4.67566123937
2^4	3.566517226753	4.66261560244
2^5	3.569062429784	4.66826768405
2^6	3.569607565715	4.66893280172
2^7	3.569724318385	4.66915172829
2^8	3.569749323294	4.66918989699
2^9	3.569754678582	4.66919936023
2^{10}	3.569755825521	4.66920056115
2^{11}	3.569756071161	4.66920357486
2^{12}	3.569756123769	4.66919786553
Per	$q = 1.4$	δ for $q = 1.4$
2^0	2.167448448345	–
2^1	3.263598572094	–
2^2	3.502251124298	–
2^3	3.553643162657	4.64376506213
2^4	3.564671363996	4.66005623010
2^5	3.567034192331	4.66737307090
2^6	3.567540282707	4.66878733057
2^7	3.567648673774	4.66911514732
2^8	3.567671887916	4.66918266467
2^9	3.567676859678	4.66919770043
2^{10}	3.567677924478	4.66920075327
2^{11}	3.567678152525	4.66920075737
2^{12}	3.567678201366	4.66920692382

of the logistic map $f_a(x)$. This implies that the phase transition happens earlier than the canonical map.

The Parrondo's paradox is a concept in which two losing approaches can be merged to produce a winning approaches. The chaotic dynamics is considered as the winning approach while the simple dynamics is taken as the losing one. We use this illustration to explain the Parrondo's paradox in the logistic map.

It is possible to produce the paradoxical effect by adjusting the deformed parameter q to increase the chaotic part of the system. The simple and chaotic dynamics are explained here using the boundary where transition takes place from simple to chaotic region. When the parameter $a < a_\infty$ then the map has simple dynamics and if $a > a_\infty$ then the map has chaotic region.

The logistic map has simple dynamics for the parameter $a < a_\infty = 3.5699\ldots$. The q-logistic map of Type-A (eq. (6)) has the periodic points of period 2^n similar to the logistic map which is discussed in §2. So, there is no paradox in q-logistic map of Type-A, when the transition happens from simple to chaotic region.

The q-logistic map of Type-B $G_{a,q}(x)$ has simple dynamics for the parameter $a < a_\infty = 3.095495\ldots$ when $q = 0.01$. There are many q values for which a_∞ of $G_{a,q}(x)$ is less than a_∞ of the logistic map but in the range $q \in (0.01, 1.5)$, the minimum a_∞ is obtained at $q = 0.01$. We observe that $a \in (3.095495, 3.5699)$, the logistic map $f_a(x)$ and the deformation $[x]$ have simple dynamics (losing condition) but their composition $f_a \circ [x]$ using the scheme-B which is $G_{a,q}(x)$ has chaotic dynamics (winning condition).

In figure 5, black colour shows the Li–Yorke chaos with positive topological entropy and the rest of the region shows simple behaviour with zero topological entropy. As q decreases from 1 to 0.01, the chaotic region expands and the maximum range of chaos is obtained for $a \in (3.095495, \ldots, 4)$ when $q = 0.01$.

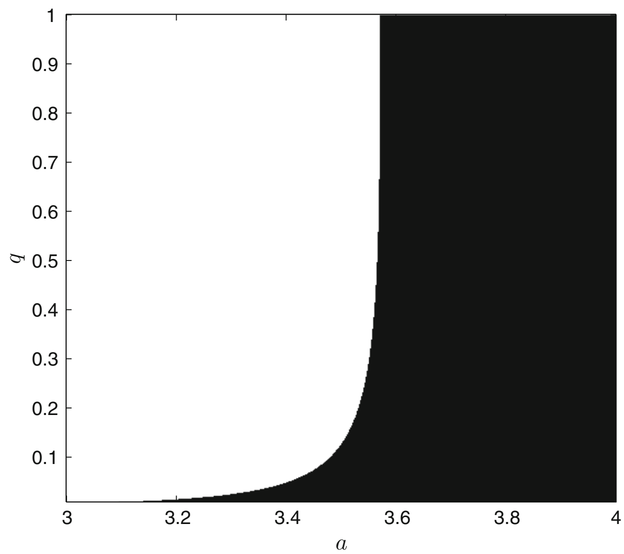

Figure 5. Topological entropy of q-logistic map of Type-B in the parameter plane for $q \in (0.01, 1)$ and $a \in (3, 4)$ where black colour shows positive entropy.

3.2 *Stochastically stable chaos*

The sensitive dependence on initial condition (SDIC) is one way out of several ways for a map to be chaotic. The SDIC is a weak condition in the sense that it can be implied by the existence of a period 3-cycle. In this case, we have sensitive dependence on an invariant uncountable set, which could be of measure zero. So period 3-cycle could coexist with a stable periodic attractor whose basin of attraction has a full measure. Another way for f to be chaotic is when f admits an absolutely continuous invariant probability (ACIP). It is equivalent to having a positive Lyapunov exponent for almost all initial conditions.

A map $G : I = [0, 1] \to I$ is called the unimodal map if G has the unique critical point $c \in I$ such that $G'(x) > 0$ for all $x \in [0, c]$ and $G'(x) < 0$ for all $x \in (c, 1]$ and the end points are mapped to zero.

The set-up shown now is based on [7]. We define a special class of $\mathcal{MU}$-map. The C^2-unimodal map $G_a : I \to I$ of parameter a is called $\mathcal{MU}$-map if

1. Each G_a has a non-degenerate critical point which says second derivative at the critical point is non-zero ($D^2 f(c) \neq 0$). We may expect that the critical point c does not depend on the parameter a.
2. Each G_a has a fixed point on the boundary which is repelling.
3. The map $(x, a) \to (G_a(x), D_x G_a(x), D_x^2 G_a(x))$ is C^1.
4. Suppose G_{a^*} is the Misiurewicz map at the parameter value $a = a^*$. The forward orbit of $G_{a^*}(c)$ does not lie in the neighbourhood U of the critical point and $G_{a^*}(c)$ is mapped onto the unstable periodic cycle P^* in finite number of steps which implies G_{a^*} has no stable periodic attractors. For each a near a^*, there exists a periodic point $\xi_a \in I$ such that $G_{a^*}(c) = \xi_{a^*}$ and the map $a \to \xi_a$ is differentiable. The kneading sequence of ξ_a under G_a is the same for each a near a^*.
5. $\frac{\mathrm{d}}{\mathrm{d}a}(\xi_a - G_a(c)) \neq 0$.

Theorem 1. [7] *Consider a family $G_a \in \mathcal{MU}$. Then there exist constants $\beta > 0$ and $K > 0$ and a subset E of the parameter space which contains a^* as the density point such that*

$$|D_x G_a^n(G_a(c))| \geq K \mathrm{e}^{n\beta} \text{for all } a \in E \text{ and all } n \geq 1.$$

By Corollary 19 [8], if $G_a \in \mathcal{MU}$ and has negative Schwarzian derivative then for all $a \in E$, the map G_a admits an absolutely continuous invariant probability measure (ACIP) with a density in L^2 and the Lyapunov exponent of G_a is positive for almost all initial conditions for $a \in E$.

The q-logistic map of Type-B is a unimodal map which satisfies conditions 1–3 of $\mathcal{MU}$ family. Now we prove the other two conditions from the following Proposition.

Proposition 1

For each $q_n \in (0, 2)\backslash\{1\}$ and $n \in \mathbb{N}$, there exist a sequence $\{a_n^*\}$ such that $G_{a_n^*, q_n}(x)$ is Misiurewicz map and satisfy condition 5 for the class of $\mathcal{MU}$-map.

Proof. The q-logistic map $G_{a,q}(x)$ acts as one-parameter family for fixed q. We propose an algorithm to calculate the sequences of parameters $\{a_n^*\}$ corresponding to $q_n \in (0, 2) \setminus \{1\}$. For a given q_n, first we calculate the fixed points $x \in (0, 1)$ and then we find the interval I_a of parameter $a \in [0, 4]$ for which fixed point x_+^* is unstable (x_+^* is the same as in the previous discussion of the fixed point of $G_{a,q}(x)$). Then, for each $q_n \in (0, 2)\backslash\{1\}$, we compute the Misiurewicz parameter in such a way that the third iterate of the map $G_{a,q_n}(x)$ at the critical point falls onto the unstable fixed point x_+^*. We calculate the Misiurewicz parameters by using the following algorithm:

Step 1: For each $q_n \in (0, 2) \setminus \{1\}$ and for a given a, we calculate the fixed points of $G_{a,q_n}(x)$ which are the roots of the equation $G_{a,q_n}(x) - x = 0$ by using Newton–Raphson method.

Step 2: Repeat Step 1 for each $a \in (0, 4)$. For every a we can get at most three fixed points.

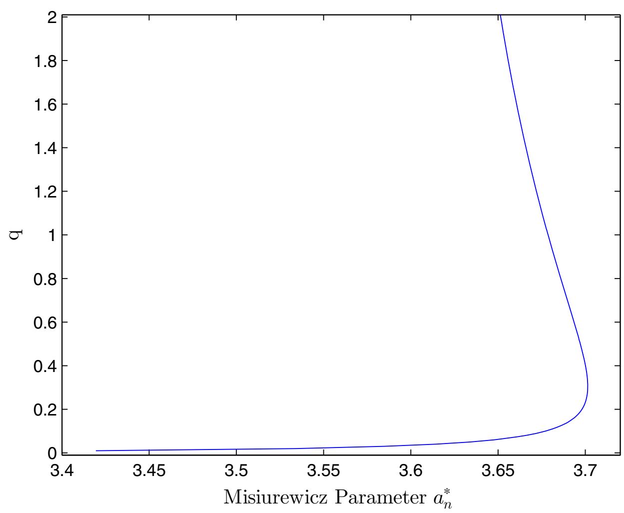

Figure 6. The Misiurewicz parameters a_n^* for 200 points of the parameter $q \in (0, 2)\backslash\{1\}$.

Step 3: We find the region I_a in which x_+^* is unstable by verifying the condition

$$|G'_{a,q_n}(x_+^*)| > 1.$$

Step 4: By computations using MATLAB, we obtain Misiurewicz parameters $a_n^* \in I_a$ which satisfy the equation $G^3_{a,q_n}(c) - x_+^* = 0$ for $a = a_n^*$ where c is the critical point of $G_{a,q}$.

Step 5: Repeat Steps 1–4 for each $q_n \in (0, 2)$ and we get the sequences $\{a_n^*\}$ of Misiurewicz parameters.

We employ the above algorithm by taking 200 points of the parameter $q \in (0, 2)\backslash\{1\}$ and $a \in (0, 4)$, then we obtain a sequence $\{a_n^*\}$ of Misiurewicz parameters which is shown in figure 6.

We have checked condition (5) numerically. As we have

$$G_{a^*,q}(c) = \xi_{a^*} \quad \text{and} \quad \frac{\partial}{\partial a} G_{a_n^*,q} \neq 0$$

then by implicit function theorem for any $q \in (0, 2)\backslash\{1\}$, there is a unique $a_n^*(q)$ which satisfies condition (5) of the $\mathcal{MU}$-map. □

By Proposition 1 and Corollary 19 [8], for fixed $q_n \in (0, 2)\backslash\{1\}$ there exists non-empty, open sets E_n containing q_n such that for all $q \in E_n$, there exists a positive measure set Γ_a such that if $a \in \Gamma_a$ then the map $G_{a,q}$ admits an absolutely continuous invariant probability measure which is strongly stochastically stable in the sense of Baladi and Viana [9]. For $a \in \Gamma_a$, the map $G_{a,q}$ also has positive Lyapunov exponent for almost all initial conditions.

4. Conclusions

The q-logistic map of Type-A has concavity in the part of x-space where the function is monotonically decreasing. However, the canonical logistic map is always convex [2]. The birth and decay of 2^n-periodic points of q-logistic map for any deformed parameter q occurs at the same value of a as in the logistic map which was proved in §2.1. For each q and varying the parameter a, the Lyapunov exponent of the q-logistic map is similar to the Lyapunov exponent of the logistic map. As concavity is observed in the q-logistic map of Type-A, there is no qualitative changes in the dynamical properties of the map. Hence, we conclude that the deformation scheme on the logistic map introduced in [2] is identical to the canonical logistic map and therefore no interesting dynamics happens.

The deformation using eq. (3) is meaningful as the dynamical behaviour of q-logistic map of Type-B is different from the canonical logistic map. $G_{a,q}(x)$ has coexisting fixed point with other periodic and chaotic attractors. Parrondo's paradox examined as transition from simple to chaotic region occurs earlier in $G_{a,q}(x)$ than in the logistic map $f_a(x)$. We also showed that the generic family $G_{a,q}$ passing through the Misiurewicz points a_n^* has positive Lyapunov exponent almost everywhere and it admits ACIP in the open sets containing a_n^*. For this class of maps, this is equivalent to having chaos which is strongly stochastically stable.

References

[1] R Jaganathan and S Sinha, *Phys. Lett. A* **338**, 277 (2005)
[2] S Banerjee and R Parthasarathy, *J. Phys. A* **44**, 045104 (2011)
[3] M Misiurewicz and W Szlenk, *Stud. Math.* **67**, 45 (1980)
[4] T-Y Li and J A Yorke, *Am. Math. Mon.* **82**, 985 (1975)
[5] F Blanchard, E Glasner, S Kolyada and A Maass, *J. für die Reine und Angew. Math.* **547**, 51 (2002)
[6] L Block, J Keesling, S Li and K Peterson, *J. Stat. Phys.* **55**, 929 (1989)
[7] W D Melo and S V Strien, *One-dimensional dynamics* (Springer Science & Buisiness Media, 2012) Vol. 25
[8] H Thunberg, *SIAM Rev.* **43**, 3 (2001)
[9] V Baladi and M Viana, *Ann. Sci. de l'Ecole Norm. Superieure* **29**, 483 (1996)