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Non-inertial effects on Klein–Gordon oscillator under a scalar potential using the Kaluza–Klein theory

FAIZUDDIN AHMED

National Academy, Gauripur 783 331, India
E-mail: faizuddinahmed15@gmail.com; faiz4U.enter@rediffmail.com

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Abstract. The Klein–Gordon (KG) oscillator under uniform rotation in the presence of a scalar potential $S(r)$ introduced by modifying the mass term $m \rightarrow m + S(r)$ in the background of a magnetic cosmic string space–time using the Kaluza–Klein theory (KKT) is analysed. We see that the energy eigenvalues and eigenfunction depend on the global parameters characterising the space–time, the scalar potential and the gravitational analogue of the Aharonov–Bohm effect. Furthermore, the angular frequency of the oscillator depends on the quantum numbers of the system which shows a quantum effect.

Keywords. Kaluza–Klein theory; relativistic wave equation; Aharonov–Bohm effect; special functions.

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1. Introduction

The relativistic quantum dynamics of a scalar and spin-half particle in the space–time background produced by topological defects using the Kaluza–Klein theory (KKT) [1–5] have been investigated in relativistic quantum mechanics. Using the KKT, several researchers have analysed the gravitational analogue of the Aharonov–Bohm effect [6–8]. This effect has been studied, for example, in the five-dimensional cosmic string space–time [9–11], and in the five-dimensional Minkowski space–time [12–14] background. In the context of the KKT, a few researchers have studied the Klein–Gordon oscillator and/or generalised Klein–Gordon (KG) oscillator with or without potential in the relativistic quantum mechanics. Klein–Gordon oscillator on curved background in the five-dimensional cosmic string space–time [15], the Klein–Gordon oscillator with a Cornell-type scalar potential in the five-dimensional Minkowski space–time [16], the generalised Klein–Gordon oscillator with a linear scalar potential in the five-dimensional cosmic string space–time [17], the generalised KG oscillator with a Cornell-type scalar potential in the five-dimensional cosmic string space–time [18] and linear confinement of the generalised KG oscillator with a uniform magnetic field in the five-dimensional magnetic cosmic string space–time [19] are some examples. Further, the KKT has been investigated in other branches of physics. For example, in Khaler fields [20], with torsion [21,22], in the Grassmannian context [23–25], in the description of geometric phases in graphene [26], in Kaluza—Klein reduction of a quadratic curvature model [27] and in the presence of fermions [28,29].

The effects of uniform rotation on a relativistic scalar particle in various space–time backgrounds have been investigated by several researchers. Some examples are: the effects of rotation on a Dirac particle [30], on a neutral particle [31], on the Dirac oscillator [32], on a scalar field in the space–time with a space-like dislocation and a spiral dislocation [33], on a scalar boson in the cosmic string space–time [34], on a spin-0 scalar particle in the cosmic string space–time [35], on the DKP equation with a magnetic cosmic string [36], on a scalar field in the space–time with a magnetic screw dislocation [37], on the Dirac oscillator in cosmic string space–time [38], on the KG oscillator in the cosmic string space–time with a space-like dislocation [39], the KG oscillator in the cosmic string space–time [41], on the scalar field under Lorentz symmetry violation [42], on a scalar field induced by the topology associated with a time-dislocation space–time [43], on the scattering problem of a non-relativistic particle in the cosmic string space–time [44], on the Dirac field in a space–time with a spiral dislocation [45], on the scalar field in the space–time with the distortion of a vertical line to a vertical

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spiral [46], on the Dirac oscillator in the cosmic string space–time background [47], on a charged half-spin particle in the presence of a uniform magnetic field and a mixed potential in the cosmic string space–time [48], on the Casimir energy in the space–time with one extra compactified dimension [49]. In the context of KKT, the effects of rotation on a scalar field subject to the Aharonov–Bohm effect was studied in [50], on spin-0 scalar charged particles in the cosmic string space–time with Coulomb potentials [51]. The effects of rotating reference frames on other physical systems have been investigated in [52–56]. The effects of rotation on the KG oscillator in the context of Kaluza–Klein have not yet been investigated.

In this work, we solve the KG oscillator with a scalar potential of Cornell-type in the five-dimensional cosmic string space–time under the effects of a uniform rotation using KKT, and analyse an analogue of the Aharonov–Bohm and Sagnac-type effects. We see that the energy eigenvalues and wave function depend on the parameters characterising the space–time, scalar potential and the angular velocity of the rotating frame.

2. Rotating frame effects on KG oscillator under a scalar potential in KKT

In the context of KKT [1–5], the metric with a magnetic quantum flux (Φ) passing along the symmetry axis of the string in five dimensions is given by [9]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2 r^2\,\mathrm{d}\phi^2 + \mathrm{d}z^2 + (\mathrm{d}y + \kappa A_\phi\,\mathrm{d}\phi)^2, \tag{1}$$

where the gauge field is given by

$$A_\phi = \kappa^{-1}\frac{\Phi}{2\pi} \tag{2}$$

such that $\vec{B} = \vec{\nabla}\times\vec{A} = \kappa^{-1}\,\Phi\,\delta^2(\vec{r})$ [9]. Here $y = x^4$ is the extra fifth spatial coordinate having ranges $0 < y < 2\pi a$ where a is the radius of the compact dimension of y and κ is the Kaluza constant [9]. The wedge parameter $\alpha = (1-4\,\mu)$ [57,58] where μ is the linear mass density of the string. In gravitation and cosmology, we assume that the values of the parameter α are in the range $0 < \alpha \leq 1$.

To introduce a uniform rotation in the above space–time (1), we consider the transformation $\phi \to \phi + \omega\,t$ [31–39,41,59], where ω is the velocity of constant rotation of the rotating frame which gives us the following line element:

$$\begin{aligned}\mathrm{d}s^2 = &-[1-\omega^2\,(\alpha^2\,r^2+\kappa^2\,A_\phi^2)]\,\mathrm{d}t^2 + \mathrm{d}r^2\\ &+\mathrm{d}z^2 + \mathrm{d}y^2 + 2\,\kappa A_\phi\,\mathrm{d}y\,\mathrm{d}\phi\\ &+(\alpha^2\,r^2+\kappa^2\,A_\phi^2)\,\mathrm{d}\phi^2 + 2\,\omega\,\kappa A_\phi\,\mathrm{d}y\,\mathrm{d}t\\ &+2\,(\alpha^2\,r^2+\kappa^2\,A_\phi^2)\,\omega\,\mathrm{d}t\,\mathrm{d}\phi.\end{aligned} \tag{3}$$

As a consequence of the rotating frame and in order for the component g_{00} to remain negative, we have imposed the restriction on the radial coordinate

$$0 \leq r < \frac{\sqrt{1-\kappa^2 A_\phi^2\,\omega^2}}{\alpha\,\omega}.$$

We can note that, in addition to the velocity of rotation of the uniformly rotating frame, the above inequality is determined by the parameter related to the quantum flux Φ of KKT. We can also note that, for

$$r > \frac{\sqrt{1-\kappa^2 A_\phi^2\,\omega^2}}{\alpha\,\omega},$$

the particle is placed outside the light cone [60].

The relativistic quantum dynamics of spin-0 scalar particle with a scalar potential $S(r)$ by modifying the mass term $m \to m + S(r)$ [11–15,17–19,61,62] in the five-dimensional space–time is described by

$$\left[\frac{1}{\sqrt{-g}}\,\partial_M(\sqrt{-g}\,g^{MN}\,\partial_N) - (m+S)^2\right]\Psi = 0, \tag{4}$$

where $M, N = 0, 1, 2, 3, 4$ and $g = -\alpha^2\,r^2$ is the determinant of the metric tensor with g^{MN} its inverse.

To couple an oscillator with the KG field, the following change in the radial momentum operator is considered [15,16,37,39,41,63]:

$$\vec{p} \to \vec{p} - im\Omega r\hat{r} \quad \text{or} \quad \partial_r \to \partial_r + m\,\Omega\,r, \tag{5}$$

such that $\vec{p}^{\,2} \to (\vec{p} - i\,m\,\Omega\,r\,\hat{r})(\vec{p} + i\,m\,\Omega\,r\,\hat{r})$.

By considering the line element (3) into eq. (4) and finally using (5), we obtain the following differential equation:

$$\begin{aligned}\Bigg[&-\left(\frac{\partial}{\partial t} - \omega\,\frac{\partial}{\partial\phi}\right)^2 + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r}\\ &-2\,m\,\Omega - m^2\,\Omega^2\,r^2 + \frac{1}{\alpha^2\,r^2}\left(\frac{\partial}{\partial\phi} - \kappa\,A_\phi\,\frac{\partial}{\partial y}\right)^2\\ &+\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} - (m+S)^2\Bigg]\Psi = 0.\end{aligned} \tag{6}$$

As the line element (2) is independent of t, ϕ, z, y, one can choose the following ansatz for the function Ψ:

$$\Psi(t, r, \phi, z, y) = \mathrm{e}^{i\,(-E\,t + l\,\phi + k\,z + q\,y)}\,\psi(r), \tag{7}$$

where E is the total energy of the particle, $l = 0, \pm 1, \pm 2, \ldots \in \mathbf{Z}$, and k, q are constants.

Substituting the ansatz (7) into eq. (6), we obtain the following equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\lambda_0-m^2\,\Omega^2\,r^2 -\frac{(l-\kappa q A_\phi)^2}{\alpha^2 r^2}-(m+S)^2\right]\psi(r)=0, \tag{8}$$

where $\lambda_0=(E+l\,\omega)^2-k^2-q^2-2\,m\,\Omega$.

In this work, we have chosen various types of potentials, especially focussing on Cornell- and Coulomb-type potentials which are as follows.

Case A: Interactions with Cornell-type potential

The Cornell-type potential consisting of linear plus Coulomb-like term is a particular case of the quark–antiquark interaction [64,65]. The Coulomb-like potential is responsible at small distances or short-range interactions whereas linear potential leads to the confinement of quark. This type of potential is given by [39,40,66]

$$S(r)=\frac{\eta_c}{r}+\eta_L\,r, \tag{9}$$

where $\eta_c>0$, $\eta_L>0$ are the potential parameters.

Substituting the gauge field (3) and the scalar potential (9) into eq. (8), we obtain the following radial wave equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\lambda-\frac{j^2}{r^2}-\Omega_0^2\,r^2-\frac{a}{r}-br\right]\psi(r)=0, \tag{10}$$

where

$$\begin{aligned}
&\lambda=(E+l\,\omega)^2-k^2-q^2-m^2-2\,\eta_c\,\eta_L-2\,m\,\Omega,\\
&\Omega_0=\sqrt{m^2\,\Omega^2+\eta_L^2},\\
&j=\sqrt{\frac{\left(l-\frac{q\,\Phi}{2\pi}\right)^2}{\alpha^2}+\eta_c^2},\\
&a=2\,m\,\eta_c,\\
&b=2\,m\,\eta_L.
\end{aligned} \tag{11}$$

Introducing a new variable $\rho=\sqrt{\Omega_0}\,r$, eq. (10) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}+\zeta-\frac{j^2}{\rho^2}-\rho^2-\frac{\eta}{\rho}-\theta\,\rho\right]\psi(\rho)=0, \tag{12}$$

where

$$\zeta=\frac{\lambda}{\Omega_0},\quad \eta=\frac{a}{\sqrt{\Omega_0}},\quad \theta=\frac{b}{\Omega_0^{\frac{3}{2}}}. \tag{13}$$

We must now impose the condition that the radial wave function $\psi(\rho)$ should be well behaved at the origin, as it is a singular point of eq. (12). In this case, for $\lim_{\rho\to 0}\psi(\rho)=0$ the solution is $\psi(\rho)\sim\rho^j$. Besides, let us consider the particular case where the frequency of rotation is very small, $\omega\ll 1$, such that in the vicinity of the fixed point

$$r_0=\frac{\sqrt{1-\kappa^2 A_\phi^2\,\omega^2}}{\alpha\,\omega},$$

and we have $\lim_{\omega\to 0} r\to\infty$ which implies $\rho\to\infty$.

Suppose the possible solution to eq. (12) is

$$\psi(\rho)=\rho^j\,\mathrm{e}^{-\frac{1}{2}(\rho+\theta)\,\rho}\,H(\rho). \tag{14}$$

Substituting eq. (14) into eq. (12), we obtain

$$H''(\rho)+\left[\frac{\gamma}{\rho}-\theta-2\,\rho\right]H'(\rho) +\left[-\frac{\beta}{\rho}+\Theta\right]H(\rho)=0, \tag{15}$$

where

$$\begin{aligned}
&\gamma=1+2\,j,\\
&\Theta=\zeta+\frac{\theta^2}{4}-2\,(1+j),\\
&\beta=\eta+\frac{\theta}{2}\,(1+2\,j).
\end{aligned} \tag{16}$$

Equation (15) is the biconfluent Heun's differential equation [11–19,37,39–41,66–68] and $H(\rho)$ is the Heun polynomial.

Equation (15) can be solved by the Frobenius method. We consider the power series solution [69]

$$H(\rho)=\sum_{i=0}^{\infty}c_i\,\rho^i. \tag{17}$$

Substituting the above power series solution into eq. (15), we obtain the following recurrence relation for the coefficients:

$$c_{n+2}=\frac{1}{(n+2)(n+2+2\,j)}\times\left[\{\beta+\theta\,(n+1)\}\,c_{n+1}-(\Theta-2\,n)\,c_n\right] \tag{18}$$

and the various coefficients are

$$c_1 = \left(\frac{\eta}{\gamma} + \frac{\theta}{2}\right) c_0,$$
$$c_2 = \frac{1}{4\,(1+j)}\,[(\beta+\theta)\,c_1 - \Theta\,c_0]. \tag{19}$$

As we are interested in solutions of bound states, to obtain a finite degree polynomial for the biconfluent Heun function $H(\rho)$, we must truncate the power series, and this is possible under the following conditions [11–19,37,39–41,66]:

$$\Theta = 2\,n, \quad (n = 1, 2, ...)$$
$$c_{n+1} = 0 \tag{20}$$

such that the wave function $\psi(\rho)$ is well behaved both at the origin $\rho \to 0$ and at $r \to \infty$.

By analysing the condition $\Theta = 2n$, we get the following second degree expression of the energy eigenvalues $E_{n,l}$:

$$\frac{\lambda}{\Omega_0} + \frac{\theta^2}{4} - 2\,(1+j) = 2\,n$$
$$\Rightarrow E_{n,l} = \pm\sqrt{k^2 + q^2 + 2\,m\,\Omega + 2\,\eta_c\,\eta_L + 2\,\Omega_0\left(n + 1 + \sqrt{\frac{(l - \frac{q\,\Phi}{2\pi})^2}{\alpha^2} + \eta_c^2}\right) - \frac{m^2\,\eta_L^2}{\Omega_0^2}} - \omega\,l. \tag{21}$$

Now, we impose additional recurrence condition $c_{n+1} = 0$ to find the individual energy levels and wave functions one by one as done in [11–19,37,39–41,66]. For $n = 1$, we have $\Theta = 2$ and $c_2 = 0$ which implies from eq. (19) that

$$c_1 = \frac{2}{\beta+\theta}\,c_0 \Rightarrow \left(\frac{\eta}{1+2\,j} + \frac{\theta}{2}\right)$$
$$= \frac{2}{\beta+\theta}(\Omega_0^{1,l})^3 - \left(\frac{a^2}{2\,(1+2\,j)}\right)(\Omega_0^{1,l})^2$$
$$-a\,b\left(\frac{1+j}{1+2\,j}\right)\Omega_0^{1,1} - \frac{b^2}{8}\,(3+2\,j) = 0 \tag{22}$$

a constraint on the parameter $\Omega_0^{1,l}$. This third degree algebraic equation has at least one real root and it is exactly this solution that gives us the allowed value of the oscillator frequency $\Omega_{1,l}$ for the lowest state of the system defined by the radial mode $n = 1$. The angular frequency of the oscillator is given by

$$\Omega_{n,l} = \frac{1}{m}\sqrt{(\Omega_0^{n,l})^2 - \eta_L^2}. \tag{23}$$

We can see from eq. (23) that the oscillator frequency Ω depends on the quantum numbers $\{n, l\}$ of the quantum system, the parameters associated with the Cornell-type scalar potential and the parameters associated with the background governed by a KKT. In addition, for each relativistic energy level, we have a different relation of the angular frequency of the oscillator to these parameters. For this reason, we have labelled the parameters Ω, Ω_0.

Therefore, the ground-state energy level for $n = 1$ is given by

$$E_{1,l} = \pm\sqrt{k^2 + q^2 + 2\,m\,\Omega_{1,l} + 2\,\eta_c\,\eta_L + 2\,\Omega_0^{1,l}\left(2 + \sqrt{\frac{(l - \frac{q\,\Phi}{2\pi})^2}{\alpha^2} + \eta_c^2}\right) - \left(\frac{m\,\eta_L}{\Omega_0^{1,l}}\right)^2} - \omega\,l \tag{24}$$

and the radial wave function is

$$\psi_{1,l} = \rho^{\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2} + \eta_c^2}}\,\mathrm{e}^{-\frac{1}{2}\left[\frac{2\,m\,\eta_L}{(\Omega_0^{1,l})^{\frac{3}{2}}} + \rho\right]\rho}\,(c_0 + c_1\,\rho)\,, \tag{25}$$

where

$$c_1 = \frac{1}{\sqrt{\Omega_0^{1,l}}}\left[\frac{2\,\eta_c}{\left(1 + 2\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2} + \eta_c^2}\right)} + \frac{1}{\Omega_0^{1,l}}\right] m c_0. \tag{26}$$

Then by substituting the real solution of $\Omega_0^{1,l}$ from eq. (22) into eq. (24) together with (25)–(26), one can obtain the ground-state energy level and the corresponding eigenfunction for the radial mode $n = 1$.

Case B: Interactions with Coulomb-type potential

The Coulomb-type potential is given by

$$S(r) = \frac{\eta_c}{r}. \tag{27}$$

This type of potential has been widely used in relativistic quantum mechanics, to study position-dependent mass systems [13,18,70,71].

Substituting the gauge field (3) and the scalar potential (27) into eq. (8), we obtain the following radial wave equation:

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\lambda_0-\frac{j^2}{r^2}-m^2\,\Omega^2\,r^2-\frac{a}{r}\right]\psi(r)=0. \tag{28}$$

Transforming a new variable $\rho=\sqrt{m\,\Omega}\,r$, we get

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+\frac{\lambda_0}{m\,\Omega}-\frac{j^2}{\rho^2}-\rho^2-\frac{a}{\rho\sqrt{m\,\Omega}}\right] \times\psi(\rho)=0. \tag{29}$$

Let the possible solution to eq. (29) be

$$\psi(\rho)=\rho^j\,e^{-\frac{\rho^2}{2}}\,H(\rho). \tag{30}$$

Equation (29) becomes

$$H''(\rho)+\left[\frac{1+2\,j}{\rho}-2\,\rho\right]H'(\rho) +\left[-\frac{a}{\sqrt{m\,\Omega}\,\rho}+\frac{\lambda_0}{m\,\Omega}-2\,(1+j)\right]H(\rho)=0. \tag{31}$$

Using the power series solution, we obtain the following recurrence relation:

$$c_{n+2}=\frac{1}{(n+2)(n+2+2\,j)} \times\left[\frac{a}{\sqrt{m\,\Omega}}\,c_{n+1}-\left(\frac{\lambda_0}{m\,\Omega}-2-2\,j-2\,n\right)c_n\right] \tag{32}$$

and the various coefficients are

$$c_1=\left[\frac{a}{\sqrt{m\,\Omega}\,(1+2\,j)}\right]c_0,$$
$$c_2=\frac{1}{4\,(1+j)}\left[\frac{a}{\sqrt{m\,\Omega}}\,c_1-\left(\frac{\lambda_0}{m\,\Omega}-2-2\,j\right)c_0\right]. \tag{33}$$

We must truncate the power series by imposing the following two conditions [12–14,17,18,66]:

$$\frac{\lambda_0}{m\,\Omega}-2-2\,j=2\,n,\quad (n=1,2,...)$$
$$c_{n+1}=0. \tag{34}$$

By analysing the first condition, we get the following second degree expression of the energy eigenvalues $E_{n,l}$:

$$\frac{\lambda_0}{m\,\Omega}-2\,(1+j)=2\,n$$
$$\Rightarrow E_{n,l}=-\omega\,l\pm\sqrt{k^2+q^2+2\,m\,\Omega\left(n+2+\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2}+\eta_c^2}\right)}. \tag{35}$$

Following similar procedure as done earlier, we have obtained the ground-state energy level and wave function for the radial mode $n=1$ as

$$E_{1,l}=\pm\sqrt{k^2+q^2+2\,m\,\Omega_{1,l}\left(3+\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2}+\eta_c^2}\right)}-\omega\,l,$$
$$\psi_{1,l}(\rho)=\rho^{\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2}+\eta_c^2}}\,e^{-\frac{\rho^2}{2}}\,(c_0+c_1\,\rho), \tag{36}$$

where

$$c_1=\frac{1}{\sqrt{\frac{1}{2}+\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2}+\eta_c^2}}}\,c_0 \tag{37}$$

with the constraint on the oscillator frequency Ω as

$$\Omega_{1,l}=\left[\frac{2\,m\,\eta_c^2}{\left(1+2\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2}+\eta_c^2}\right)}\right]c_0. \tag{38}$$

We can see in both Cases A and B that the oscillator frequency Ω depends on the quantum numbers $\{n,l\}$ of the relativistic quantum system which shows a quantum effect.

3. Conclusions

We have investigated the rotating effects on the KG oscillator with a scalar potential in the magnetic cosmic string space–time using the KKT. We have started our discussion through the restriction on the radial coordinate r that arises from the uniformly rotating frame, and the topology of the space–time. In this paper, we discuss a particular scenario in which the angular velocity of the rotating frame is very small, $\omega \ll 1$, such that in the vicinity of the fixed point

$$r_0 = \frac{\sqrt{1 - \kappa^2 A_\phi^2\, \omega^2}}{\alpha\, \omega},$$

we have $\lim_{\omega \to 0} r \to \infty$ which implies that $\rho \to \infty$. In Case A, we have considered a Cornell-type scalar potential and derived the biconfluent Heun's differential equation for a suitable wave function ψ as eq. (14) considered in the radial wave equation. It is noted that the possible solution, eq. (14), for the radial wave equation (12) is only possible for the aforementioned particular case where the angular velocity of the rotating frame is very small, $\omega \ll 1$. In this particular case, the wave function $\psi(\rho)$ is well-behaved and vanishes in the limit $\lim_{\rho \to \infty} \psi(\rho) \to 0$. Otherwise, one cannot consider eq. (14) as the possible solution for the radial wave equation (12). Using the power series expansion method, we have solved the Heun differential equation and finally truncated the power series solution to obtain a finite degree polynomial of the function $H(\rho)$. After that, we analyse the truncating condition $\Theta = 2\,n$, and non-compact expression of the energy eigenvalues $E_{n,l}$ by eq. (21) is obtained. This expression is not the general expression of the energy eigenvalues as one needed to impose another condition $c_{n+1} = 0$ on the eigenvalue problem. By analysing the recurrence condition $c_{n+1} = 0$ for each radial mode, for example, for the radial mode $n = 1$, the ground-state energy level $E_{1,l}$ by eq. (24) and the eigenfunction eqs (25)–(26) with the restriction on the angular frequency of the oscillator (23) is obtained, and others are in the same way. We see that the presence of a Cornell-type potential in the quantum system, the angular velocity of the rotating frame as well as the parameters that govern the background of KKT modified the energy spectrum and the wave function. In Case B, we have considered a Coulomb-type scalar potential and following similar procedure, we have obtained the non-compact expression of the energy eigenvalues $E_{n,l}$ by eq. (35). For the radial mode $n = 1$, we have obtained the ground-state energy level and the wave function (36)–(37) with the restriction (38) on the angular frequency of the oscillator.

Equations (21) and (35) give us the non-compact expression of the energy eigenvalues of the KG oscillator that interacts with Cornell- and Coulomb-type scalar potentials, respectively in the magnetic cosmic string space–time in a uniformly rotating frame. The contributions to the relativistic energy level that stem from the topology of the magnetic cosmic string is given by the effective angular momentum,

$$l \to l_0 = \frac{\left(l - \frac{q\,\Phi}{2\pi}\right)}{\alpha}.$$

So, the energy level and eigenfunction depend on the geometric quantum phase [6] and we have

$$E_{n,l}(\Phi + \Phi_0) = E_{n,l\mp\tau}(\Phi)$$

where

$$\Phi_0 = \pm\frac{2\pi}{q}\,\tau \quad \text{with} \quad \tau = 0, 1, 2, \ldots .$$

This dependence of the energy level on the geometric quantum phase gives rise to a relativistic analogue of the Aharonov–Bohm effect for bound states. Besides, we can observe a Sagnac-type effect [30,72–74] due to the presence of a coupling between the angular velocity of the rotating frame ω and the angular momentum quantum number l.

References

[1] Th Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys) K* **1**, 966 (1921)
[2] Th Kaluza, *Int. J. Mod. Phys. D* **27**, 1870001 (2018)
[3] O Klein, *Mag. Phys*. **37**, 895 (1926)
[4] O Klein, *Nature* **118**, 516 (1927)
[5] T Muta, *An introduction to Kaluza–Klein theories* edited by H C Lee (World Scientific Publishing Co. Pvt. Ltd, Singapore, 1984)
[6] Y Aharonov and D Bohm, *Phys. Rev*. **115**, 485 (1959)
[7] V B Bezerra, *J. Math. Phys*. **30**, 2895 (1989)
[8] M Peskin and A Tonomura, *Lect. Notes Phys.* (Springer, Berlin, Germany, 1989) Vol. 340
[9] C Furtado, F Moraes and V B Bezerra, *Phys. Rev. D* **59**, 107504 (1999)
[10] C Furtado, V B Bezerra and F Moraes, *Mod. Phys. Lett. A* **15**, 253 (2000)
[11] F Ahmed, *Mod. Phys. Lett. A* **36**, 2150004 (2021)
[12] E V B Leite, H Belich and K Bakke, *Adv. High Energy Phys*. **2015**, 925846 (2015)
[13] E V B Leite, H Belich and R L L Vitória, *Adv. High Energy Phys*. **2019**, 6740360 (2019)
[14] E V B Leite, R L L Vitória and H Belich, *Mod. Phys. A* **34**, 1950319 (2019)

[15] J Carvalho, A M de M Carvalho, E Cavalcante and C Furtado, *Eur. Phys. J. C* **76**, 365 (2016)
[16] E V B Leite, H Belich and R L L Vitória, *Braz. J. Phys*. **50**, 744 (2020)
[17] F Ahmed, *Eur. Phys. J. C* **80**, 211 (2020)
[18] F Ahmed, *Adv. High Energy Phys*. **2020**, 8107025 (2020)
[19] F Ahmed, *Sci. Rep*. **11**, 1742 (2021)
[20] I M Benn and R W Tucker, *J. Phys. A : Math. Gen*. **16**, L123 (1983)
[21] G German, *Class. Quantum Grav*. **2**, 455 (1985)
[22] Y-S Wu and A Zee, *J. Math. Phys.* **25**, 2696 (1984)
[23] P Ellicott and D J Toms, *Class. Quantum Grav*. **6**, 1033 (1989)
[24] R Delbourgo, S Twisk and R B Zhang, *Mod. Phys. Lett. A* **3**, 1073 (1988)
[25] R Delbourgo and R B Zhang, *Phys. Rev. D* **38**, 2490 (1988)
[26] K Bakke, A Yu Petrov and C Furtado, *Ann. Phys*. **327**, 2946 (2012)
[27] S Baskal and H Kuyrukcu, *Gen. Relat. Gravit*. **45**, 359 (2013)
[28] S Ichinose, *Phys. Rev. D* **66**, 104015 (2002)
[29] A Macias and H Dehnen, *Class. Quantum Grav*. **8**, 203 (1991)
[30] F W Hehl and W-T Ni, *Phys. Rev. D* **42**, 2045 (1990)
[31] K Bakke, *Phys. Lett. A* **274**, 4642 (2010)
[32] P Strange and L H Ryder, *Phys. Lett. A* **380**, 3465 (2016)
[33] R L L Vitória and K Bakke, *Eur. Phys. J. C* **78**, 175 (2018)
[34] L B Castro, *Eur. Phys. J. C* **76**, 61 (2016)
[35] L C N Santos and C C Barros Jr, *Eur. Phys. J. C* **78**, 13 (2018)
[36] M Hosseinpour and H Hassanabadi, *Eur. Phys. J. Plus* **130**, 236 (2015)
[37] R L L Vitória, *Eur. Phys. J. C* **79**, 844 (2019)
[38] K Bakke, *Gen. Relativ. Grav.* **45**, 1847 (2013)
[39] F Ahmed, *Chin. J. Phys.* **66**, 587 (2020)
[40] E R F Medeiros and E R B de Mello, *Eur. Phys. J. C* **72**, 2051 (2012)
[41] F Ahmed, *Int. J. Geom. Meth. Mod. Phys*. **17**, 2050138 (2020)
[42] K Bakke and H Belich, *Int. J. Mod. Phys. A* **35**, 20500023 (2020)
[43] K Bakke, V B Bezerra and R L L Vitória, *Int. J. Mod. Phys. A* **35**, 2050129 (2020)
[44] H F Mota and K Bakke, *Gen. Relativ. Grav.* **49**, 104 (2017)
[45] A V D M Maia and K Bakke, *Eur. Phys. J. C* **79**, 551 (2019)
[46] K Bakke, *Eur. Phys. J. Plus* **134**, 546 (2019)
[47] K Bakke, *Eur. Phys. J. Plus* **127**, 82 (2012)
[48] B-Q Wang, Z-W Long, C-Y Long and S-R Wu, *Int. J. Mod. Phys. A* **33**, 1850158 (2018)
[49] L C N Santos and C C Barros Jr, *Int. J. Mod. Phys. A* **33**, 1850122 (2018)
[50] E V B Leite, H Belich and R L L Vitória, *Mod. Phys. Lett. A* **35**, 2050283 (2020)
[51] F Ahmed, *Int. J. Mod. Phys. A* **35**, 2050101 (2020)
[52] J R Letaw *et al*, *Phys. Rev. D* **22**, 1345 (1980)
[53] J R Letaw, *Phys. Rev. D* **23**, 1709 (1981)
[54] J S Bell *et al*, *Nucl. Phys. Rev. B* **212**, 131 (1983)
[55] E T Akhmedov *et al*, *JETP Lett*. **86**, 615 (2007)
[56] E T Akhmedov *et al*, *Int. J. Mod. Phys. A* **22**, 4797 (2007)
[57] A Vilenkin, *Phys. Rep*. **121**, 263 (1985)
[58] A Vilenkin and E P S Shellard, *Strings and other topological defects* (Cambridge University Press, Cambridge, 1994)
[59] L D Landau and E M Lifshitz, *The classical theory of fields, Course of theoretical physics* (Elsevier, 1980) Vol. 2
[60] K Bakke, *Ann. Phys. (N.Y.)* **346**, 51 (2014)
[61] W Y P Hwang and T Y Wu, *Relativistic quantum mechanics and quantum fields* (World Scientific Publishing Co. Ptv. Ltd., Singapore, 1991)
[62] W Greiner, *Relativistic quantum mechanics: Wave equations* (Springer-Verlag, Berlin, Germany, 2000)
[63] A Boumali and N Messai, *Can. J. Phys*. **92**, 1460 (2014)
[64] M K Bahar and F Yasuk, *Adv. High Energy Phys*. **2013**, 814985 (2013)
[65] C Alexandrou, P de Forcrand and O Jahn, *Nucl. Phys. B (Proc. Suppl.)* **119**, 667 (2003)
[66] F Ahmed, *Adv. High Energy Phys*. **2020**, 5691025 (2020)
[67] A Ronveaux, *Heun's differential equations* (Oxford University Press, Oxford, 1995)
[68] S Y Slavyanov and W Lay, *Special functions: A unified theory based in singularities* (Oxford University Press, New York, 2000)
[69] G B Arfken and H J Weber, *Mathematical methods for physicists* (Elsevier Academic Press, London, 2005)
[70] A L C de Oliveira and E R Bezerra de Mello, *Class. Quantum Grav*. **23**, 5249 (2006)
[71] M S Cunha, C R Muniz, H R Christiansen and V B Bezerra, *Eur. Phys. J. C* **76**, 512 (2016)
[72] M G Sagnac, *C. R. Acad. Sci. (Paris)* **157**, 708 (1913)
[73] M G Sagnac, *C R. Acad. Sci. (Paris)* **157**, 1410 (1913)
[74] E J Post, *Rev. Mod. Phys*. **39**, 475 (1967)