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Nonlinear dynamical analysis of a time-fractional Klein–Gordon equation

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Abstract. In the present work, an enhanced perturbation analysis to solve a time-fractional Klein–Gordon equation (KG equation) and obtain an analytic approximate periodic solution is examined. The Riemann–Liouville fractional derivative is utilised. A travelling wave solution is adopted throughout the perturbation method by including two small perturbation parameters. The amplitude equation is formulated in the form of a cubic–quintic complex nonlinear Schrödinger equation. The solution of this equation leads to a transcendental frequency equation. An approximate solution to this frequency equation is performed. The stability criteria are derived. The procedure adopted here is very significant and powerful for solving many nonlinear partial differential equations (NLPDEs) arising in nonlinear science and engineering.

Keywords. Fractional nonlinear Klein–Gordon equation; homotopy perturbation method; multiple-scales method; cubic–quintic nonlinear Schrödinger equation; stability analysis.

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1. Introduction

Many researchers considered the fractional calculus hypothesis as a new branch of mathematics. The beginnings of the hypothesis of differential calculus and differential integration are due to the emergence of the calculus hypothesis. Therefore, the differentiation and integration of the fractional order may be defined as a distinct extension and generalisation of classical differential calculus and differential integration, see Podlubny [1]. In the past decades, fractional integrals and derivatives have been widely used in many fields. They are the most widely used in the fields of applied mathematics. But up to this point, and despite these studies, the usage of the fractional integrals and derivatives is not sufficient, especially with non-linear fractional differential equations (NLFDEs) which are the generalisation of the traditional differential equations; for instance, see Guo [2]. The NLFDEs play important roles, because of their prospective implementations in various scientific and technological fields, especially in solid-state physics, mathematical physics, mechanics, plasma physics, signal processing, bio-engineering, optical fibres, geochemistry, stochastic dynamical systems, nonlinear optics, economics, business and others, see refs [1–3]. These kinds of equations play more and more important roles in the fields of fluid mechanics and receive more and more attention.

In the last few decades, many researchers were interested in solving a nonlinear fractional differential equation by interposing several explicit, effective and powerful approaches. Researchers have introduced different methods, see refs [4–6]. Recent developments in both applied and theoretical sciences have widely adopted the use of fractional derivatives and integrals. Over the past decades, the fractional calculus theory has attracted much more consideration and interest due to its powerful applications. It has been attracting the interest of many investigators because of its applications in numerous areas of engineering and science, see Diethelm [7]. Numerical and analytical techniques have been used to study such equations. Additionally, many engineers and scientists in various branches of sciences like mathematics, biology, physics, particularly in branches of engineering like fluid mechanics followed the same approach. It can be seen that seeking periodic solutions of NLFDEs remain a significant problem that needs new techniques to develop exact and approximate solutions. The stability analysis of a

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periodic solution is very essential and crucial for a nonlinear dynamical system, see Peletan *et al* [8]. El-Dib and Elgazery [9] demonstrated a periodic solution of the time-fractional nonlinear oscillator based on the sense of Riemann–Liouville. Shen and El-Dib [10] derived periodic solution of a fractional sine-Gordon equation with the Riemann–Liouville fractional derivative by the homotopy perturbation method (HPM). Also, Elgazery [11] used the Riemann–Liouville fractional calculus to obtain an approximate periodic solution of the Newell–Whitehead–Segel (NWS) equation. Some other stability results in fractional-order problems were obtained by Li and Rhang [12]. Besides, fractional calculus has many applications in health science. A fractional-order model for the spread of human immunodeficiency virus (HIV) infection, leading to a disease called the acquired immunodeficiency syndrome (AIDS) was introduced by Babaei *et al* [13] to study numerically the impact of screening of infected individuals on the spread of human immunodeficiency virus via a predictor–corrector method. Recently, using the shifted Legendre basis with the spectral collocation method, Tuan *et al* [14] proposed a numerical study of fractional rheological models and fractional Newell–Whitehead–Segel equation.

The KG equation is considered as one of the most important mathematical models in the quantum field theory, see refs [15,16]. It was first considered as a quantum wave equation by Schrödinger in his search for an equation describing the de Broglie waves. Furthermore, it is called the relativistic Schrödinger equation. It behaves like a Schrödinger equation when constituent particles are treated at low energy or velocity. This equation, as well as the Dirac equation, has been studied a lot in recent years, see refs [17,18]. The KG equation arises in engineering and many physical applications. It describes the extension of biological membranes, the propagation of waves, the water wave evolution as well as other nonlinear waves arising in different physical systems, e.g. nonlinear optical waves, hydromagnetic waves and plasma waves, see El-Dib *et al* [19]. The nonlinear waves in fluid-filled viscoelastic tubes, nonlinear stability of surface waves and many important physical phenomena have been introduced in El-Dib *et al* [20]. Moreover fractional diffusion equations are always used for describing the abnormal KG phenomenon of the liquid in the medium. Additionally, some numerical solutions for the fractional KG equation have been obtained. A numerical solution for the fractional KG equation using the wavelet method is presented by Hariharan [21]. Kurulay [22] presented an analytic approximate solution of the fractional KG equation via the homotopy analysis method. Lyu and Vong [23] used the implicit difference scheme to calculate a numerical solution for nonlinear time-fractional KG equation. Furthermore, a numerical solution of the nonlinear KG equation with a time-fractional equation using the spectral collocation method has been proposed by Yang *et al* [24]. Recently, He and El-Dib introduced a modified HPM with an exponential decay parameter to solve the damping Duffing equation and the KG equation in refs [25] and [26], respectively. This modification yields a more effective outcome for the nonlinear oscillators and helps to overcome the shortcoming of the traditional approach.

El-Dib [27] suggested a new combination between the present two methodologies, the homotopy perturbation and the multiple scales. The multiple time-scale methodology is well-known within the perturbation theory. It is effective for week nonlinear oscillators. However, for all-powerful nonlinear oscillators, the combination between the multiple scales methodology and the homotopy perturbation one yields associate good surprising results as a few iterations are sufficient to achieve an accurate approximate solution. El-Dib has applied this new modification in the harmonic Duffing equation and discussed the stability behaviour in some famous resonance cases such as harmonic, subharmonic and superharmonic ones. This technique is successfully used to improve computational efficiency as well as the accuracy of the nonlinear KG equation, see El-Dib [28]. An extremely correct periodic temporal solution has been derived from three orders of perturbation. The amplitude equation that is obligatory as a homogeneous condition is of the fourth-order cubic-quintic nonlinear Schrödinger equation. Moreover, El-Dib [29] presented a new research in the homotopy case as a way to construct a homotopy equation in a generalised form for the partial differential equations. In this proposal, coupled homotopy expanding parameters were used. Therefore, there exist two homotopy outer and inner perturbation expansions. Accordingly, this technique yields a generalised rapid convergent solution.

As previously mentioned, there is increasing importance in the area of fractional derivatives. Therefore, the present work aims to extend the proposal given by El-Dib [28]. Many researchers focussed on the solitary wave solutions of the KG equation. Consequently, the main purpose is to derive an analytic approximate periodic solution for the nonlinear KG equation by including a temporal-fractional damping term. Furthermore, analytical approaches to the KG equation are rare and this paper will apply the HPM, see refs [30–32] coupled with the multiple scale method [27,28,33–39] to analyse it. Anjum and Ain [40] applied He's fractional derivative for the time-fractional Camassa–Holm equation by employing a fractional complex transform to convert the time-fractional Camassa–Holm differential equation into its partial differential equation, then apply the HPM to obtain a fairly accurate solution. The rest of the paper is organised as follows: Section 2 is devoted to introducing a perturbation technique with a travelling wave solution. The amplitude equation and the stability discussion, in detail, are given in §3. The obtained results are summarised as concluding remarks in §4. This section gives the main outcomes of the influences of various physical parameters in the analysis of linear as well as nonlinear stability of the problem at hand.

2. An enhanced perturbation method with a travelling wave solution

The enhanced perturbation method is a method that couples the homotopy method [41] with the multiple scale method [42] and is characterised by two perturbation parameters. Here, it should be noticed that there are major differences between the combined multiple scales with the homotopy technique and the classical multiple scales method. The advantages of the combined approach are

- The combined technique does not need a small parameter, while the traditional method needs it.
- Instead of the classical technique, a few iterations in the combined approach are sufficient to provide an accurate approximate solution.
- In contrast to the classical approach, to obtain advantageous outcomes, it is possible to subtract and add any term, and the choice of the zero-order equation is arbitrary.

Because of these advantages, we shall apply the approach of the temporal-spatial multiple-scales coupling with the homotopy perturbation for solving the following cubic nonlinear wave equation having a time-fractional order:

$$y_{tt} + P y_{xx} + m^2 y = \lambda D_t^{\alpha} y + Q y^3; \quad y = y(x,t), \tag{1}$$

where the coefficients P, m^2, λ and Q are real physical constants. The parameter λ represents the fractional temporal damped coefficient, m refers to the natural frequency and Q stands for the cubic stiffness parameter. D_t^{α} is the Riemann–Liouville time-fractional derivative of the function $y(x,t)$ of order $0 < \alpha \le 1$ which is defined as follows [1]:

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f(\gamma)}{(t-\gamma)^{\alpha}} \mathrm{d}\gamma; \quad 0 < \alpha \le 1; \quad t > a. \tag{2}$$

Here, we shall apply the temporal-spatial multiple scales as well as the wave-train approach to find an analytic approximate periodic solution of eq. (1). For this objective, the wave-train operator L is selected as

$$L = \partial_{tt} + P\partial_{xx} + m^2, \tag{3}$$

where ∂_{xx} and ∂_{tt} are the spatial second-order and total temporal derivatives, respectively.

Therefore, the homotopy equation with two perturbed parameters ρ and ε may be constructed in the form

$$\left(\partial_{tt} + P\partial_{xx} + m^2\right) y - \rho \left(Q y^3 + \varepsilon \lambda D_t^{\alpha} y\right) = 0; \quad \rho, \varepsilon \in [0,\ 1]. \tag{4}$$

Apply the methodology of the three time scales T_0, T_1, T_2, and the three spatial scales X_0, X_1, X_2 such that $T_n = \rho^n t$ and $X_n = \rho^n x$, $n = 0, 1, 2$. According to the methodology of the three scales, it is convenient to expand the function $y(x,t)$ up to three orders (zero, first and second-order) as follows:

$$\begin{aligned} y(x,t;\rho,\varepsilon) = {} & y_0(X_0, X_1, X_2, T_0, T_1, T_2; \varepsilon) \\ & + \rho y_1(X_0, X_1, X_2, T_0, T_1, T_2; \varepsilon) \\ & + \rho^2 y_2(X_0, X_1, X_2, T_0, T_1, T_2; \varepsilon). \end{aligned} \tag{5}$$

It is enough to specify y_0, y_1 and y_2 only. It is worth noting that there is no difficulty if we want to have higher order of approximation. The method is straightforward, but is not advancing effectively when we want to obtain y_3 and y_4 in the present methodology. This is due to the use of three time and spatial scales, only. To obtain higher ranking, other scales are needed. Thus, if we want to obtain y_3, it is convenient to apply the methodology up to four time scales T_0, T_1, T_2, T_3 or four spatial scales X_0, X_1, X_2, X_3. At this stage, the solvability condition will be a differential equation in a partial derivative D_3. The same conclusion will be obtained, if we further want to obtain y_4, and we focus to apply the methodology up to five time scales T_0, T_1, T_2, T_3, T_4, or five spatial scales X_0, X_1, X_2, X_3, X_4.

Therefore, the following spatial temporal partial derivatives and time-fractional derivative transformation as the multiple scale method can be used [34,39,43]:

$$(\partial_x, \partial_t) = (D_{X_0} + \rho D_{X_1} + \rho^2 D_{X_2}, D_{T_0} + \rho D_{T_1} + \rho^2 D_{T_2}), \tag{6}$$

$$\begin{aligned} (\partial_{xx}, \partial_{tt}) = {} & (D_{X_0}^2 + 2\rho D_{X_0} D_{X_1} + \rho^2 (D_{X_1}^2 + 2 D_{X_0} D_{X_2}), \\ & D_{T_0}^2 + 2\rho D_{T_0} D_{T_1} + \rho^2 (D_{T_1}^2 + 2 D_{T_0} D_{T_2})), \end{aligned} \tag{7}$$

$$D_t^{\alpha} = (D_{T_0} + \rho D_{T_1} + \rho^2 D_{T_2})^{\alpha}. \tag{8}$$

By applying Taylor expansion, we have

$$D_t^{\alpha} = D_{T_0}^{\alpha} + \rho\,\alpha D_{T_0}^{\alpha-1} D_{T_1} + \frac{1}{2}\rho^2\alpha[(\alpha-1)D_{T_0}^{\alpha-2}D_{T_1}^2 + 2D_{T_0}^{\alpha-1}D_{T_2}] + \dots, \tag{9}$$

where

$$D_{T_n} = \frac{\partial}{\partial T_n} \quad \text{and} \quad D_{X_n} = \frac{\partial}{\partial X_n}.$$

Employing eqs (5)–(9) into the homotopy eq. (4), one gets

$$\rho^0 : \left(D_{T_0}^2 + P D_{X_0}^2 + m^2\right) y_0 = 0, \tag{10}$$

$$\rho^1 : \left(D_{T_0}^2 + P D_{X_0}^2 + m^2\right) y_1 = -2\left(D_{T_0}D_{T_1} + P D_{X_0}D_{X_1}\right) y_0 + \varepsilon\lambda D_{T_0}^{\alpha} y_0 + Q y_0^3, \tag{11}$$

$$\rho^2 : \left(D_{T_0}^2 + P D_{X_0}^2 + m^2\right) y_2 = -2\left(D_{T_0}D_{T_1} + P D_{X_0}D_{X_1}\right) y_1 - \left(D_{T_1}^2 + 2D_{T_0}D_{T_2}\right) y_0 - P\left(D_{X_1}^2 + 2D_{X_0}D_{X_2}\right) y_0 + \varepsilon\lambda\left(D_{T_0}^{\alpha} y_1 + \alpha D_{T_0}^{\alpha-1} D_{T_1} y_0\right) + 3Q y_0^2 y_1. \tag{12}$$

The solution of eq. (10) can be sought in the form of the wave train solution as

$$y_0 = A(T_1, T_2, X_1, X_2)\mathrm{e}^{i(\omega T_0 + kX_0)} + \bar{A}(T_1, T_2, X_1, X_2)\mathrm{e}^{-i(\omega T_0 + kX_0)}, \tag{13}$$

where the frequency ω and the wavenumber k, which are assumed to be positive, are related by the relation

$$m^2 = \omega^2(\varepsilon) + Pk^2. \tag{14}$$

As the periodic solution response is of special interest in this study, the fractional-order derivatives of the periodic solutions can be derived approximately as follows, see refs [1,3,44]:

$$D_{-\infty}^{\alpha}\mathrm{e}^{i\omega t} = (i\omega)^{\alpha}\,\mathrm{e}^{i\omega t} = \left(\omega^{\alpha}\mathrm{e}^{\frac{1}{2}\pi\alpha i}\right)\mathrm{e}^{i\omega t}. \tag{15}$$

Inserting eq. (13) into eq. (11), using (15), for obtaining uniform solution we need to eliminate the source producing the secular terms. Therefore, one finds

$$i\left(\omega D_{T_1} + kPD_{X_1}\right)A - \tfrac{1}{2}\varepsilon\lambda\omega^{\alpha}\mathrm{e}^{\frac{1}{2}\pi\alpha i}A - \tfrac{3}{2}QA^2\bar{A} = 0. \tag{16}$$

According to condition (16), the uniform first-order solution is given as

$$y_1 = -\frac{Q}{8m^2}\left(A^3\mathrm{e}^{3i(\omega T_0 + kX_0)} + \bar{A}^3\mathrm{e}^{-3i(\omega T_0 + kX_0)}\right). \tag{17}$$

Substituting eqs (13) and (17) into eq. (12), one finds

$$\left(D_{T_0}^2 + PD_{X_0}^2 + m^2\right)y_2 = \Big[-\left(D_{T_1}^2 + 2i\omega D_{T_2}\right)A - P\left(D_{X_1}^2 + 2ikD_{X_2}\right)A + \alpha\varepsilon\lambda D_{T_0}^{\alpha-1}D_{T_1}A - \frac{3Q^2}{8m^2}A^3\bar{A}^2\Big]\mathrm{e}^{i(\omega T_0 + kX_0)} - \frac{3Q^2}{8m^2}\left(A^5\mathrm{e}^{5i(\omega T_0 + kX_0)} + 2\bar{A}A^4\mathrm{e}^{3i(\omega T_0 + kX_0)}\right) + \frac{Q}{4m^2}\left(3i\omega D_{T_1} + 3ikPD_{X_1} - \frac{1}{2}\varepsilon(3i\omega)^{\alpha}\lambda\right)A^3\mathrm{e}^{3i(\omega T_0 + kX_0)} + \text{c.c.} \tag{18}$$

The cancellation of the secular terms requires

$$2i\left(\omega D_{T_2} + kPD_{X_2}\right)A + D_{T_1}^2 A + PD_{X_1}^2 A + i\alpha\varepsilon\lambda\omega^{\alpha-1}\mathrm{e}^{\frac{1}{2}\pi\alpha i}D_{T_1}A + \frac{3Q^2}{8m^2}A^3\bar{A}^2 = 0. \tag{19}$$

At this stage, the solution of the second-order problem becomes

$$y_2 = \frac{3Q^2}{192m^4}A^5\mathrm{e}^{5i(\omega T_0 + kX_0)} - \frac{Q}{32m^4}\left(3QA^4\bar{A} + \frac{1}{2}\left(3 - 3^{\alpha}\right)\varepsilon(i\omega)^{\alpha}\lambda A^3\right) \times\mathrm{e}^{3i(\omega T_0 + kX_0)} + \text{c.c.}, \tag{20}$$

where the solvability condition (16) is used.

It is worth noting that, only three iteration processes are enough and the solution can be satisfied because all the parameters of the problem are represented within this approximate solution.

One can substitute eqs (13), (17) and (20) into (5), and setting $\rho \to 1$, one obtains the approximate solution in the form

$$y(x,t) = A\mathrm{e}^{i(\omega t + kx)} - \frac{Q}{32m^4}\left(3QA\bar{A} + 4Qm^2 + \frac{1}{2}\left(3 - 3^{\alpha}\right)\varepsilon(i\omega)^{\alpha}\lambda\right)A^3\mathrm{e}^{3i(\omega T_0 + kX_0)} + \frac{3Q^2}{192m^4}A^5\mathrm{e}^{5i(\omega T_0 + kX_0)} + \text{c.c}; \quad A = A(x,t). \tag{21}$$

The complete solution will be obtained when the unknown amplitude $A(x,t)$ is completely determined.

3. The amplitude equation and the stability discussion

Two solvability conditions (16) and (19) will be used to construct the amplitude equation of $A(x,t)$. It is possible to obtain a better form of the solvability condition (19) free of the parts $D_{T_1}A$ and $D_{T_1}^2 A$. This can be accomplished with the help of solvability condition (16). Therefore, one gets

$$\begin{aligned} &2i\left(\omega D_{T_2}+kPD_{X_2}\right)A \\ &+i\varepsilon(1-\alpha)kP\lambda\omega^{\alpha-2}\mathrm{e}^{\frac{1}{2}\pi\alpha i}D_{X_1}A \\ &+\frac{Pm^2}{\omega^2}D_{X_1}^2A+\frac{1}{4}\alpha\varepsilon\lambda^2\omega^{2\alpha-2}\mathrm{e}^{\pi\alpha i}A \\ &+\left[\frac{3}{2}Q\lambda\omega^{\alpha-2}\left(\cos\frac{1}{2}\pi\alpha+\alpha\mathrm{e}^{\frac{1}{2}\pi\alpha i}\right)\right. \\ &\left.+3i\frac{kPQ}{\omega^2}D_{X_1}\right]A^2\bar{A} \\ &+\frac{3Q^2}{8\omega^2m^2}\left(\omega^2-6m^2\right)A^3\bar{A}^2=0. \end{aligned} \tag{22}$$

Adding the first-order solvability condition (16) multiplied by ρ to eq. (22) multiplied by ρ^2 using transformations (6) and (7), at the final state let $\rho\to 1$, the result is

$$\begin{aligned} &i\frac{\partial A}{\partial t}+i\frac{kP}{\omega}\left(1+\frac{1}{2}\varepsilon(1-\alpha)\lambda\omega^{\alpha-2}\mathrm{e}^{\frac{1}{2}\pi\alpha i}\right)\frac{\partial A}{\partial x} \\ &+\frac{Pm^2}{2\omega^3}\frac{\partial^2 A}{\partial x^2}+\frac{1}{2}\varepsilon\lambda\omega^{\alpha-1}\mathrm{e}^{\frac{1}{2}\pi\alpha i} \\ &\times\left(\frac{1}{4}\alpha\varepsilon\lambda\omega^{\alpha-2}\mathrm{e}^{\frac{1}{2}\pi\alpha i}-1\right)A \\ &+\frac{3Q}{2\omega}\left(i\frac{kP}{\omega^2}\frac{\partial}{\partial x}+\frac{1}{2}\varepsilon\lambda\omega^{\alpha-2}\left(\cos\frac{1}{2}\pi\alpha+\alpha\mathrm{e}^{\frac{1}{2}\pi\alpha i}\right)\right. \\ &\left.-1\right)A^2\bar{A}+\frac{3Q^2}{16\omega^3m^2}\left(\omega^2-6m^2\right)A^3\bar{A}^2=0. \end{aligned} \tag{23}$$

Equation (23) represents a cubic–quintic complex nonlinear Schrödinger equation. A similar equation is obtained by El-Dib and Mady [37] for studying the three-dimensional nonlinear Kelvin–Helmholtz (KH) instability of the rotating magnetic fluids.

To discuss the stability behaviour, we proceed to find the wave travelling description. The wave variable

$$\theta=\frac{k}{\omega}(Vx-Pt)$$

is introduced so that $A(x,t)\Rightarrow A(\theta)$, where the localised wave solution $A(\theta)$ travels with the phase velocity $V=\omega/k$. Consequently, we make the following changes:

$$\frac{\partial}{\partial t}\Rightarrow-\frac{Pk}{\omega}\frac{\mathrm{d}}{\mathrm{d}\theta},\quad \frac{\partial}{\partial x}\Rightarrow\frac{\mathrm{d}}{\mathrm{d}\theta},\quad \frac{\partial^2}{\partial x^2}\Rightarrow\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}. \tag{24}$$

Accordingly, the nonlinear Schrödinger eq. (23) will change to the following nonlinear ordinary complex damping second-order differential equation:

$$\begin{aligned} &\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}+i(1-\alpha)\frac{\varepsilon\lambda k}{m^2}\omega^{\alpha}\mathrm{e}^{\frac{1}{2}\pi\alpha i}\frac{\mathrm{d}A}{\mathrm{d}\theta} \\ &+\frac{\varepsilon\lambda}{Pm^2}\omega^{\alpha+2}\mathrm{e}^{\frac{1}{2}\pi\alpha i}\left(\frac{1}{4}\alpha\varepsilon\lambda\omega^{\alpha-2}\mathrm{e}^{\frac{1}{2}\pi\alpha i}-1\right)A \\ &+\frac{3Q}{Pm^2}\left(ikP\frac{\mathrm{d}}{\mathrm{d}\theta}+\frac{1}{2}\varepsilon\lambda\omega^{\alpha}\left(\cos\frac{1}{2}\pi\alpha+\alpha\mathrm{e}^{\frac{1}{2}\pi\alpha i}\right)\right. \\ &\left.-\omega^2\right)A^2\bar{A}+\frac{3Q^2}{8Pm^4}\left(\omega^2-6m^2\right)A^3\bar{A}^2=0. \end{aligned} \tag{25}$$

In the following subsection, we shall study and solve the stability behaviour of the above amplitude equation in the special case of the absence of the fractional part of eq. (1)

3.1 *The implication in the case of $\varepsilon\to 0$*

Seeking the case of $\varepsilon\to 0$, yields the homotopy eq. (4) free from the fraction term. The limiting case of eq. (25), as $\varepsilon\to 0$, has the simplest form

$$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=\frac{3Q}{Pm^2}\left(\omega^2-ikP\frac{\mathrm{d}}{\mathrm{d}\theta}\right)A^2\bar{A}+\frac{3Q^2\left(\omega^2-6m^2\right)}{8Pm^4}A^3\bar{A}^2, \tag{26}$$

where eq. (26) represents a cubic–quintic nonlinear Duffing equation. As the zero solution satisfies eq. (26), we proceed, and since the trivial solution is valid, one makes a perturbation about the trivial solution as

$$A(\theta)=\text{zero}+A_d(\theta), \tag{27}$$

where A_d represents a small deviation from the zero solution. Employing eq. (27) into eq. (26), we obtain

$$\frac{\mathrm{d}^2A_d}{\mathrm{d}\theta^2}=\frac{3Q}{Pm^2}\left(\omega^2-ikP\frac{\mathrm{d}}{\mathrm{d}\theta}\right)A_d^2\bar{A}_d+\frac{3Q^2\left(\omega^2-6m^2\right)}{8Pm^4}A_d^3\bar{A}_d^2. \tag{28}$$

Suppose that the disturbance function A_d has a linear solution in the form

$$A_d=\frac{Pk}{\omega^2}\mathrm{e}^{i\sigma\theta}, \tag{29}$$

where the disturbance is measured by the characteristic exponent σ which is given by

$$\sigma^2+\frac{3PQk^2}{m^2\omega^2}\left(\omega^2+kP\sigma\right)+\frac{3P^3k^4Q^2\left(\omega^2-6m^2\right)}{8m^4\omega^4}=0. \tag{30}$$

The disturbance will be stable when the characteristic exponent σ is real. This requires that $Q>0$, $P<0$ besides the discriminant of the quadratic eq. (30) is positive. Therefore, one finds

$$\Delta=-\frac{3k^2PQ}{2m^4\omega^2}\left(5P^2k^2Q+8\omega^2m^2\right)>0. \tag{31}$$

The above condition may be satisfied when

$$PQ<0 \quad \text{and} \quad 5P^2k^2Q+8\omega^2m^2>0. \tag{32}$$

These are the conditions that controlled the stability behaviour for the following classical nonlinear KG equation:

$$y_{tt}+Py_{xx}+m^2y=Qy^3; \qquad y=y(x,t). \tag{33}$$

This stability criteria are the same as previously obtained by El-Dib [29]

In light of eqs (27), (29), (30) and (31) the amplitude function $A(\theta)$ can be sought in the form

$$A(\theta)=\frac{Pk}{\omega^2}\mathrm{e}^{i\sigma\theta}; \quad \sigma=-\frac{3k^3P^2Q}{2m^2\omega^2}\pm\tfrac{1}{2}\sqrt{\Delta}. \tag{34}$$

Accordingly, the complete primary solution of (13) becomes

$$y_0(x,t)=\frac{2Pk}{\omega^2}\cos\left((\sigma+k)x+\left(\omega-\frac{Pk\sigma}{\omega}\right)t\right). \tag{35}$$

3.2 *Solution and the stability analysis due to the amplitude equation in the non-zero ε*

To derive the stability criteria of the amplitude eq. (25) for the non-zero ε, we proceed to use the following solution:

$$A(\theta)=B\mathrm{e}^{i\Omega\theta}, \tag{36}$$

where $B$and Ω are real constants

By substituting the suggested solution (36) into the amplitude eq. (25), the resulting equation represents a relation between the amplitude B and the frequency Ω. The separation of the real and imaginary parts yields the following pair of relations:

$$\Omega^2+\left[\frac{3Q}{Pm^2}kPB^2+(1-\alpha)\frac{\varepsilon\lambda k}{m^2}\omega^\alpha\cos\frac{1}{2}\pi\alpha\right]\Omega$$
$$-\frac{\varepsilon\lambda}{Pm^2}\omega^{\alpha+2}\left(\frac{1}{4}\alpha\varepsilon\lambda\omega^{\alpha-2}\cos\pi\alpha-\cos\frac{1}{2}\pi\alpha\right)$$
$$-\frac{3Q}{Pm^2}B^2\left[\frac{1}{2}\varepsilon\lambda\omega^\alpha\left(\cos\pi\alpha+\alpha\cos\frac{1}{2}\pi\alpha\right)-\omega^2\right]$$
$$-\frac{3Q^2}{8Pm^4}\left(\omega^2-6m^2\right)B^4=0 \tag{37}$$

and

$$\Omega(1-\alpha)\frac{k}{Pm^4}+\omega s^2-\frac{3\alpha Q}{2}B^2-\tfrac{1}{2}\alpha\varepsilon\lambda\omega^\alpha\cos\tfrac{1}{2}\pi\alpha=0. \tag{38}$$

To compute the frequency Ω, one needs to combine eqs (37) and (38) in one characteristic equation. This may be accomplished by eliminating ω^α from the above equations to yield the following characteristic frequency equation:

$$a_2\Omega^2+a_1\Omega+a_0=0, \tag{39}$$

where the constants a, a_1 and a_2 are listed as follows:

$$a_2=8m^2P\left[\alpha m^2(1+\cos(\pi\alpha))+2(\alpha-1)^2k^2P\right],$$
$$a_1=8(\alpha-1)m^2kP\left[3QB^2(1+2\alpha+\cos(\pi\alpha))-4m^2+4k^2P\right],$$
$$a_0=3\alpha Q^2B^4\left[k^2P+(12\alpha+5)+(k^2P+5m^2)\cos(\pi\alpha)\right]-24m^2\left(m^2-k^2P\right)\times QB^2\left[\alpha+1-(\alpha-1)\cos(\pi\alpha)\right]+16m^2(m^2-k^2P).$$

The necessary condition for the amplitude equation (25) to behave like an oscillatory solution, is that the parameter Ω should be a real value. This requires that the discriminant of the frequency equation (39) satisfies the following condition:

$$P\left(b_2B^4Q^2+b_1B^2Q+b_0\right)<0, \tag{40}$$

where

$$b_2=3\alpha\left[k^2P+(12\alpha+5)m^2+(5m^2+k^2P)\cos(\pi\alpha)\right]\times\left[\alpha m^2(1+\cos(\pi\alpha))+2(\alpha-1)^2k^2P\right]-18k^2Pm^2(\alpha-1)^2(1+2\alpha+\cos(\pi\alpha))^2,$$
$$b_1=24m^2(m^2-k^2P)\left\{2k^2P(\alpha-1)^2(1+2\alpha+\cos(\pi\alpha))-[1+\alpha+(1-\alpha)\cos(\pi\alpha)]\times\left[\alpha m^2(1+\cos(\pi\alpha))+2(\alpha-1)^2k^2P\right]\right\},$$
$$b_0=32\alpha m^4\left(m^2-k^2P\right)^2\cos^2\left(\tfrac{1}{2}\pi\alpha\right).$$

It is worthwhile to note that condition (40) should be satisfied when $P<0$ associated with the quadratic polynomial is positive. This can be satisfied when its discriminant is negative, i.e., the stability constraint requires

$$P < 0 \quad \text{and} \quad b_1^2 - 4b_0 b_2 < 0. \tag{41}$$

These stability criteria can be arranged in the form

$$P < 0$$

and

$$\left(k^2 + \frac{\alpha m^2 \cos\left(\frac{1}{2}\pi\alpha\right)}{P\,(\alpha-1)^2}\right) \times \left(k^2 + \frac{m^2\left(3+\alpha-9\alpha^2+3(\alpha-1)^2\cos(\pi\alpha)\right)}{\alpha\left(6\alpha^2-12\alpha+5\right)P}\right) > 0. \tag{42}$$

For an arbitrary k^2, one can summarise the stability conditions as

$$P < 0$$

and

$$\left(6\alpha^2 - 12\alpha + 5\right)\left(3 + \alpha - 9\alpha^2 + 3(\alpha-1)^2\cos(\pi\alpha)\right) > 0. \tag{43}$$

It is effective to observe that the second condition in (43) gives the best values of the fractional parameter α, that yield the solution of the amplitude equation (25) to be a periodic solution, where

$$\Omega = \frac{1}{2a_2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}\right). \tag{44}$$

In terms of the natural frequency ω, one can eliminate the parameter Ω from eq. (37) with the help of eq. (38) to yield the following governing equation in the quadratic form in ω^2, which includes the transcendental parts in ω^α as

$$C_2\omega^4 - C_1\omega^2 + \left(C_0 + \varepsilon\, C_{00}\omega^\alpha + \varepsilon^2 C_{000}\omega^{2\alpha}\right) = 0, \tag{45}$$

where the coefficients $C's$ are constant and listed as follows:

$$C_2 = 8m^4 - 24\,(\alpha-1)^2\, QB^2m^2 + 3\,(\alpha-1)^2\, Q^2B^4,$$
$$C_1 = 3QB^2m^2\left[7\,(\alpha-1)^2\, QB^2 - 8m^2 + 8\alpha\,(3-\alpha)\right],$$
$$C_0 = 18(1-2\alpha+2\alpha^2)Q^2B^4m^4,$$
$$C_{00} = 12(2\alpha-1)QB^2m^4\lambda\cos\left(\tfrac{1}{2}\pi\alpha\right),$$
$$C_{000} = \alpha\left(2\alpha^2 - 3\alpha + 2 + \alpha\cos(\pi\alpha)\right)m^4\lambda^2.$$

As the characteristic equation (45) is in a perturbed form, only one approximate solution is available. Therefore, one can seek the regular perturbation expansion $\omega(\varepsilon)$ in a series of a small parameter ε, in the form

$$\omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \tag{46}$$

where ω_j; $j = 0, 1, 2, \ldots$ are unknown and we have to determine them.

Inserting expansion (46) into the fractional frequency equation (45) and putting the coefficients of each power of the small parameter ε to zero, the unknowns can be easily obtained. The first three unknowns ω_0, ω_1 and ω_2 are found to satisfy the following equation:

$$C_2\omega_0^4 - C_1\omega_0^2 + C_0 = 0, \tag{47}$$

$$\omega_1 = \frac{C_{00}\omega_0^\alpha}{2\omega_0\left(C_1 - 2C_2\omega_0^2\right)}, \tag{48}$$

$$\omega_2 = \frac{C_{000}\omega_0^{1+2\alpha} + \alpha C_{00}\omega_0^\alpha\omega_1 - \omega_0\left(C_1 - 6C_2\omega_0^2\right)\omega_1^2}{2\omega_0^2\left(C_1 - 2C_2\omega_0^2\right)}. \tag{49}$$

Quadratic equation (47) has the following roots:

$$\omega_0^2 = \frac{1}{2C_2}\left(C_1 \pm \sqrt{C_1^2 - 4C_2C_0}\right). \tag{50}$$

The second-order approximate solution of the frequency equation (45) can be formulated by substituting eqs (48)–(50) into expansion (46), and letting $\varepsilon \to 1$, yields

$$\omega = \sqrt{\frac{1}{2C_2}\left(C_1 \pm \sqrt{C_1^2 - 4C_2C_0}\right)} + \frac{\omega_0^\alpha C_{00} + \omega_0^{2\alpha}C_{000}}{2\omega_0\left(C_1 - 2C_2\omega_0^2\right)} + \frac{C_{00}\omega_0^{2\alpha}\left[(2\alpha-1)\,C_1 - 2\,(2\alpha-3)\,\omega_0^2C_2\right]}{8\omega_0^3\left(C_1 - 2C_2\omega_0^2\right)^3}. \tag{51}$$

It is useful to notice that stability occurs when the following conditions are satisfied:

$$C_2 > 0, \quad C_1 > 0, \quad C_0 > 0 \text{ and } C_1^2 - 4C_2C_0 > 0. \tag{52}$$

The last condition can be arranged in the form

$$\left(25 - 50\alpha + \alpha^2\right)Q^2B^4 + 16\left(17\alpha^2 - 3\alpha + 5\right)m^2QB^2 + 64\alpha\,(\alpha-4)\,m^4 > 0. \tag{53}$$

4. Concluding remarks

In this paper, we solve a time-fractional KG equation where the fractional derivative is the fractional Riemann–Liouville derivative. The enhanced homotopy perturbation method across the coupling with the temporal and spatial multiple scales is applied by using two different artificial small parameters. An analytic approximate periodic solution of the travelling wave description is derived. The present theoretical analysis leads to deal with an equation that covered the amplitude function like the complex nonlinear cubic–quintic Schrödinger-type equation as given in eq. (23). Across the travelling wave transform, it is converted to the nonlinear complex second-order equation of the cubic–quintic Duffing-type as given in eq. (25). This amplitude equation is used to derive the complex frequency equation. The combination of the imaginary and real parts produces a second-order frequency equation (39). The stability conditions (43) are derived from the last equation. One of these conditions gives the best values of the fractional parameter α, which produces a periodic solution. Additionally, in terms of the natural frequency, a transcendental frequency equation is derived in eq. (45) and an approximate solution is derived in eq. (51).

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