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Recovering source term of the time-fractional diffusion equation

MOHAMMAD PARTOHAGHIGHI[1], ESRA KARATAS AKGÜL[2], GERHARD-WILHELM WEBER[3], GUANGMING YAO[1] and ALI AKGÜL[2],*

[1]Department of Mathematics, Clarkson University, 8 Clarkson Ave, Potsdam, NY13699-5815, USA
[2]Department of Mathematics, Art and Science Faculty, Siirt University, 56100 Siirt, Turkey
[3]Faculty of Engineering Management, Poznan University of Technology, Poznan 60-965, Poland
*Corresponding author. E-mail: aliakgul00727@gmail.com

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Abstract. In this paper, a computational approach is suggested to obtain unknown space–time-dependent source term of the fractional diffusion equations. We assign a time-dependent source term and a linear space with the zero components which represent a series of boundary functions. In linear space, an energy border functional equation is obtained. After that, some numerical examples are provided to verify the accuracy of the method. Also, two tables are presented to display the values of solutions.

Keywords. Inverse problems; optimisation; engineering; iterative method; fractional diffusion equation; energy boundary functions.

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1. Introduction

In recent times, problems including non-integer order partial differential equations (PDEs) have drawn huge amount of interest from the researchers. Some investigations on the fractional partial equations have been worked out as in [1–17]. Some investigations on these equations have been worked out using high-order finite element methods [18], regularisation methods [19], truncation method [20], modified quasi-boundary value method [21], finite difference and spectral approximations [22], local discontinuous Galerkin method [23] and many other methods [24–26]. We consider the following problem:

$$\begin{cases} {}^{ABC}\mathcal{D}_s^{\alpha} u(y,s) = u_{yy} + H(y,s), & 0 < y < 1, 0 < s < s_f, \\ u(0,s) = k_0(s), & 0 \le s \le S, \\ u(1,s) = k_1(s), & 0 \le s \le S, \\ u(y,0) = \phi(y), & 0 \le y \le 1, \end{cases} \tag{1}$$

where $H(y,s)$ is an undetermined source term to be gained. Here, Ω_t and Ω_y are boundaries of $\Omega := \{(y,s) : a \le y \le b,\ 0 \le s \le s_f\}$ in s and y, ζ is a real parameter and κ is the average velocity.

Since eq. (1) contains $H(y,s)$ as unknown and provides no initial condition (IC), it is impossible to find $u(x,t)$ without delay. It is not required to be aware of internal data of $u(y,s)$; and the IC, final time condition. But, to obtain $H(y,s)$ we consider

$$\begin{aligned} &u(0,s) = F_0(s), \quad u(l,s) = F_l(s), \\ &u(0,s) = Q_0(s), \quad u_y(l,s) = Q_l(s), \end{aligned} \tag{2}$$

$$\begin{aligned} &H(0,s) = H_0(s), \quad H(l,s) = H_l(s), \\ &H_y(0,s) = H_0^{'}(s), \quad u(l,s) = H_l^{'}(s). \end{aligned} \tag{3}$$

We arrange our manuscript as: we depict main definitions and a fresh notion of energy border functional in §2. Section 3 shows numerical algorithm to recover $H(y,s)$. Also, in §4 two numerical instances are solved. Concluding remarks are given in §5.

2. Main definitions and functional of boundary functions

2.1 *Main definitions*

The ABC derivative is defined as

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$$
\begin{aligned}
&{}^{ABC}\mathcal{D}_s^{\alpha} f(s) \\
&\quad = \frac{N(\alpha)}{n-\alpha}\int_0^s f^{(n)}(c) E_\beta\left(\frac{-\alpha}{n-\alpha}(s-c)^{\alpha}\right)\mathrm{d}c, \\
&\quad n-1<\alpha<n,
\end{aligned} \tag{4}
$$

where $E_\beta(z)$ is the Mittag–Leffler function depicted by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

With respect to (4) for $0<\alpha\le 1$ we obtain

$$
\begin{aligned}
&{}^{ABC}\mathcal{D}_s^{\alpha} f(s) \\
&\quad = \frac{N(\alpha)}{1-\alpha}\int_0^s f'(c) E_\beta\left(\frac{-\alpha}{1-\alpha}(s-c)^{\alpha}\right)\mathrm{d}c.
\end{aligned} \tag{5}
$$

2.2 *Functional of boundary functions*

We multiply eq. (1) by $u(y,s)$ to get

$$
\begin{aligned}
u(y,s)\,{}^{ABC}\mathcal{D}_s^{\alpha} u(y,s) = {} & u(y,s)u_{yy}(y,s) \\
& + u(y,s)H(y,s).
\end{aligned} \tag{6}
$$

We then solve the above equation and employ eq. (2). Then, we reach

$$
\begin{aligned}
&\int_0^{\ell} u(y,s)\,{}^{ABC}\mathcal{D}_s^{\alpha} u(y,s)\mathrm{d}y + \int_0^{\ell} u_y^2(y,s)\mathrm{d}y \\
&\quad - \int_0^{\ell} H(y,s)u(y,s)\mathrm{d}y \\
&= Q_\ell(s)F_\ell(s) - Q_0(s)F_0(s) =: F(s).
\end{aligned} \tag{7}
$$

The obtained result is an energy equation and can be used to get $H(y,s)$. Now, we set

$$v(y,s) = u(y,s) - B_0(y,s), \tag{8}$$

where

$$
\begin{aligned}
B_0(x,t) = {} & \frac{1}{\ell^3}\left[2F_0(s) - 2F_\ell(s) + Q_0(s)\ell\right. \\
& \left. + Q_\ell(s)\ell\right] y^3 \\
& - \frac{1}{\ell^2}\left[3F_0(s) - 3F_\ell(s)\right. \\
& \left. + 2Q_0(s)\ell + Q_\ell(s)\ell\right] x^2 \\
& + Q_0(s)y + F_0(s).
\end{aligned} \tag{9}
$$

We consider for homogenisation function, resulting in a new energy functional equation with respect to $v(y,s)$ as

Table 1. Numerical results when $m=5$, $k=0.002$ and $\alpha=0.75$ for Example 1.

(y,s)	Approximate	Exact	Absolute error
(0.1,0.1)	0.0034	0.0034	6.1973e−05
(0.2,0.2)	0.0358	0.0370	0.0011
(0.3,0.3)	0.1438	0.1495	0.0057
(0.4,0.4)	0.3824	0.3985	0.0161
(0.5,0.5)	0.8050	0.8371	0.0321
(0.6,0.6)	1.4514	1.5003	0.0489
(0.7,0.7)	2.3343	2.3916	0.0573
(0.8,0.8)	3.4236	3.4711	0.0475
(0.9,0.9)	4.6290	4.6481	0.0190

$$
\begin{aligned}
&\int_0^{\ell} [v(y,s) + B_0(x,t)] \\
&\quad \times \left[{}^{ABC}\mathcal{D}_s^{\alpha} v(y,s) + {}^{ABC}\mathcal{D}_s^{\alpha} B_0(y,s)\right]\mathrm{d}x \\
&\quad + \int_0^{\ell} \left[v_y(y,s) + B_0'(y,s)\right]^2 \mathrm{d}y \\
&\quad - \int_0^{\ell} [v(y,s) + B_0(y,s)]\, H(y,s)\mathrm{d}x = F(s)
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
& v(0,s) = 0, \quad v(\ell,s) = 0, \\
& v_y(0,s) = 0, \quad v_y(\ell,s) = 0.
\end{aligned} \tag{11}
$$

We do not know $v(y,s)$. So some features need to be organised to investigate $v(y,s)$. Therefore, we obtain the limit function as

$$B_j(y) = \left(y^4 - 2\ell y^3 + \ell^2 y^2\right) y^{j-1}, \quad j \ge 1 \tag{12}$$

with

$$B_j(0) = 0, \quad B_j(\ell) = 0, \quad B_j'(0) = 0, \quad B_j'(\ell) = 0. \tag{13}$$

According to eqs (12) and (13), it is inevitable that $\beta B_j(y)$, $\beta \in \mathbb{R}$, is a function with boundary, if $B_j(y)$ is a function with boundary, and if $B_j(y)$ and $B_k(y)$ are functions with boundary, $B_j(y) + B_k(y)$ is said to be a function with boundary. The functions with boundary are closed under scalar addition and multiplication. Thus, the group of

$$\left\{B_j(y)\right\}, \quad j \ge 1 \tag{14}$$

whereas zero component set up a linear space of homogeneous functions with boundary, represented as $\mathcal{B}$.

For identifying $H(y,s)$, an approximate functional equation can be obtained in the following way.

Theorem 2.1. *We assume*

$$E_j(y) = \gamma_j B_j(y), \quad j \ge 1, \tag{15}$$

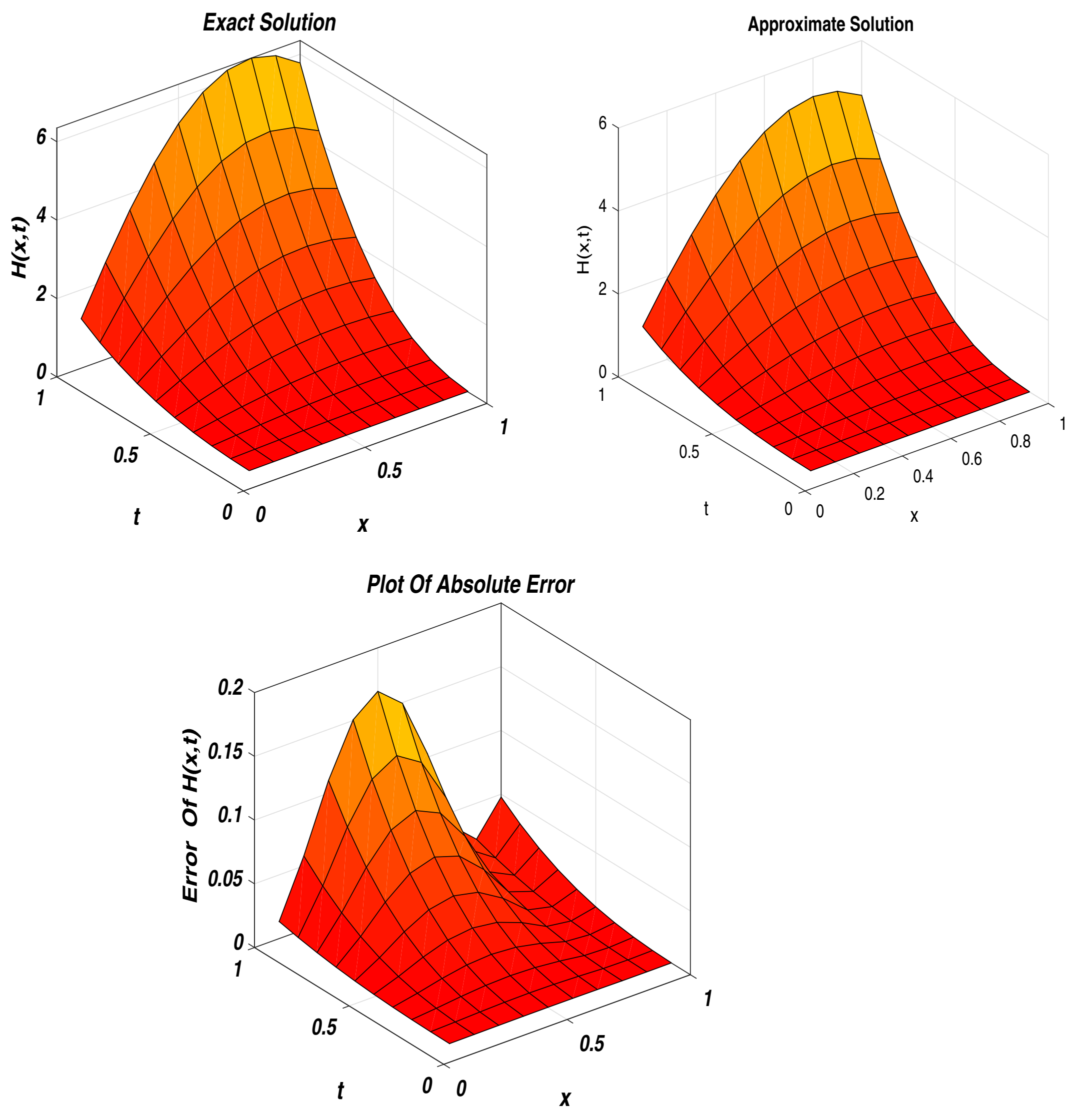

Figure 1. Numerical results when $m = 5$, $k = 0.002$ and $\alpha = 0.75$ of Example 1.

Table 2. Numerical results when $m = 5$, $k = 0.001$ and $\alpha = 0.75$ with $0 < y < 2$, $0 < s < 2$ for Example 1.

(y, s)	Approximate	Exact	Absolute error
(0,0)	0	0	0
(0.4,0.4)	0.4574	0.4572	1.9658e−04
(0.9,0.9)	4.0625	4.0657	0.0032
(1.6,0.1.6)	0.6243	0.6249	6.2485e−04
(2,2)	−31.9758	−31.9468	0.0290

such that $H(y, s)$ is a solution of $E_j(y)$:

$$\int_0^{\ell} \left[E_j(y) + B_0(y, s)\right]\left[{}^{ABC}\mathcal{D}_s^{\alpha} u(y, s) B_0(y, s) - H(y, s)\right] \mathrm{d}y + \int_0^{\ell} \left[E'_j(y) + B'_0(y, s)\right]^2 \mathrm{d}y = F(s), \tag{16}$$

where

$$a_2 = \int_0^{\ell} B'_j(y)^2 \mathrm{d}y,$$
$$a_1 = \int_0^{\ell} \left\{2B'_0(y, s), B'_j(y) + \left[D_s^{\alpha} B_0(y, s) - H(y, s)\right] B_j(y)\right\} \mathrm{d}y$$
$$a_0 = \int_0^{\ell} \left\{B'_0(y, s)^2 + B_0(y, s)\left[{}^{ABC}\mathcal{D}_s^{\alpha} B_0(y, s) - H(y, s)\right]\right\} \mathrm{d}x - F(s), \tag{17}$$

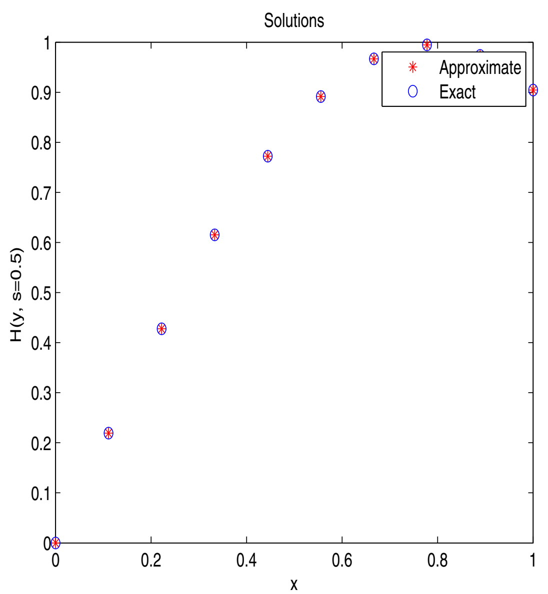

Figure 2. Numerical results at $t = 0.5$ when $m = 5$, $k = 0.002$ and $\alpha = 0.75$ for Example 1.

$$\gamma_j = \frac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_2}. \tag{18}$$

Proof. $B_j(y) \in \mathcal{B}$ is an element in $\mathcal{B}$, the scalar multiplication in eq. (14) renders $E_j(y) \in \mathcal{B}$, which fulfills

$$E_j(0) = 0, \quad E_j(\ell) = 0, \quad E'_j(0) = 0, \quad E'_j(\ell) = 0, \tag{19}$$

owed to eq. (13).

The function $E_j(y)$ already fulfills the conditions in eq. (19) as those of eq. (11) for $v(y, s)$. The power identity eq. (10) is introduced to $E_j(y)$, from which we can get approximately $v(y, s)$ by $E_j(y)$ and obtain eq. (16) which is a functional energy equation of $Ej(y)$ in $\mathcal{B}$.

Adding eq. (14) for $Ej(y)$ and

$$E'_j(y) = \gamma_j B'_j(y), \tag{20}$$

for $Ej(y)$ into eq. (15) we can attain a quadratic equation to describe the multiplier γ_j by

$$a_2\gamma_j^2 + a_1\gamma_j + a_0 = 0. \tag{21}$$

The coefficients a_0, a_1, a_2 are described in eq. (17). Therefore, the solution of γ_j is obtained in eq. (18). □

3. Numerical algorithm to recover $H(y, s)$

We shall recover the unknown source $H(y, s)$. We get $H(y, s)$ in the equation that follows:

$$H(y, s) = D(y, s) + \sum_{i=1}^{m} c_i E_i(y), \tag{22}$$

where

$$\begin{aligned} D(y, s) &= \frac{1}{\ell^3}\left[2H_0(s) - 2H_\ell(s) + H'_0(s)\ell + H'_\ell(s)\ell\right] y^3 \\ &\quad -\frac{1}{\ell^2}\Big[3H_0(s) - 3H_\ell(s) \\ &\quad +2H'_0(s)\ell + H'_\ell(s)\ell\Big]y^2 + H'_0(s)y + H_0(s). \end{aligned} \tag{23}$$

Therefore, we assume the data $H_0(s) = H(0, s)$, $H_\ell(s) = H(\ell, s)$, $H_y(0, s) = H'_0(s)$ and $H_y(\ell, s) = H'_\ell(s)$ to be ensured. Placing eq. (22) for $H(y, s)$ and $E_j(y)$, $j = 1, \ldots, m$ into eq. (16), we obtain a system to calculate c_i, $i = 1, \ldots, m$.

$$\begin{aligned} &c_i \int_0^\ell \left[E_j(y) + B_0(y, s)\right] E_i(y)\mathrm{d}y \\ &= \int_0^\ell \left[E_j(y) + B_0(y, s)\right] \Big[{}^{ABC}\mathcal{D}_s^\alpha u(y, s)B_0(y, s) \\ &\quad -D(y, s)\Big]\mathrm{d}y \\ &\quad + \int_0^\ell \left[E'_j(y) + B'_0(y, s)\right]^2 \mathrm{d}y - F(s). \end{aligned} \tag{24}$$

Here, the repeated index i is determined from $i = 1$ to $i = m$.

We have

$$H(0, s) = D(0, s) + \sum_{i=1}^{m} c_i E_i(0) = H_0(s),$$
due to $D(0, s) = H_0(s)$, $E_i(0) = 0$,

$$H(\ell, s) = D(\ell, s) + \sum_{i=1}^{m} c_i E_i(\ell) = H_\ell(s),$$
due to $D(\ell, s) = H_\ell(s)$, $E_i(\ell) = 0$,

$$H_y(0, s) = D_y(0, s) + \sum_{i=1}^{m} c_i E'_i(0) = H'_0(s),$$
due to $D_y(0, s) = H'_0(s)$, $E'_i(0) = 0$,

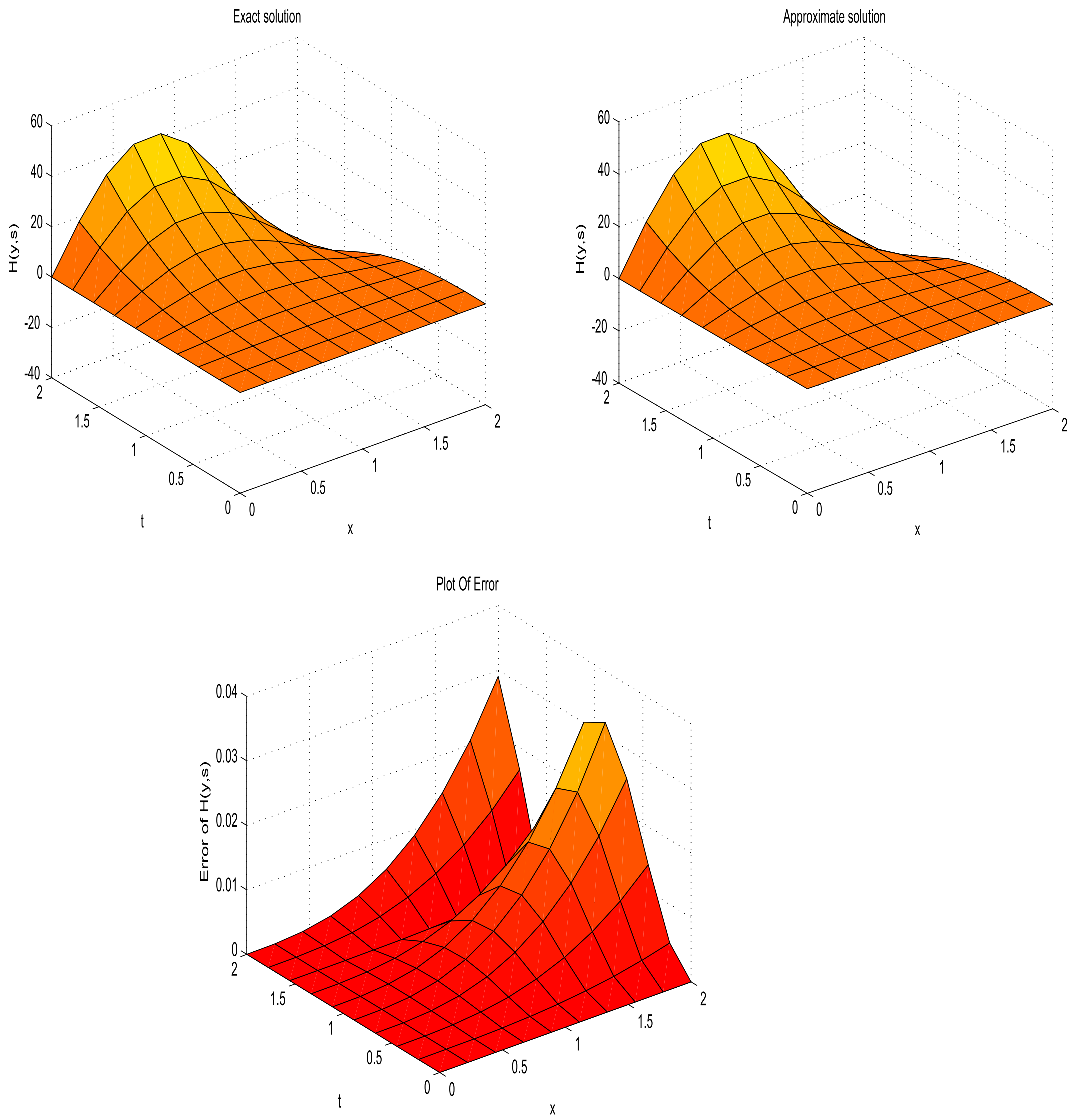

Figure 3. Numerical results when $m = 5$, $k = 0.002$ and $\alpha = 0.75$ with $0 < y < 2$, $0 < s < 2$ for Example 1.

$$H_y(\ell, s) = D_y(\ell, s) + \sum_{i=1}^{m} c_i E_i'(\ell) = H_\ell'(s),$$

$$\text{due to } D_y(\ell, s) = H_\ell'(s),\ E_i'(\ell) = 0,$$

where we have applied eqs (19) and (23). Secondly, the coefficient matrix $\int_0^\ell E_j(y)E_i(y)\mathrm{d}y$ in (24) is symmetric. In §2, it is described that the bases $E_i(y)$ in eq. (22) are time-dependent, because of the time-dependence of $\gamma_j(s)$. The numerical processes of energy border functional technique are summarised in the following, according to Theorem 2.1.

(i) We present $s \in \left(0, s_f\right]$, and given $m, \varepsilon, \gamma_j = 0, j = 0$, and an initial estimate of $\mathbf{c} = (c_1, \ldots, c_{\mathrm{rn}})^{\mathrm{T}}, \mathbf{c}^0 = \mathbf{0}$.

(ii) For $s = 0, 1, \ldots,$

$$E_j(y) = \gamma_j B_j(y),$$

$$H(y, s) = D(y, s) + \sum_{j=1}^{m} c_j^k E_j(y),$$

and we determine

$$a_2 = \int_0^\ell B_j'(y)^2 \mathrm{d}y,$$

$$a_1 = \int_0^\ell \Big\{ 2B_0'(y, s)B_j'(y) + \Big[{}^{ABC}\mathcal{D}_s^\alpha B_0(y, s) - H(y, s) \Big] \times B_j(y) \Big\} \mathrm{d}y,$$

$$a_0 = \int_0^\ell \Big\{ B_0'(y, s)^2 + \Big[{}^{ABC}\mathcal{D}_s^\alpha B_0(y, s) - H(y, s) \Big] B_0(y, s) \Big\} \mathrm{d}y - F(s).$$

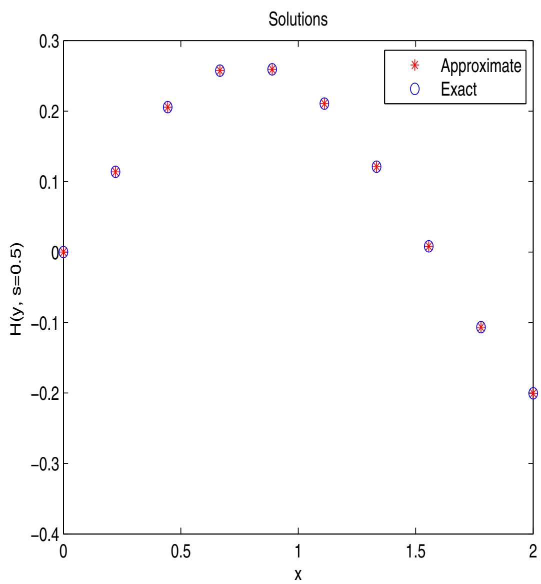

Figure 4. Numerical results at $t = 0.3$ when $m = 5$, $k = 0.002$ and $\alpha = 0.75$ with $0 < y < 2, 0 < s < 2$ for Example 1.

(iii) We compute

$$\gamma_j = \frac{-a_1 - \sqrt{\left|a_1^2 - 4a_0a_2\right|}}{2a_2},$$

$$E_j(y) = \gamma_j B_j(y),$$

$$E_j'(y) = \gamma_j B_j'(y).$$

(iv) We insert $E_j(y)$ and $E_j'(y)$ in

$$c_i \int_0^\ell \left[E_j(y) + B_0(y,s)\right] E_i(y)\mathrm{d}y$$

$$= \int_0^\ell \left[E_j(y) + B_0(y,s)\right]$$

$$\times \left[{}^{ABC}\mathcal{D}_s^\alpha B_0(y,s) - D(y,s)\right]\mathrm{d}y$$

$$+ \int_0^\ell \left[E_j'(y) + B_0'(y,s)\right]^2 \mathrm{d}y - F(s).$$

By solving these equations, we reach c_1^{k+1}. If

$$\|\mathbf{c}^{k+1} - \mathbf{c}^k\| \le \varepsilon, \tag{25}$$

our iterations end. Otherwise, we check on (ii) $a_1^2 - 4a_0a_2$ is likely to be negative, and we use $\left|a_1^2 - 4a_0a_2\right|$ to prevent from program interruption. In the above criterion, $\|\mathbf{c}^{k+1} - \mathbf{c}^k\|$ is the Euclidean norm of the vector $\mathbf{c}^{k+1} - \mathbf{c}^k$.

4. Numerical experiments

We consider

$$\hat{u}\,(0,s_i) = F_0\,(s_i)\,[1 + kR(i)],$$
$$\hat{u}\,(\ell,s_i) = F_\ell(s)[1 + kR(i)],$$
$$\hat{u}_y\,(0,s_i) = Q_0\,(s_i)\,[1 + kR(i)],$$
$$\hat{u}_y(\ell,t) = Q_\ell\,(s_i)\,[1 + kR(i)],$$
$$\hat{H}\,(0,s_i) = H_0\,(s_i)\,[1 + kR(i)],$$
$$\hat{H}\,(\ell,s_i) = H_\ell\,(s_i)\,[1 + kR(i)],$$
$$\hat{H}_y\,(0,s_i) = H_0'\,(s_i)\,[1 + kR(i)],$$
$$\hat{H}_y\,(\ell,s_i) = H_\ell'\,(s_i)\,[1 + kR(i)], \tag{26}$$

where $R(i)$ are haphazard errors between $[-1, 1]$.

Example 1. Consider the following variable:

$$u(y,s) = s^3\sin(2y), \tag{27}$$

and the function

$$H(y,s) = \sin(2y)\left(\frac{\Gamma(4)}{\Gamma(3-\alpha)}s^{3-\alpha} + 4s^3\right). \tag{28}$$

When $\alpha = 0.75$ and $m = 5$, competitive results are obtained compared to the results in [23]. In fact, the error of about 2 is obtained in [23], but by using the current method, the maximum error of about 1e−2 can be seen in table 1. The graphs of the solutions as well as the error graph can be seen in figure 1. Also, good results are obtained by changing the domain, for which the error graphs and solutions can be observed in figure 3. Moreover, the results can be seen in table 2. These results prove that our method is much more reliable than the generalised Tikhonov method in [23]. Also, the graph of approximate solutions at $s = 0.5$ and 0.3 can be seen in figures 2 and 4, respectively. Comparing our results with results in [23] shows that the presented method is more reliable than the generalised Tikhonov method for solving inverse problems.

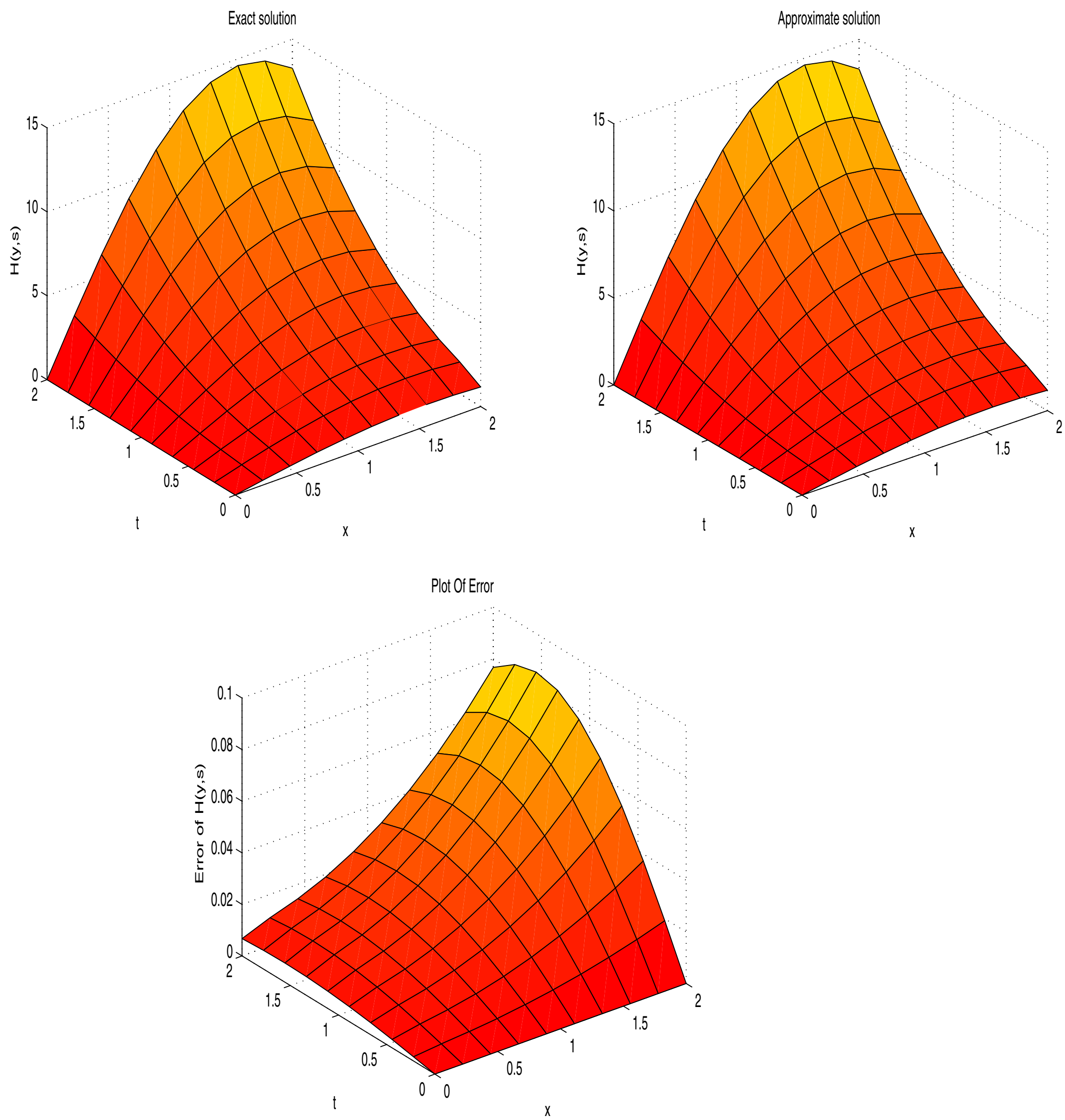

Figure 5. Numerical results when $m = 5$, $k = 0.001$ and $\alpha = 0.5$ for Example 2.

Example 2. We take into consideration

$$u_{ss}(y, s) + {}^{ABC}\,\mathcal{D}_s^{\alpha} u(y, s) - u_{yy}(y, s) = H(y, s) \quad (29)$$

and

$$u(y, s) = \left(\frac{1-\alpha}{N(\alpha)} + \frac{2\alpha}{\Gamma(\alpha+3)N(\alpha)} s^{\alpha}\right) s^2 \sin(y). \quad (30)$$

where

$$N(\alpha) = 1 - \alpha + \frac{1}{\Gamma(\alpha)}.$$

By using $\alpha = 0.5$ and $m = 5$, satisfactory results are obtained. The maximum absolute error is about 1E−2. The graphs related to the error can be observed in figure 5 as well as graphs of exact and numerical solutions. Also, the graph of approximate solution at $s = 0.4$ is provided in figure 6 by selecting $m = 5$, $k = 0.001$ and $\alpha = 0.75$ with $0 < y < 2$, $0 < s < 2$. Moreover, the values of the results can be seen in table 3.

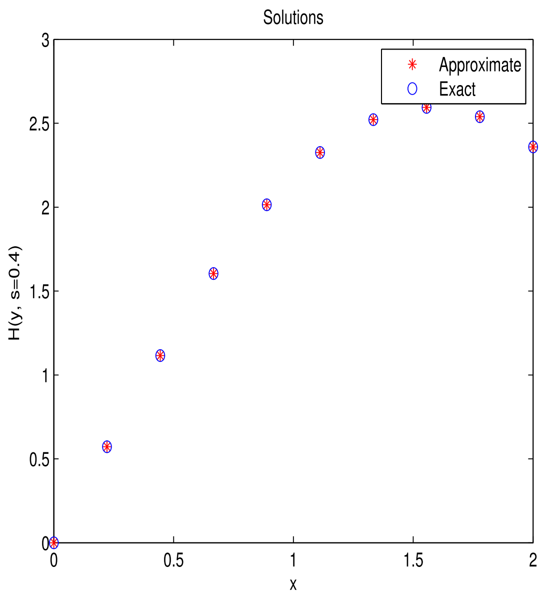

Figure 6. Numerical results at $t = 0.4$ when $m = 5$, $k = 0.001$ and $\alpha = 0.5$ for Example 2.

Table 3. Numerical results when $m = 5$, $k = 0.001$ and $\alpha = 0.75$ with $0 < y < 2$, $0 < s < 2$ for Example 2.

(y, s)	Approximate	Exact	Absolute error
(0,0)	0	0	0
(0.4,0.4)	1.1716	1.1786	0.0070
(0.9,0.9)	3.6473	3.6690	0.0217
(1.6,0.1.6)	9.6949	9.7517	0.0568
(2,2)	13.2079	13.2847	0.0768

5. Conclusions

In this work, we proposed a powerful numerical method to obtain the source term of the time-fractional diffusion equation including AB derivative. The proposed approach is based on the energy boundary functions. Some numerical experiments were presented to show the accuracy of the method. The graphs of solutions and errors are plotted as well as tabulated, and the obtained results are presented.

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