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On the behaviour of functions at the boundary conditions in the domain of the generalised momentum operators

JAFARI MATEHKOLAEE MEHDI and RASTEGARZADEH GOHAR*

Faculty of Physics, Semnan University, Semnan 35131-19111, Iran
*Corresponding author. E-mail: grastegar@semnan.ac.ir

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Abstract. In this paper we have investigated the general condition of self-adjointness of the generalised momentum operators and we have shown that it highly depends on the metric of the space. We have also discussed the domain of the generalised momentum operators at boundary conditions.

Keywords. Generalised momentum operators; curved space; momentum operator's domain.

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1. Introduction

The terminology of momentum operators in curved space was used in [1] for the first time, and terminology of the generalised momentum operators was first mentioned in [2]. There are many ways to represent momentum operators in curved space [3–8]. These operators are defined in curved space as follows:

$$P_i = -i\hbar\left(\frac{\partial}{\partial x^i} + \frac{1}{2}\Gamma^j_{ji}\right) = -i\hbar\frac{1}{\sqrt[4]{g}}\frac{\partial}{\partial x^i}\sqrt[4]{g}, \tag{1}$$

where

$$\Gamma^j_{ji}(x) = \frac{\partial}{\partial x^i}\ln\left(\sqrt{g(x)}\right)$$

is the Christofell symbol and g is the metric of the curved space under consideration. Recently, the inverse momentum operator in curved space has been obtained [9].

In general, operator (1) is not self-adjoint but it can be symmetric [10]. At infinite range, operator (1) is assumed to be self-adjoint [1]. At finite range, self-adjointness of this operator highly depends on the metric of the space [10]. In §2, we precisely indicate this dependency.

According to the definition of momentum operator in standard quantum mechanics and using integration by parts, one can get

$$(h, Pf) - (Ph, f) = [-i\hbar^*(x)f(x)]^{+\infty}_{-\infty}. \tag{2}$$

As $f(x)$ and $h(x)$ are square integrable, one usually concludes that these functions vanish for $x \to \pm\infty$. But we should note that, not all square-integrable functions vanish or even tend to a finite value at infinity. There are some examples in [11] which explicitly confirm this point. Now a question can be raised. Is there any function which is square-integrable but does not tend to zero at infinity, but is in the domain of momentum operator?

The momentum operator is not generally self-adjoint if we define it on the basis of classical derivative concept [12]. For this reason, this operator is defined based on the weak derivative [13]. As a matter of fact, the domain of the momentum operator, in this situation, coincides with the Sobolev space $H^1(R)$ [13]. Also, every function in Sobolev space also tends to zero at infinity [14]. It is also discussed in ref. [15] that every absolute continuous function in Hilbert space $L^2(R)$ tends to zero at infinity. For a generalised space, we have to change the mentioned question slightly: Is there any function in the domain of the generalised momentum operator which is square-integrable but does not tend to zero at boundary conditions? In §3, we try to answer this question.

2. Self-adjointness of the generalised momentum operator

We consider the functions ψ and φ in the domain of the generalised momentum operator and with the finite range of $a \leq x \leq b$. Using the definition of (1) and

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again integrating by parts, one can obtain

$$(\varphi, P\psi) - (P\varphi, \psi) = -i\big(\sqrt{g(b)}\varphi^*(b)\psi(b) - \sqrt{g(a)}\varphi^*(a)\psi(a)\big). \tag{3}$$

P is self-adjoint if and only if

$$\big(\sqrt{g(b)}\varphi^*(b)\psi(b) - \sqrt{g(a)}\varphi^*(a)\psi(a)\big) = 0. \tag{4}$$

So we immediately conclude that the functions ψ and φ should be linear. This means that $\psi(a) = \lambda\psi(b)$ and $\varphi(a) = \lambda\varphi(b)$ where λ is a constant complex number. By applying these equations into (4) one can obtain

$$\lambda = \left(\frac{g(b)}{g(a)}\right)^{1/4} q. \tag{5}$$

Here q is only a phase factor. Therefore, we conclude that

$$\psi(a) = \left(\frac{g(b)}{g(a)}\right)^{1/4} \mathrm{e}^{2\pi i\theta}\psi(b) \quad \text{and} \quad 0 \le \theta < 1. \tag{6}$$

The generalised momentum operator (1) is self-adjoint with respect to the boundary condition (6). Note that, according to (6) for each value of θ one can get a different self-adjoint generalised momentum operator P_θ. This result obviously confirms the prediction of ref. [10] concerning the self-adjointness of the generalised momentum operators in finite range.

3. The domain of generalised momentum operator at boundary conditions

To define the self-adjoint momentum operator on the manifold, in ref. [1], it has been proposed that, a surface with the topological properties of an infinite plane, but in specific region, is shaped like a portion of a sphere. Domingos and Caldeira [10] have divided the domain of operator (1) into three intervals: infinite, semi-infinite and finite. For the infinite interval, the analysis of ref. [10] is similar to [1] and for the semi-infinite interval, one we would have a symmetric operator which has no symmetric extensions. In the case of finite intervals, the discussion would largely depend on the metric of the space.

There is an open window to discuss the case with a new approach. We can assume the situation in which the determinant of the metric is eliminated and there would be no sign of the curvature of the space. So the important point is the boundary of the space, otherwise there is no boundary condition. For example, consider a simple one-dimensional space. It could be a circle with no boundary. The sphere is also a two-dimensional analog of the circle so that it has no boundaries. If there is a boundary at infinity, then we apply the argument in the following way:

At first, we choose $\sqrt[4]{g} = A$, then eq. (1) would be as follows:

$$D_i = -i\hbar\frac{1}{A}\frac{\partial}{\partial x_i}A, \tag{7}$$

where h is the complex conjugate of f, $h = f^*$. Note that if we consider f as a square-integrable function, then h will also be square-integrable, and we can write

$$(h, D_i f) = \int_a^b (A^2\,\mathrm{d}^n x)(f)\left(\frac{1}{A}\right) D_i(Af), \tag{8}$$

where A^2 is the square of the determinant of the metric and $A^2\,\mathrm{d}^n x$ is the integration measure. Note that, as f is in the domain of the momentum operator, then $D_i f$ is also square-integrable which means that $(h, D_i f)$ is well-defined. Moreover, it is worthwhile to notice that the domain of momentum operator must be dense. Otherwise, the adjoint operator is not well-defined. The integral is equal to

$$\frac{1}{2}\int_a^b \mathrm{d}^n x\, D_i[(Af)^2] = \frac{1}{2}(\varphi^2(b) - \varphi^2(a)), \tag{9}$$

where $Af = \varphi$. Now we use the Cauchy–Schwarz inequality which holds in curved space (for a simple argument, see the Appendix).

$$\left(\int_a^b \mathrm{d}x|\varphi(x)|^2\right)\cdot\left(\int_a^b \mathrm{d}x|\varphi^*(x)|^2\right) \ge \frac{1}{4}|\varphi^2(b) - \varphi^2(a)|^2. \tag{10}$$

If we assume that a and b tend to infinity, then we obviously have

$$\varphi^2(b) - \varphi^2(a) \to 0. \tag{11}$$

Equation (11) shows that the limit of $\varphi(x)$ exists at infinity. So we can write $\lim_{x\to\infty}\varphi(x) = C$, where C is a constant and as $\varphi(x)$ is square-integrable we immediately conclude that the constant value C must be equal to zero. Also, if we consider $a \to -\infty$ and $b \to +\infty$, then we can conclude from (8) and (9) that the limits of both $\varphi^2(a)$ and $\varphi^2(b)$ exist. So φ^2 and then $|\varphi|^2$ tend to some limits for $\pm\infty$. If either of those limits is nonzero, then the integral of $|\varphi|^2$ will be infinite. But, as φ is square-integrable, this cannot happen. Therefore, $|\varphi|^2$ and φ tend to zero at $\pm\infty$.

4. Conclusion

Our investigation indicates that the self-adjointness of the generalised momentum operators depends on the metric of the space and it is explicitly confirmed by eq. (6). By referring to different examples of metrics through eq. (6), the self-adjoint generalised momentum operators can be examined in detail. Also, if there is a boundary at infinity, our argument indicates that the wave function vanishes at this point.

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Appendix A

The Cauchy–Schwarz inequality is a result of the inner product being positive-definite which is part of the definition of the inner product. At first, consider the scalar product in curved space

$$(f, h) = \int_a^b \sqrt{(g)}\, f^*(x) h(x) \mathrm{d}^n x. \tag{A.1}$$

Suppose

$$A = (f, f), B = (h, h), C = (h, f). \tag{A.2}$$

The inequality is $AB \geq |C|^2$. For

$$B = 0 \Leftrightarrow h = 0 \Rightarrow C = 0. \tag{A.3}$$

The inequality is satisfied. Therefore, we have to prove only for the case when $B \neq 0$.

$$\int_a^b \sqrt{(g)} |Bf(x) - Ch(x)|^2 = B(BA - |C|^2) \geq 0. \tag{A.4}$$

As $B \neq 0$ and $B > 0$ we have $BA \geq |C|^2$.

References

[1] B S DeWitt, *Phys. Rev.* **85**(**4**), 653 (1952)
[2] P D Robinson and J O Hirschfelder, *J. Math. Phys.* **4**(**3**), 338 (1963)
[3] G R Gruber, *Am. J. Phys.* **40**(**11**), 1702 (1972)
[4] B Leaf, *Am. J. Phys.* **47**(**9**), 811 (1979)
[5] P V-Gonzalez and J C-Parra, *Am. J. Phys.* **49**(**8**), 754 (1981)
[6] Y Zhan, *Phys. Lett. A* **128**(**9**), 451 (1988)
[7] C Wang, *Am. J. Phys.* **57**(**1**), 87 (1989)
[8] M Carreau, *Phys. Rev. D* **40**(**6**), 1956 (1989)
[9] M J Matehkolaee, *Pramana – J. Phys.* **93**(**6**): 84 (2019)
[10] J M Domingos and M H Caldeira, *Found. Phys.* **14**(**2**), 147 (1984)
[11] F Gieres, *Rep. Prog. Phys.* **63**(**12**), 1893 (2001)
[12] G Teschi, in *Mathematical methods in quantum mechanics, with applications to Schrodinger operators*, 2 Edn (American Mathematical Society, 2014)
[13] M Valter, *Int. J. Geom. Methods Mod. Phys.* **13**, 1630011-1 (2016)
[14] H Brezis, *Functional analysis, Sobolev spaces and partial differential equations* (Springer, 2011)
[15] D M Gitman, I V Tyutin and B L Voronov, *Self-adjoint extensions in quantum mechanics, General theory and applications to Schrodinger and Dirac equations with singular potentials* (Birkhauser, 2012)