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On new symmetries and exact solutions of Einstein's field equation for perfect fluid distribution

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Abstract. Some new infinite-dimensional generalised Lie symmetries of Einstein's field equations for perfect fluid distribution are found by using the Lie symmetry analysis. The reduced ordinary differential equations are solved to obtain new non-trivial exact solutions. The software MAPLE is used for computation and MAPLE code is given to facilitate the research in this field.

Keywords. Einstein's field equations; Lie symmetries; perfect fluid; exact solutions.

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1. Introduction

The mathematical model consists of differential equations [1], which are solved to find solutions using different methods, and supplemented by physical meanings. In general relativity, the problems arise due to the effect of gravitational field on other fields and matter. The simplest assumption is that the Universe is homogeneous and isotropic on very large scales of the Universe. Einstein's field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ govern the dynamics of the Universe, where $G_{\mu\nu}$ is the Einstein tensor that describes the geometrical sector, $T_{\mu\nu}$ is the energy–momentum of the matter distribution and $8\pi G$ is the Einstein's gravitational constant. The stress-energy tensor $T_{\mu\nu}$ represents the cosmological fluid consisting of dust, radiation and vacuum energy. These Einstein's field equations describe the interaction of matter with gravitation as a tensor equation.

There are different methods to determine the optical solutions [2–9] of nonlinear partial differential equations but the Lie symmetry technique [10–13] is a powerful tool. The non-trivial exact solutions of Einstein's field equations are found by using Lie symmetries in [14–16] etc.

In the present paper, the symmetries of the derived equation are presented. In §2, the field equations are derived. In §3, more generalised symmetries are found by using Lie symmetry analysis. Some assumptions are considered to find new exact solutions. Further, their graphical representations are shown by particularising the arbitrary functions. In §4, the conclusions are summarised.

2. Einstein's field equation for perfect fluid distribution

The metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - (1-u)\mathrm{d}y^2 - (1+u)\mathrm{d}z^2 + 2v\,\mathrm{d}y\,\mathrm{d}z, \tag{1}$$

where u and v are functions of x and t only. On taking $G = 1$, the Einstein's field equations for perfect fluid distribution are as follows:

$$R_{ik} = -8\pi\left[(p+\rho)v_i v_k - \frac{1}{2}g_{ik}(\rho - p)\right], \quad g^{ik}v_i v_k = 1, \tag{2}$$

where ρ and p are density and pressure and v_i is the flow vector. If $v_y = 0$ and $v_z = 0$, we obtain the following relations:

$$(1-u)^{-1}R_{yy} = (1+u)^{-1}R_{zz} = -v^{-1}R_{yz}, \tag{3}$$

$$((1-u)R_{tt} - R_{yy})((1-u)R_{xx} + R_{yy} = (1-u)^2)R_{tx}^2. \tag{4}$$

From these relations and letting $u = v$, we obtain the following differential equation:

$$(1 - 2u^2)(u_{tt} - u_{xx}) + 2u(u_t^2 - u_x^2) = 0. \tag{5}$$

Some symmetries and exact solutions of eq. (5) have been already discussed in [14]. In the present work, new general Lie symmetries, reductions and exact solutions of eq. (5) are found. The reduction of the Einstein's field equations using infinitesimals is followed by the determination of exact solutions.

3. Symmetries and exact solutions of Einstein's field equation for perfect fluid distribution

In this section, more general Lie symmetries than the symmetries found in [14], are computed. To determine Lie symmetries of eq. (5), let us take the following Lie group of point transformations:

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2),\\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2),\\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2). \end{aligned} \tag{6}$$

Equation (5) is invariant under these transformations. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}. \tag{7}$$

The second prolongation of this operator is given by

$$\begin{aligned} X^{(2)} = X + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}}\\ +\eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}}. \end{aligned} \tag{8}$$

This prolonged operator is applied to eq. (5). Then the resulting equation is reduced by eliminating the dependence between all partial derivatives of u. The over-determined system of infinitesimals is obtained by equating the coefficients of all partial derivatives of u to zero. After some simplifications, the following infinitesimals are obtained:

$$\begin{aligned} \eta &= (c_1 \ln(u\sqrt{2} + \sqrt{(2u^2-1)})\\ &\quad + F_1(t+x) + F_2(t-x))\sqrt{(2u^2-1)},\\ \tau &= G_1(t+x) - G_2(t-x) + c_2,\\ \xi &= G_1(t+x) + G_2(t-x), \end{aligned}$$

where

$\{F_1(t+x), F_2(t-x), G_1(t+x), G_2(t-x)\}$

is a set of arbitrary functions, and c_1, c_2 are arbitrary constants.

Now, we shall discuss new reductions and new exact solutions of eq. (5). Let us consider

$$c_1 = c_2 = 0, \quad G_1(t+x) = aF_1(t+x)$$

and

$$G_2(t-x) = aF_2(t-x),$$

where a is an arbitrary constant. The characteristic form is given by

$$\begin{aligned} &\frac{\mathrm{d}x}{(aF_1(t+x) + aF_2(t-x))}\\ &\quad = \frac{\mathrm{d}t}{(aF_1(t+x) - aF_2(t-x))}\\ &\quad = \frac{\mathrm{d}u}{\sqrt{(2u^2-1)}(F_1(t+x) + F_2(t-x))}. \end{aligned}$$

The forms of similarity variable and similarity solution are as follows:

$$\begin{aligned} \alpha &= \int \frac{F_1(t+x) - F_2(t-x)}{F_1(t+x)F_2(t-x)}\mathrm{d}x\\ &\quad - \int \frac{F_1(t+x) + F_2(t-x)}{F_1(t+x)F_2(t-x)}\mathrm{d}t, \end{aligned} \tag{9}$$

$$u = \frac{2\mathrm{e}^{2(\sqrt{2}x + F(\alpha))} + 1}{4\mathrm{e}^{\sqrt{2}x + F(\alpha)}}. \tag{10}$$

On substituting this expression of u in eq. (5), the reduced equation in one variable is obtained as follows:

$$\frac{\mathrm{d}^2F(\alpha)}{\mathrm{d}\alpha^2} = 0,$$

which gives

$$F(\alpha) = r_1\alpha + r_2,$$

where r_1 and r_2 are arbitrary constants.

From expression (10), the exact solution of eq. (5) is given by

$$u(x,t) = \frac{2\mathrm{e}^{2(\sqrt{2}x + r_1\alpha + r_2)} + 1}{4\mathrm{e}^{\sqrt{2}x + r_1\alpha + r_2}}, \tag{11}$$

where

$$\begin{aligned} \alpha &= \int \frac{F_1(t+x) - F_2(t-x)}{F_1(t+x)F_2(t-x)}\mathrm{d}x\\ &\quad - \int \frac{F_1(t+x) + F_2(t-x)}{F_1(t+x)F_2(t-x)}\mathrm{d}t. \end{aligned} \tag{12}$$

In expression (9), $F_1(t+x)$ and $F_2(t-x)$ are arbitrary functions. The different expressions of $F_1(t+x)$

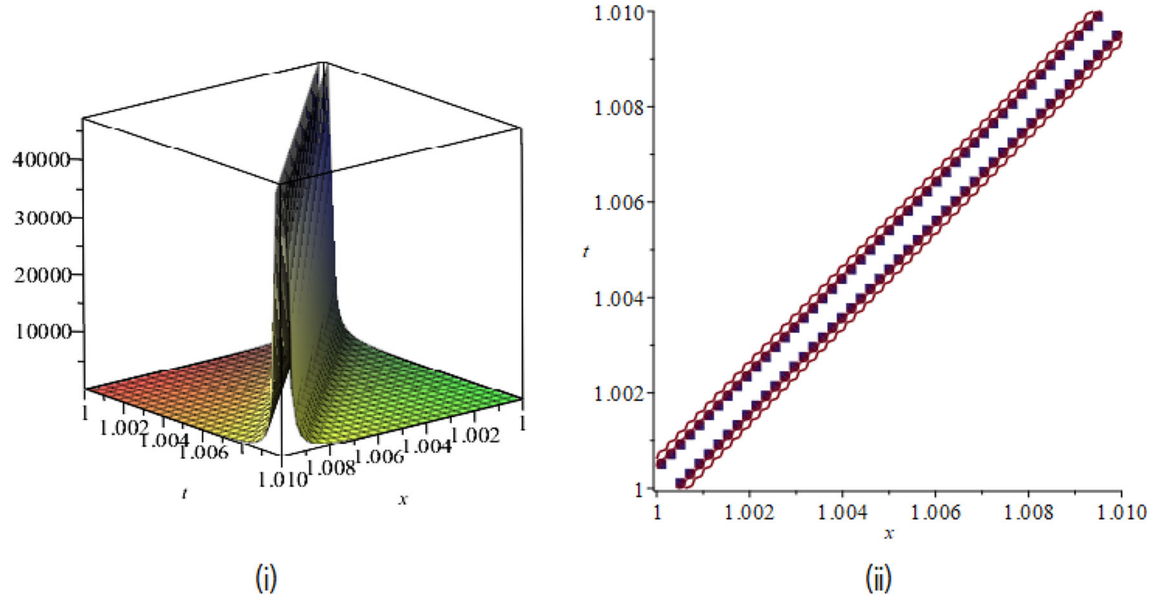

Figure 1. 3D plot and contour plot of the fluid solution to field equation (5) with $F_1(t+x) = t+x$, $F_2(t-x) = t-x$, $a = 1$, $r_1 = -2$ and $r_2 = 2$.

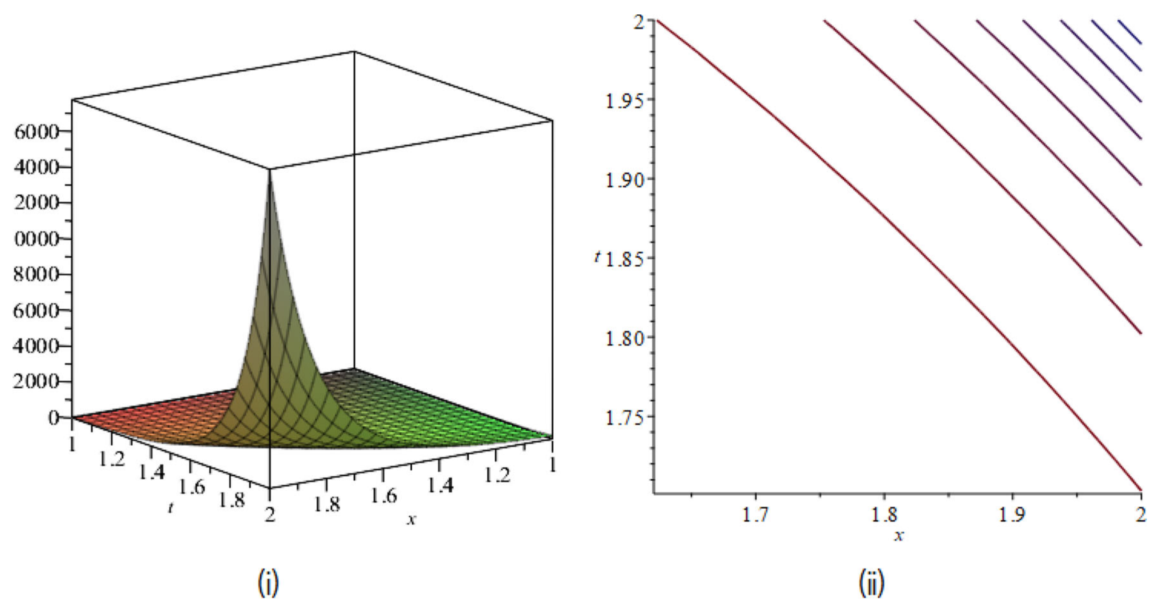

Figure 2. 3D plot and contour plot of the fluid solution to field equation (5) with $F_1(t+x) = 1/(t+x)$, $F_2(t-x) = 1/(t-x)$, $a = 1$, $r_1 = 2$ and $r_2 = 2$.

and $F_2(t-x)$ produce various exact solutions u from eq. (11). Some of the solutions are considered and graphically represented in figures 1–3.

Figure 1 shows that the behaviour of the solution is symmetric about the oblique asymptote $x = t$. Figure 2 shows that u increases sharply by increasing x and t. Figure 3 shows that u is a smooth function free of singularities.

4. Conclusion

In this paper, the more general Lie symmetries of eq. (5) are obtained by using Lie symmetry analysis. Attempts are made to find new exact solutions which are represented graphically by considering some cases. The solutions are verified with the aid of MAPLE software to prove their authenticity.

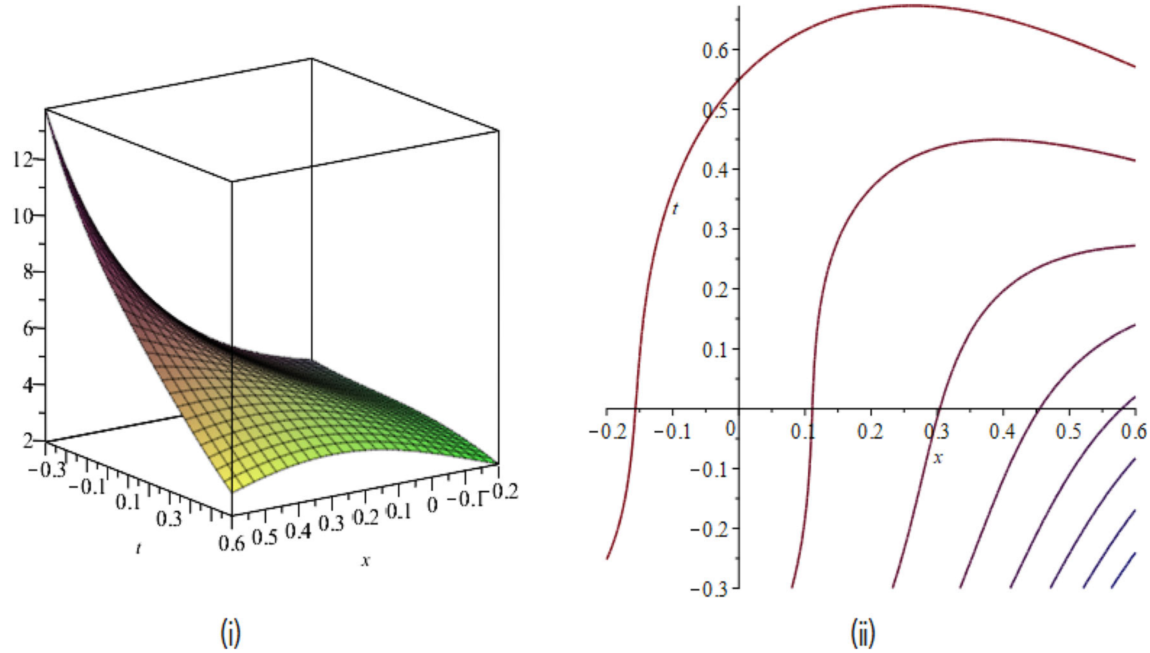

Figure 3. 3D plot and contour plot of the fluid solution to field equation (5) with $F_1(t+x) = 1/(t+x)^2$, $F_2(t-x) = 1/(t-x)^2$, $a = 1$, $r_1 = 2$ and $r_2 = 2$.

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Appendix A

MAPLE code [17,18] for Lie symmetries:

```
with(PDEtools):
depvar := u(x,t):
declare(u(x,t)):
alias(u = u(x,t)):
pde1 := (-2*u^2 + 1)*(diff(u,t,t)
  -(diff(u,x,x)))
  +2*u*((diff(u,t))^2 - (diff(u,x))^2) = 0:
detsys := DeterminingPDE(pde1):
pdsolve(detsys) :
```

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