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Group-invariant solutions to $SL'(2)$-motion equation

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Abstract. By using the theory of group-invariant solutions, the symmetries of the motion equation of plane curve in $SL'(2)$ geometry is presented. Group-invariant solutions associated with one-dimensional optimal system are obtained and classified.

Keywords. $SL'(2)$-motion equation of plane curve; Lie symmetry methods; optimal system; group-invariant solutions.

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1. Introduction

The hyperbolic curvature flow has significant applications in the growth of biological tissues, evolving foams, physics, etc. For instance, it was shown in [1] that the cell-based mathematical models of tissue growth to account for the mechanistic influence of curvature on cell evolution can be reduced to a specific type of hyperbolic curvature flow, in which the normal acceleration of the interface is the function of the mean curvature flow. Hyperbolic mean curvature flow has been successfully applied in evolving foams [2] and physics [3]. From the geometric point of view, the hyperbolic mean curvature flow was considered in [4–19].

Motivated by a $SL'(2)$-geometry invariant flow [20, 21],

$$\gamma_t = \mathbf{N},$$

where γ is the plane curve and $\mathbf{N}$ is the group normal of plane curve γ. In this paper, we consider an associated acceleration motion equation arising from $SL'(2)$-geometry,

$$\gamma_{tt} = \mathbf{N}. \tag{1}$$

A $SL'(2)$-motion in the form

$$\gamma_{tt} = f\mathbf{T} + g\mathbf{N}, \tag{2}$$

in which $\mathbf{T}$ and $\mathbf{N}$ are respectively the group tangent and normal and f, g are functions depending on the group curvature k and its derivatives with respect to the group arc-length s.

Assume that $\gamma = (x, u(x,t))$ is a graph over the x-axis. The group arc-length and curvature are given respectively by

$$ds = \sqrt{u_x - u^2}dx, \quad k = \frac{u_{xx} - 6uu_x + 4u^3}{(\sqrt{u_x - u^2})^3}.$$

One can verify that its tangent and normal are respectively equal to

$$\mathbf{T} = \frac{(1, u_x)}{\sqrt{u_x - u^2}}, \quad \mathbf{N} = \sqrt{u_x - u^2}(0, 1).$$

By

$$\gamma = (x, u(x,t)),$$
$$\gamma_t = (x_t, u_t + x_t u_x),$$
$$\gamma_{tt} = (x_{tt}, u_{tt} + 2u_{xt}x_t + x_t^2 u_{xx} + u_x x_{tt}),$$

we see that (2) is split into two equations, i.e.,

$$u_{tt} + 2u_{xt}x_t + x_t^2 u_{xx} + u_x x_{tt} = g\sqrt{u_x - u^2} + f\frac{u_x}{\sqrt{u_x - u^2}} \tag{3}$$

and

$$x_{tt} = \frac{f}{\sqrt{u_x - u^2}}. \tag{4}$$

In this paper, we fix $f = 0$, $g = 1 (\gamma_{tt} = \mathbf{N})$. Then, $x_{tt} = 0$, i.e., $x_t = c$, and (2) is rewritten as

$$c^2 u_{xx} + 2cu_{xt} + u_{tt} = \sqrt{u_x - u^2}. \tag{5}$$

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When $c = 0$, we get the nonlinear partial differential equation

$$u_{tt} = \sqrt{u_x - u^2}. \tag{6}$$

It is well known that the group-invariant solutions play important roles in the study of the properties of solutions which satisfy the geometry flows [22–26]. Based on Lie symmetry analysis approach and symbolic computing methods [27–29], all the symmetry Lie algebra of a motion equation in $SL'(2)$-geometry are obtained, and an optimal system of one-dimensional subalgebras of the Lie algebra is established by Lie group analysis.

This paper is organised as follows. In §2, the Lie symmetries of eq. (6) are obtained and in §3, an optimal system of one-dimensional subalgebras of the Lie algebra of eq. (6) are constructed. Section 4 concentrates on the reduced equations and some group-invariant solutions. The last section gives the conclusion.

2. The Lie symmetries of eq. (6)

We introduce a one-parameter Lie group of the vector field of infinitesimal transformation

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{7}$$

in which $\xi(x,t,u)$, $\eta(x,t,u)$ and $\phi(x,t,u)$ are the infinitesimal generators. The second-order prolongation of V is

$$pr^{(2)}(V) = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}. \tag{8}$$

According to invariance conditions for eq. (6) with respect to eq. (7), we have

$$pr^{(2)}(V)(\Delta)|_\Delta = 0, \tag{9}$$

where

$$\Delta = u_{tt} - \sqrt{u_x - u^2}. \tag{10}$$

Substitute eqs (7) and (8) into eq. (9),

$$\phi^{tt} - \frac{\phi^x}{2\sqrt{u_x - u^2}} + \frac{\phi^u}{\sqrt{u_x - u^2}} = 0, \tag{11}$$

in which ϕ^x, ϕ^u, ϕ^{tt} are given by

$$\begin{aligned} \phi^x &= D_x(\phi - u_x\xi - u_t\eta) + \xi u_{xx} + \eta u_{xt}, \\ \phi^u &= D_u(\phi - u_x\xi - u_t\eta) + \xi u_{xu} + \eta u_{tu}, \\ \phi^{tt} &= D_{tt}(\phi - u_x\xi - u_t\eta) + \xi u_{xtt} + \eta u_{ttt}, \end{aligned} \tag{12}$$

in which D_x is the total differentiation with respect to x, D_u is the total differentiation with respect to u, D_{tt} is the second-order total differentiation with respect to t.

Substitute eq. (11) into eq. (10), the determining systems of eq. (6) is

$$\begin{aligned} &\xi_t = 0, \quad \xi_u = 0, \quad \xi_{xxx} = 0, \quad \eta_t = 0, \\ &\eta_u = 0, \quad \eta_x = 0, \\ &\phi = -\xi_x u - \frac{1}{2}\xi_{xx}. \end{aligned} \tag{13}$$

Solving the determining systems, we obtain

$$\begin{aligned} &\xi = c_1x^2 + c_2x + c_3, \\ &\phi = -(2c_1x + c_2)u - c_1, \quad \eta = c_4, \end{aligned} \tag{14}$$

in which c_1, c_2, c_3 and c_4 are arbitrary constants.

Therefore, eq. (6) has a four-dimensional Lie algebra generated by four generators

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \\ V_4 &= x^2\frac{\partial}{\partial x} - (2xu+1)\frac{\partial}{\partial u}. \end{aligned}$$

The corresponding one-parameter transformation groups G_i by using V_i are

$$\begin{aligned} &G_1 : (x,t,u) \to (x+\varepsilon, t, u), \\ &G_2 : (x,t,u) \to (x, t+\varepsilon, u), \\ &G_3 : (x,t,u) \to (x\mathrm{e}^{\varepsilon}, t, u\mathrm{e}^{-\varepsilon}), \\ &G_4 : (x,t,u) \to \left(\frac{x}{1-\varepsilon x}, t, \frac{(2xu+1)\mathrm{e}^{-2x\varepsilon} - 1}{2x}\right). \end{aligned} \tag{15}$$

Then if $\omega = f(x,t)$ is a solution to (6), then there exists another solution given by

$$\begin{aligned} \omega^{(1)} &= f(x-\varepsilon, t), \\ \omega^{(2)} &= f(x, t-\varepsilon), \\ \omega^{(3)} &= f(\mathrm{e}^{-\varepsilon}x, t)\mathrm{e}^{-\varepsilon}, \\ \omega^{(4)} &= \left[f\left(\frac{x}{1+\varepsilon x}, t\right) + \frac{1+\varepsilon x}{2x}\right]\mathrm{e}^{-\frac{2x}{1+\varepsilon x}\varepsilon} - \frac{1+\varepsilon x}{2x}. \end{aligned} \tag{16}$$

3. One-dimensional optimal system of symmetric Lie algebra for eq. (6)

Firstly, the commutator and the adjoint representation of Lie algebra are calculated.

From $[V_i, V_j] = V_iV_j - V_jV_i$, the commutator of Lie algebra is shown in table 1.

From

$$\mathrm{Ad}\exp_{(\varepsilon V_i)}(V_j) = \mathrm{e}^{-\varepsilon V_i} V_j \mathrm{e}^{\varepsilon V_i} = V_j - \varepsilon[V_i, V_j] + \frac{1}{2!}\varepsilon^2[V_i,[V_i,V_j]] - \frac{1}{3!}\varepsilon^3[V_i,[V_i,[V_i,V_j]]] + \cdots$$

the adjoint representation of Lie algebra is shown in table 2.

Theorem 1. *One-dimensional optimal system of symmetric Lie algebra for (6) is*

$$V_2,\ \alpha V_2 + V_1,\ \beta V_2 + V_3,\ \lambda V_2 + V_4,\ \sigma V_1 + \tau V_2 + V_4. \tag{17}$$

Proof. The four-dimensional Lie algebra generated by $\{V_1, V_2, V_3, V_4\}$ is denoted as h_1. Let the general vector of h_1 be $V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4$. To simplify it, consider the following:

V_1 is adjoint to V,

$$\begin{aligned}\widetilde{V}_1 &= \mathrm{Ad}\exp_{(\varepsilon_1 V_1)}(V) \\ &= a_1\mathrm{Ad}\exp_{(\varepsilon_1 V_1)}(V_1) + \cdots + a_4\mathrm{Ad}\exp_{(\varepsilon_1 V_1)}(V_4) \\ &= (a_1 - a_3\varepsilon_1 + a_4\varepsilon_1^2)V_1 + a_2V_2 + (a_3 - 2a_4\varepsilon_1)V_3 + a_4V_4 \\ &= \tilde{a}_1V_1 + \tilde{a}_2V_2 + \tilde{a}_3V_3 + \tilde{a}_4V_4.\end{aligned} \tag{18}$$

Case 1. When $a_3^2 - 4a_4a_1 > 0$

(1) $a_4 \neq 0$, take

$$\varepsilon_1 = \frac{a_3 + \sqrt{a_3^2 - 4a_1a_4}}{2a_4},$$

so that $\tilde{a}_1 = a_1 - a_3\varepsilon_1 + a_4\varepsilon_1^2 = 0$, and $\widetilde{V}_1$ becomes

$$\widetilde{V}_2 = a_2V_2 - \sqrt{a_3^2 - 4a_1a_4}V_3 + a_4V_4. \tag{19}$$

Next we act on $\widetilde{V}_2$ by $\mathrm{Ad}\exp_{(\varepsilon_2 V_4)}(\widetilde{V}_2)$,

$$\begin{aligned}\widetilde{V}_3 &= \mathrm{Ad}\exp_{(\varepsilon_2 V_4)}(\widetilde{V}_2) \\ &= a_2V_2 - \sqrt{a_3^2 - 4a_1a_4}V_3 + [a_4 - (\sqrt{a_3^2 - 4a_1a_4}\varepsilon_2)]V_4.\end{aligned} \tag{20}$$

Take

$$\varepsilon_2 = \frac{a_4}{\sqrt{a_3^2 - 4a_1a_4}},$$

so that $a_4 - (\sqrt{a_3^2 - 4a_1a_4}\varepsilon_2) = 0$, and $\widetilde{V}_3$ becomes

$$\widetilde{V}_4 = a_2V_2 - \sqrt{a_3^2 - 4a_1a_4}V_3. \tag{21}$$

It is equivalent to

$$\widetilde{V}_{41} = -\frac{a_2}{\sqrt{a_3^2 - 4a_1a_4}}V_2 + V_3. \tag{22}$$

So the original vector V is equivalent to $\beta V_2 + V_3$.

(2) $a_4 = 0$, $a_3^2 > 0$. $\widetilde{V}_1$ becomes

$$\widetilde{V}_5 = (a_1 - a_3\varepsilon_1)V_1 + a_2V_2 + a_3V_3.$$

Take $\varepsilon_1 = a_1/a_3$, so that $a_1 - a_3\varepsilon_1 = 0$ and $\widetilde{V}_5$ becomes

$$\widetilde{V}_6 = a_2V_2 + a_3V_3.$$

It is equivalent to

$$\widetilde{V}_{61} = \frac{a_2}{a_3}V_2 + V_3.$$

So the original vector V is equivalent to $\beta V_2 + V_3$.

Case 2. When $a_3^2 - 4a_4a_1 = 0$

(1) When $a_1 = a_3 = a_4 = 0$, V becomes $\widetilde{V}_7 = a_2V_2$. So the original vector V is equivalent to V_2.

(2) When $a_1 = a_3 = 0, a_4 \neq 0$, V becomes $\widetilde{V}_8 = a_4\varepsilon_1^2V_1 + a_2V_2 - 2a_4\varepsilon_1V_3 + a_4V_4$. When $\varepsilon_1 = 0$, $\widetilde{V}_8 = a_2V_2 + a_4V_4$, it is equivalent to

$$\widetilde{V}_{81} = \left(\frac{a_2}{a_4}\right)V_2 + V_4.$$

So the original vector V is equivalent to $\lambda V_2 + V_4$.

(3) When $a_3 = a_4 = 0$, $a_1 \neq 0$, V becomes $\widetilde{V}_9 = a_1V_1 + a_2V_2$. It is equivalent to

$$\widetilde{V}_{91} = \left(\frac{a_2}{a_1}\right)V_2 + V_1.$$

So the original vector V is equivalent to $\alpha V_2 + V_1$.

(4) When $a_1,\ a_2,\ a_3,\ a_4 \neq 0$, take $\varepsilon_1 = \frac{a_3}{2a_4}$. $\widetilde{V}_1$ becomes $\widetilde{V}_{10} = a_2V_2 + a_4V_4$. It is equivalent to

$$\widetilde{V}_{10-1} = \frac{a_2}{a_4}V_2 + V_4.$$

Table 1. The commutator.

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	V_1	$2V_3$
V_2	0	0	0	0
V_3	$-V_1$	0	0	V_4
V_4	$-2V_3$	0	$-V_4$	0

Table 2. The adjoint representation.

Ad	V_1	V_2	V_3	V_4
V_1	V_1	V_2	$V_3-\varepsilon V_1$	$V_4-2\varepsilon V_3+\varepsilon^2 V_1$
V_2	V_1	V_2	V_3	V_4
V_3	$\mathrm{e}^{\varepsilon}V_1$	V_2	V_3	$\mathrm{e}^{-\varepsilon}V_4$
V_4	$V_1+2\varepsilon V_3+\varepsilon^2 V_4$	V_2	$V_3+\varepsilon V_4$	V_4

So the original vector V is equivalent to λV_2+V_4.

Case 3. When $a_3^2-4a_4a_1<0$. Take $\varepsilon_1=\frac{a_3}{2a_4}(a_4\neq 0)$, so that $\tilde{a}_3=a_3-2a_4\varepsilon_1=0$. $\widetilde{V}_1$ becomes

$$\begin{aligned}\widetilde{V}_{11}&=(a_1-a_3\varepsilon_1+a_4\varepsilon_1^2)V_1+a_2V_2+a_4V_4\\&=\left(a_1-a_3\frac{a_3}{2a_4}+a_4\frac{a_3^2}{4a_4^2}\right)\\&\quad\times V_1+a_2V_2+a_4V_4\\&=-\frac{a_3^2-4a_1a_4}{4a_4}V_1+a_2V_2+a_4V_4.\end{aligned} \tag{23}$$

It is equivalent to

$$\tilde{V}_{11-1}=-\frac{a_3^2-4a_1a_4}{4a_4^2}V_1+\frac{a_2}{a_4}V_2+V_4. \tag{24}$$

So the original vector V is equivalent to $\sigma V_1+\tau V_2+V_4$. □

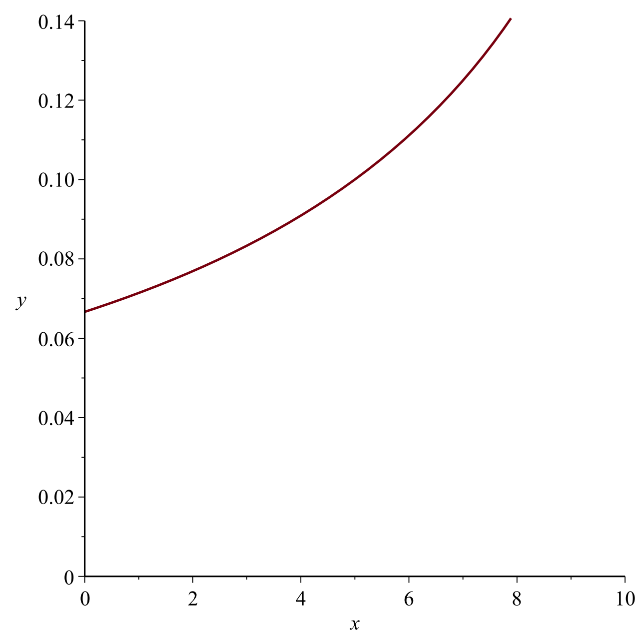

Figure 1. $u(x,t)$ of eq. (26) for $c_1=15$.

4. Group-invariant solutions of eq. (6)

In this section,we shall reduce the equation and obtain the group-invariant solutions of eq. (6).

4.1 $V_2=\dfrac{\partial}{\partial t}$

Its corresponding characteristic equation is

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0}.$$

Solving this equation, we get the invariance and invariant solution respectively as

$$\xi=x,\quad u=P(\xi).$$

Substituting the above equation into the original eq. (6), we get the following ordinary differential equation:

$$P_\xi-P^2=0. \tag{25}$$

By solving the above, the general solution can be obtained as

$$P=\frac{1}{c_1-\xi}.$$

Then the group-invariant solution of eq. (6) is

$$u=\frac{1}{c_1-x}. \tag{26}$$

This is shown in figure 1 ($c_1=15$).

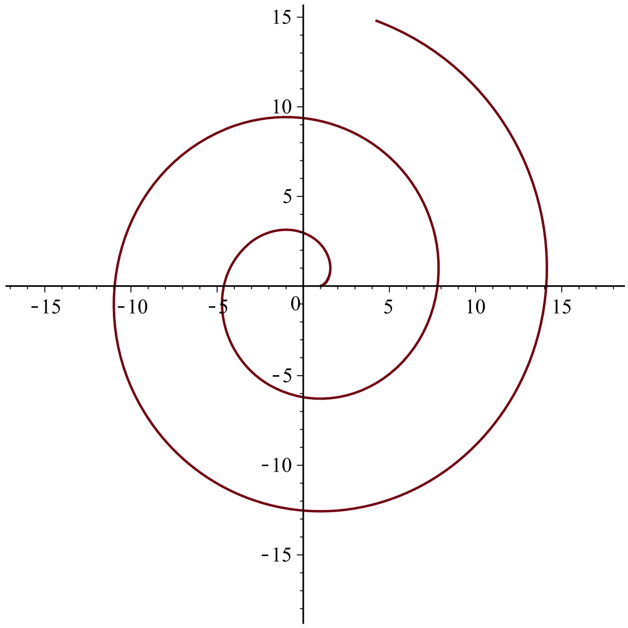

Figure 2. $\alpha P_\xi - P^2 > 0$.

4.2 $V_1 + \alpha V_2 = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial t} (\alpha \neq 0)$

Its corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dt}{\alpha} = \frac{du}{0}.$$

Solving the above equation, we get the invariance and invariant solution respectively as

$$\xi = \alpha x - t, \quad u = P(\xi).$$

Substituting the above equation into eq. (6), we get the following ordinary differential equation:

$$P_{\xi\xi} - \sqrt{\alpha P_\xi - P^2} = 0. \tag{27}$$

Because $\alpha P_\xi - P^2 \geq 0$ $(\alpha > 0)$, $P_{\xi\xi} \geq 0$ in (27), we are talking about three cases:

Case 1. When $\alpha P_\xi - P^2 > 0$, $P_{\xi\xi} > 0$. So $P_\xi > 0$. This means that the initial velocity and acceleration are in the same direction. Therefore, $P(\xi)$ is extended. This is shown in figure 2.

Case 2. When $\alpha P_\xi - P^2 = 0$, $P_{\xi\xi} = 0$. So $P = C_1$, i.e., $u = C_1$.

Case 3. When $\alpha P_\xi - P^2 < 0$, the equation has no solution.

4.3 $\beta V_2 + V_3 = \beta \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}$

Its corresponding characteristic equation is

$$\frac{dx}{x} = \frac{dt}{\beta} = \frac{du}{-u}.$$

Solving the above equation, we get the invariance and invariant solution respectively as

$$\xi = \beta \ln x - t, \quad u = \frac{P(\xi)}{x}.$$

Substituting the above equation into eq. (6), we get the following ordinary differential equation:

$$P_{\xi\xi} - \sqrt{\beta P_\xi - P - P^2} = 0. \tag{28}$$

Because $\beta P_\xi - P - P^2 \geq 0$ $(\beta > 0)$, $P_{\xi\xi} \geq 0$ in (28), we are talking about three cases:

Case 1. When $\beta P_\xi - P - P^2 > 0$, $P_{\xi\xi} > 0$

$$P_\xi > \frac{\beta}{P + P^2}.$$

Let $P + P^2 > 0$, $P_\xi > 0$. The figure is similar to figure 2.

Case 2. When $\beta P_\xi - P - P^2 = 0$, $P_{\xi\xi} = 0$

$$P = C_2, \quad u = \frac{C_2}{x}.$$

This is shown in figure 3 ($C_2 = 5$).

Case 3. When $\beta P_\xi - P - P^2 < 0$, the equation has no solution.

4.4 $\lambda V_2 + V_4 = \lambda \frac{\partial}{\partial t} + x^2 \frac{\partial}{\partial x} - (2xu + 1) \frac{\partial}{\partial u}$

Its corresponding characteristic equation is

$$\frac{dx}{x^2} = \frac{dt}{\lambda} = \frac{du}{-(2xu + 1)}.$$

Solving the above equation, we get the invariance and invariant solution respectively as

$$\xi = \frac{\lambda}{x} + t, \quad u = \frac{P(\xi) - x}{x^2}.$$

Substituting the above equation into the original eq. (6), we get the following ordinary differential equation:

$$P_{\xi\xi}^2 + \lambda P_\xi + P^2 = 0. \tag{29}$$

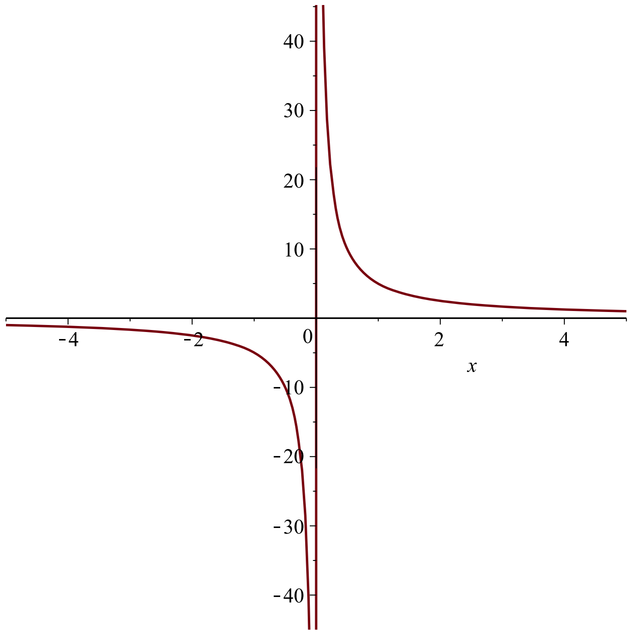

Figure 3. $u(x,t)$ of $u = C_2/x$ for $C_2 = 5$.

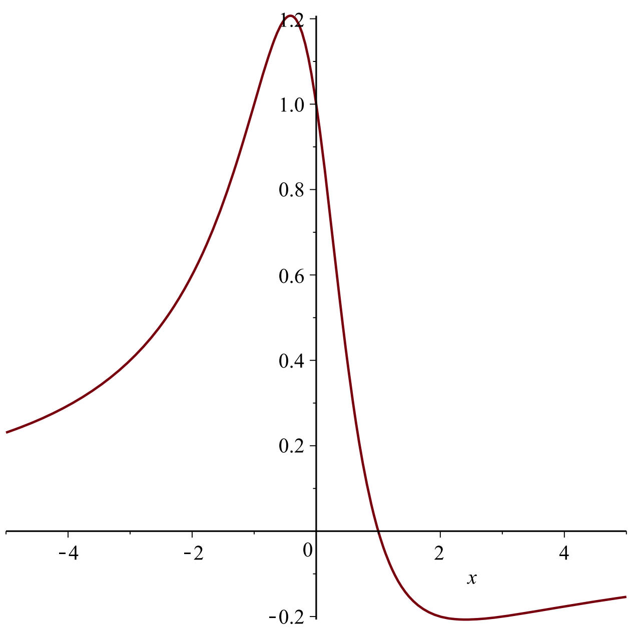

Figure 4. $u(x,t)$ of $u=(1-x)/(x^2+1)$ for $\sigma=1$, $C_3=1$.

4.5

$$\sigma V_1 + \tau V_2 + V_4 = (\sigma + x^2)\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} - (2xu+1)\frac{\partial}{\partial u}$$

Its corresponding characteristic equation is

$$\frac{dx}{\sigma + x^2} = \frac{dt}{\tau} = \frac{du}{-(2xu+1)}.$$

Solving this equation, we get the invariance and invariant solution respectively as

$$\xi = \tau \arctan\frac{x}{\sqrt{\sigma}} - \sqrt{\sigma}t, \quad u = \frac{P(\xi) - x}{x^2 + \sigma}.$$

Substituting the above equation into the original eq. (6), we get the following ordinary differential equation:

$$\sigma P_{\xi\xi} - \sqrt{\sigma\tau P_\xi - \sigma - P^2} = 0. \tag{30}$$

Because $\sigma\tau P_\xi - \sigma - P^2 \geq 0$ $(\sigma > 0, \tau > 0)$, $P_{\xi\xi} \geq 0$ in (30), we are talking about three cases:

Case 1. When $\sigma\tau P_\xi - \sigma - P^2 > 0$, $P_{\xi\xi} > 0$. So $P_\xi > 0$. This means that the initial velocity and acceleration are in the same direction. Therefore, $P(\xi)$ is extended. This is similar to figure 2.

Case 2. When $\sigma\tau P_\xi - \sigma - P^2 = 0$, $P_{\xi\xi} = 0$. So

$$P = C_3, \quad u = \frac{C_3 - x}{x^2 + \sigma}.$$

This is shown in figure 4 (when $\sigma = 1$, $C_3 = 1$).

Case 3. When $\sigma\tau P_\xi - \sigma - P^2 < 0$, the equation has no solution.

5. Conclusion

In this paper, we studied the $SL'(2)$-motion equation of geometric plane curve by applying Lie symmetry methods. Based on this method, the optimal system of the obtained symmetry and reduced equations are derived. Furthermore, group-invariant solutions are obtained.

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