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Non-classical Lie symmetry and conservation laws of the nonlinear time-fractional Kundu–Eckhaus (KE) equation

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Abstract. This work provides an analytical investigation of the time-fractional Kundu–Eckhaus (KE) equation. We use the symmetry of the Lie group as an appropriate tool which deals with the wide class of fractional-order differential equations in Riemann–Liouville sense. In the current work, firstly, we employ classical Lie symmetries to obtain similarity reductions of nonlinear generalised time-fractional KE equation. At the final step, we find relevant exact solutions for the extracted generators.

Keywords. Lie symmetry; Kundu–Eckhaus equation; time-fractional partial differential equation; classical symmetry; non-classical symmetry.

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1. Introduction

In the past century, the theory of fractional calculus and providing various fractional-order derivatives and integral operators have gained considerable importance among researchers. In fact, they found the fractional-order operators as powerful instruments which can be used to give more accurate descriptions of some non-local phenomena than the classical integer-order models in different fields of applied science and engineering. Well-known fields of fractional calculus include control, porous media, electrical networks, electrochemistry, signal processing, spread of disease, etc. [1–4]. One extremely difficult issue, that occurs when dealing with such fractional equations, is that the analytical solutions of such problems are generally hard to find. Therefore, many researchers have to use some numerical methods in this field such as, the radial basis functions (RBF) [5,6], homotopy perturbation method (HPM) [7], homotopy analysis method (HAM) [8], variational method [9], geometric methods [10,11] and semianalytic collocation methods [12–14]. These algorithms cannot be applied easily to solve time-fractional partial differential equations (FPDEs). For instance, perturbation methods such as HPM and HAM are based on small or large parameters. Hence, they cannot deal with the strong nonlinearity in the problems. These difficulties restrict applications of the mentioned approximate methods to solveF DEs.

Considering the aforementioned points, this work is devoted to Lie symmetry analysis method. Symmetry group method is among the most powerful tools for studying differential equations and has an important role in analysing different types of differential equations [15–22].

In this paper, we have considered the following time-fractional Kundu–Eckhaus (KE) equation [23]:

$$i\frac{\partial^\alpha q}{\partial t^\alpha} + aq_{xx} + b|q|^4 q + c(|q|^2)_x q = 0,\quad 0 < \alpha < 1, \tag{1}$$

where $q = q(x,t)$ is an unknown complex-valued function, a, b and c are real valued constants and $\frac{\partial^\alpha}{\partial t^\alpha}(\cdot)$ shows the operator of time-fractional derivative of order $\alpha \in (0,1]$ in Riemann–Liouville sense, which is described in [2] as

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$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \\ \times \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\mathrm{d}\tau, 0<\alpha<1, \\ \frac{\mathrm{d}}{\mathrm{d}t} f(t), \qquad \alpha = 1, \end{cases}$$

where $\Gamma(\cdot)$ denotes the well-known gamma function. Fractional KE equation is a complex nonlinear model applied to describe various real-world problems in physics and has gained considerable attention of many researchers. This equation was first introduced by Kundu in 1984. In fact, it is a special case of the well-known nonlinear Schrödinger equation [24]. A linearisation process for the nonlinear quantum mechanical equation was proposed by Calogero and Eckhaus [25]. Also, the KE equation is used to represent the optical soliton emission of ultrashort pulses in nonlinear fibres with high reflectivity. This equation has been studied by researchers, e.g. González-Gaxiola [26] who used the Laplace–Adomian decomposition method to solve that. Using the Hirota method, dark and light soliton solutions with a nonlinear quantum cube sense were introduced for KE equation [27]. If $\alpha = 1$, eq. (1) represents the classical model of the KE equation that is of nonlinear Schrödinger type [28].

In this work, we shall perform Lie symmetry analysis for eq. (1). Suppose

$$q(x,t) = u(x,t) + iv(x,t). \tag{2}$$

By substituting (2) into (1), and separating real and imaginary parts, the following equations are obtained:

$$\begin{cases} F_1: \frac{\partial^\alpha}{\partial t^\alpha} u + a v_{xx} \\ \quad + b(u^2+v^2)^2 v + 2cuvu_x + 2cv^2 v_x = 0, \\ F_2: -\frac{\partial^\alpha}{\partial t^\alpha} v + a u_{xx} \\ \quad + b(u^2+v^2)^2 u + 2cuvv_x + 2cu^2 u_x = 0. \end{cases} \tag{3}$$

In the next section, Lie group analysis is briefly described for the TFPDEs.

2. Lie symmetry analysis and symmetry reduction of the KE equation

Preliminaries of Lie symmetry analysis for TFPDE are summarised below. Here, the following TFPDE system of order $\alpha \in (0, 1)$ is considered:

$$\Delta_1 \equiv \frac{\partial^\alpha u}{\partial t^\alpha} - \delta_1(x,t,u,v,u_x,v_x,\ldots) = 0,$$
$$\Delta_2 \equiv \frac{\partial^\alpha v}{\partial t^\alpha} - \delta_2(x,t,u,v,u_x,v_x,\ldots) = 0, \tag{4}$$

where x, t are independent variables, and u, v depend on x, t.

Suppose that eq. (4) is invariant with respect to the following one-parameter Lie symmetry transformations:

$$\hat{x} = x + \varepsilon\xi(x,t,u,v) + O(\varepsilon^2),$$
$$\hat{t} = t + \varepsilon\tau(x,t,u,v) + O(\varepsilon^2),$$
$$\hat{u} = u + \varepsilon\eta(x,t,u,v) + O(\varepsilon^2),$$
$$\hat{v} = v + \varepsilon\phi(x,t,u,v) + O(\varepsilon^2),$$
$$\frac{\partial^\alpha \hat{u}}{\partial \hat{t}^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\eta^{\alpha,t}(x,t,u,v) + O(\varepsilon^2),$$
$$\frac{\partial^\alpha \hat{v}}{\partial \hat{t}^\alpha} = \frac{\partial^\alpha v}{\partial t^\alpha} + \varepsilon\phi^{\alpha,t}(x,t,u,v) + O(\varepsilon^2),$$
$$\frac{\partial^j \hat{u}}{\partial \hat{x}^j} = \frac{\partial^j u}{\partial x^j} + \varepsilon\eta^{j,x}(x,t,u,v) + O(\varepsilon^2), \quad j \in \mathbb{N},$$
$$\frac{\partial^j \hat{v}}{\partial \hat{x}^j} = \frac{\partial^j v}{\partial x^j} + \varepsilon\phi^{j,x}(x,t,u,v) + O(\varepsilon^2), \quad j \in \mathbb{N}, \tag{5}$$

where the group parameter is shown by ε, and its corresponding infinitesimal generator is

$$V = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \phi(x,t,u,v)\frac{\partial}{\partial v}. \tag{6}$$

This vector field must be creating a symmetry of (4) iff the following is true in invariance conditions:

$$Pr^{(\alpha,2)}V(\Delta_1)\Big|_{\Delta_1=0} = 0,$$
$$Pr^{(\alpha,2)}V(\Delta_2)\Big|_{\Delta_2=0} = 0. \tag{7}$$

It is noteworthy that the extended operator of fractional prolongation $Pr^{(\alpha,2)}V$ is

$$Pr^{(\alpha,2)}V = V + \eta^{\alpha,t}\frac{\partial}{\partial(\partial_t^\alpha u)} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}},$$
$$Pr^{(\alpha,2)}V = V + \phi^{\alpha,t}\frac{\partial}{\partial(\partial_t^\alpha v)} + \phi^x\frac{\partial}{\partial v_x} + \phi^{xx}\frac{\partial}{\partial v_{xx}}, \tag{8}$$

where $\eta^{j,x}$ and $\phi^{j,x}$ represent integer-order extended infinitesimals which are defined as

$$\eta^{j,x} = D_x\eta^{j-1,x} - (D_x\xi)\frac{\partial^j u}{\partial x^j} - (D_x\tau)\frac{\partial}{\partial t}\left(\frac{\partial^{j-1}u}{\partial x^{j-1}}\right),$$
$$\phi^{j,x} = D_x\phi^{j-1,x} - (D_x\xi)\frac{\partial^j v}{\partial x^j} - (D_x\tau)\frac{\partial}{\partial t}\left(\frac{\partial^{j-1}v}{\partial x^{j-1}}\right), \quad (9)$$

and the operator of total derivative along x, i.e., D_x is

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + \cdots . \quad (10)$$

Also, the α-order extended infinitesimal operator $\eta^{\alpha,t}$ is

$$\eta^{\alpha,t} = D_t^\alpha\eta + \xi_1 D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha\,(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u),$$

where D_t^α represents the total α-order fractional derivative in the course of time. Applying the Leibnitz formula and the chain rule [2,29], the α-order extension of infinitesimal $\eta^{\alpha,t}$ can be obtained explicitly as

$$\eta^{\alpha,t} = \frac{\partial^\alpha\eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha\eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) + \mu_1 + \mu_2, \quad (11)$$

where

$$\mu_1 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{r=2}^{m}\sum_{s=0}^{r-1}\binom{\alpha}{n}\binom{n}{m}\binom{r}{s} \times \frac{t^{n-\alpha}}{r!\Gamma(n-\alpha+1)} \times (-1)^s u^s\frac{\partial^m(u^{r-s})}{\partial t^m}\frac{\partial^{n-m+r}\eta}{\partial t^{n-m}\partial u^r},$$
$$\mu_2 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{r=2}^{m}\sum_{s=0}^{r-1}\binom{\alpha}{n}\binom{n}{m}\binom{r}{s} \times \frac{t^{n-\alpha}}{r!\Gamma(n-\alpha+1)} \times (-1)^s v^s\frac{\partial^m(v^{r-s})}{\partial t^m}\frac{\partial^{n-m+r}\phi}{\partial t^{n-m}\partial v^r}.$$

In the same manner, $\phi^{\alpha,t}$ can be obtained as follows:

$$\phi^{\alpha,t} = \frac{\partial^\alpha\phi}{\partial t^\alpha} + (\phi_v - \alpha D_t(\tau))\frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha\phi_v}{\partial t^\alpha} + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n\phi_v}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(v) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(v_x) + \lambda_1 + \lambda_2, \quad (12)$$

where

$$\lambda_1 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{r=2}^{m}\sum_{s=0}^{r-1}\binom{\alpha}{n}\binom{n}{m}\binom{r}{s} \times \frac{t^{n-\alpha}}{r!\Gamma(n-\alpha+1)} \times (-1)^s u^s\frac{\partial^m(u^{r-s})}{\partial t^m}\frac{\partial^{n-m+r}\eta}{\partial t^{n-m}\partial u^r},$$
$$\lambda_2 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{r=2}^{m}\sum_{s=0}^{r-1}\binom{\alpha}{n}\binom{n}{m}\binom{r}{s} \times \frac{t^{n-\alpha}}{r!\Gamma(n-\alpha+1)} \times (-1)^s v^s\frac{\partial^m(v^{r-s})}{\partial t^m}\frac{\partial^{n-m+r}\phi}{\partial t^{n-m}\partial v^r}.$$

Furthermore, since η and ϕ are linear with respect to variable u and v, then derivatives $\partial^n\eta/\partial u^n$ and $\partial^n\phi/\partial v^n$ for $n \geq 2$ are vanished and we immediately conclude that $\mu_1 = \mu_2 = \lambda_1 = \lambda_2 = 0$.

2.1 *Classical symmetries*

The invariance condition under Lie group transformations for eq. (4) is

$$Pr^{(\alpha,2)}V\left(\frac{\partial^\alpha u}{\partial t^\alpha} + av_{xx} + b(u^2+v^2)^2 v + 2cuvu_x + 2cv^2v_x\right)\Big|_{(3)} = 0,$$
$$Pr^{(\alpha,2)}V\left(\frac{\partial^\alpha v}{\partial t^\alpha} + au_{xx} + b(u^2+v^2)^2 u + 2cuvv_x + 2cu^2u_x\right)\Big|_{(3)} = 0.$$

According to the above, the following vector fields are obtained:

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = 2\alpha x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u} - \alpha v\frac{\partial}{\partial v}. \quad (13)$$

Case 1: The corresponding characteristic equations for vector field V_1 is

$$\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{0}=\frac{\mathrm{d}v}{0}$$

and invariant solutions for this vector field are $u(x,t)=f(t)$ and $v(x,t)=g(t)$. Then, eq. (3) is reduced to the following fractional ordinary differential equation:

$$\begin{aligned}&\frac{\partial^\alpha}{\partial t^\alpha}f+b(f^2+g^2)^2g=0,\\&-\frac{\partial^\alpha}{\partial t^\alpha}g+b(f^2+g^2)^2f=0.\end{aligned}\tag{14}$$

If we assume that $f(t)=ig(t)$, one of the solutions of the above system will be $f(t)=g(t)=c_1t^{\alpha-1}$.

Case 2: The corresponding characteristic equations for vector field V_2 is

$$\frac{\mathrm{d}x}{2\alpha x}=\frac{\mathrm{d}t}{4t}=\frac{\mathrm{d}u}{-\alpha u}=\frac{\mathrm{d}v}{-\alpha v}.$$

Hence, invariant solutions of vector field V_2 are

$$\begin{cases}u(x,t)=t^{-\frac{\alpha}{4}}F(\zeta),\\ v(x,t)=t^{-\frac{\alpha}{4}}G(\zeta),\quad \zeta=xt^{-\frac{\alpha}{2}}.\end{cases}\tag{15}$$

Theorem 1. *Transformation* (15) *reduce the governing equation* (3) *to the following FODEs:*

$$\begin{cases}\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}F\right)(\zeta)+aG''+b(F^2+G^2)^2G\\ \quad +2cFGF'+2cG^2G'=0,\\[2ex] \left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}G\right)(\zeta)-aF''-b(F^2+G^2)^2F\\ \quad -2cFGF'-2cF^2F'=0.\end{cases}\tag{16}$$

It must be noted that $\mathcal{P}_{\beta}^{\tau,\alpha}\mathcal{F}$ *shows the Erdélyi–Kober fractional derivative operator which is defined as* [30]

$$\left(\mathcal{P}_{\beta}^{\tau,\alpha}\mathcal{F}\right):=\prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\beta}\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)\left(\mathcal{K}_{\beta}^{\tau+\alpha,n-\alpha}\mathcal{F}\right)(\zeta),$$

$$n=\begin{cases}[\alpha]+1, & \alpha\notin\mathbb{N},\\ \alpha, & \alpha\in\mathbb{N},\end{cases}$$

where

$$\left(\mathcal{K}_{\beta}^{\tau,\alpha}\mathcal{F}\right):=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^\infty(s-1)^{\alpha-1}s^{-(\tau+\alpha)}\mathcal{F}\left(\zeta s^{\frac{1}{\beta}}\right)\mathrm{d}s, & \alpha>0,\\ \mathcal{F}(\zeta), & \alpha=0,\end{cases}$$

denotes the Erdélyi–Kober fractional integral operators for $\mathcal{F}(\zeta)$.

Proof. Let $\alpha\in(n-1,n)$, $n\in\mathbb{N}$. The Rieman–Lioville fractional derivative of eq. (15) can be written as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}s^{-\frac{\alpha}{4}}F(xs^{-\frac{\alpha}{2}})\,\mathrm{d}s\right].\tag{17}$$

Let

$$w=\frac{t}{s}.$$

Then

$$\mathrm{d}s=\frac{-t}{w^2}\mathrm{d}w.$$

Equation (17) becomes

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\frac{5\alpha}{4}}}{\Gamma(n-\alpha)}\times\int_1^\infty(w-1)^{n-\alpha-1}w^{\frac{5\alpha}{4}-n-1}F(xt^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}})\,\mathrm{d}w\right]\\ \frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}F\right)(\zeta)\right].\end{aligned}$$

Furthermore, consider $\zeta=xt^{-\alpha/2}$, $\rho\in(0,\infty)$. We have

$$t\frac{\partial}{\partial t}\rho(\zeta)=t\frac{\partial\zeta}{\partial t}\frac{\mathrm{d}\rho(\zeta)}{\mathrm{d}\zeta}=-\frac{\alpha}{2}\zeta\frac{\mathrm{d}\rho(\zeta)}{\mathrm{d}\zeta}.$$

Hence,

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}F\right)(\zeta)\right]\\ &=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{5\alpha}{4}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}F\right)(\zeta)\right)\right]\\ &=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{5\alpha}{4}-1}\left(n-\frac{5\alpha}{4}-\frac{\alpha}{2}\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)\times\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}F\right)(\zeta)\right]\\ &\ \ \vdots\\ &=t^{-\frac{5\alpha}{4}}\Pi_{j=0}^{n-1}\left(1-\frac{5\alpha}{4}+j-\frac{\alpha}{2}\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)\times\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}F\right)(\zeta)\\ &=t^{-\frac{5\alpha}{4}}\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}F\right)(\zeta).\end{aligned}$$

Thus,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\frac{5\alpha}{4}}\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{5\alpha}{4},\alpha} F\right)(\zeta).$$

Similarly, we have

$$\frac{\partial^\alpha v}{\partial t^\alpha} = t^{-\frac{5\alpha}{4}}\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{5\alpha}{4},\alpha} G\right)(\zeta).$$

So, the proof is completed. □

To find one of the solutions of eq. (16), assuming that $G(\zeta) = iF(\zeta)$, then eq. (16) can be written as follows:

$$\begin{cases} \left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{5\alpha}{4},\alpha} F\right)(\zeta) + aG'' = 0, \\ \left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{5\alpha}{4},\alpha} G\right)(\zeta) - aF'' = 0. \end{cases} \tag{18}$$

In order to obtain the exact solution of eq. (18) by using power series method we assume that,

$$\begin{aligned} F(\zeta) &= \sum_{i=0}^{\infty} a_i \zeta^i, \\ G(\zeta) &= \sum_{i=0}^{\infty} b_i \zeta^i, \\ F''(\zeta) &= \sum_{i=0}^{\infty} (i+2)(i+1) a_{i+2} \zeta^i, \\ G''(\zeta) &= \sum_{i=0}^{\infty} (i+2)(i+1) b_{i+2} \zeta^i, \\ \left(\mathcal{P}_{\beta}^{\tau,\alpha} F\right)(\zeta) &= \sum_{i=0}^{\infty} a_i \frac{\Gamma(\tau - \frac{i}{\beta} + 1)}{\Gamma(\tau - \frac{i}{\beta} + 1 - \alpha)} \zeta^i, \\ \left(\mathcal{P}_{\beta}^{\tau,\alpha} G\right)(\zeta) &= \sum_{i=0}^{\infty} b_i \frac{\Gamma(\tau - \frac{i}{\beta} + 1)}{\Gamma(\tau - \frac{i}{\beta} + 1 - \alpha)} \zeta^i. \end{aligned} \tag{19}$$

Substituting eq. (19) in (18) yields

$$\begin{aligned} &\sum_{i=0}^{\infty} a_i \frac{\Gamma(2 - \frac{5\alpha}{4} - \frac{\alpha i}{2})}{\Gamma(2 - \frac{9\alpha}{4} - \frac{\alpha i}{2})} \zeta^i \\ &\quad + a \sum_{i=0}^{\infty} (i+2)(i+1) b_{i+2} \zeta^i = 0, \\ &\sum_{i=0}^{\infty} b_i \frac{\Gamma(2 - \frac{5\alpha}{4} - \frac{\alpha i}{2})}{\Gamma(2 - \frac{9\alpha}{4} - \frac{\alpha i}{2})} \zeta^i \\ &\quad - a \sum_{i=0}^{\infty} (i+2)(i+1) a_{i+2} \zeta^i = 0. \end{aligned} \tag{20}$$

Comparing coefficients in eq. (20), for $i = 0$ we get

$$b_2 = -\frac{\Gamma(2 - \frac{5\alpha}{4})}{2a\Gamma(2 - \frac{9\alpha}{4})} a_0, \qquad a_2 = \frac{\Gamma(2 - \frac{5\alpha}{4})}{2a\Gamma(2 - \frac{9\alpha}{4})} b_0.$$

Hence, a_i and b_i for $i \geq 1$ can be obtained easily from the following recursive relation:

$$b_{i+2} = -\frac{\Gamma(2 - \frac{5\alpha}{4} - \frac{\alpha i}{2})}{a(i+2)(i+1)\Gamma(2 - \frac{9\alpha}{4} - \frac{\alpha i}{2})} a_i,$$

$$a_{i+2} = \frac{\Gamma(2 - \frac{5\alpha}{4} - \frac{\alpha i}{2})}{a(i+2)(i+1)\Gamma(2 - \frac{9\alpha}{4} - \frac{\alpha i}{2})} b_i,$$

where a_0, a_1 and b_0, b_1 are real arbitrary constants. Thus,

$$\begin{aligned} F(\zeta) &= a_0 + a_1\zeta + a_2\zeta^2 + \sum_{i=1}^{\infty} a_{i+2}\zeta^{i+2} \\ &= a_0 + a_1\zeta + \frac{\Gamma(2 - \frac{5\alpha}{4})}{2a\Gamma(2 - \frac{9\alpha}{4})} b_0 \zeta^2 \\ &\quad + \sum_{i=1}^{\infty} \frac{\Gamma(2 - \frac{5\alpha}{4} - \frac{\alpha i}{2})}{a(i+2)(i+1)\Gamma(2 - \frac{9\alpha}{4} - \frac{\alpha i}{2})} b_i \end{aligned}$$

and

$$\begin{aligned} G(\zeta) &= b_0 + b_1\zeta + b_2\zeta^2 + \sum_{i=1}^{\infty} b_{i+2}\zeta^{i+2} \\ &= b_0 + b_1\zeta - \frac{\Gamma(2 - \frac{5\alpha}{4})}{2a\Gamma(2 - \frac{9\alpha}{4})} a_0 \zeta^2 \\ &\quad - \sum_{i=1}^{\infty} \frac{\Gamma(2 - \frac{5\alpha}{4} - \frac{\alpha i}{2})}{a(i+2)(i+1)\Gamma(2 - \frac{9\alpha}{4} - \frac{\alpha i}{2})} a_i. \end{aligned}$$

Substituting the evaluated functions $F(\zeta)$ and $G(\zeta)$ with $\zeta = xt^{-\frac{\alpha}{2}}$ in eq. (15) yields $u(x,t)$ and $v(x,t)$. Also, from $q(x,t) = u(x,t) + iv(x,t)$ the exact solution of the main problem (1) can be obtained as follows:

$$q(x,t) = t^{-\frac{\alpha}{4}} F(xt^{-\frac{\alpha}{2}}) + it^{-\frac{\alpha}{4}} G(xt^{-\frac{\alpha}{2}}).$$

In order to show the convergence behaviour of the obtained exact solution at $t = 1$ and $a = 1$, we increase the value of α from 0.85 to 1 in figure 1. In this figure, we show the real part, imaginary part and $\mathcal{L}_2$ norm of $q(x, 1)$ for different values of α.

2.2 *Non-classical symmetries*

To investigate the non-classical state of the Lie group, besides $Pr^{(\alpha,2)}(V) = 0$, we must also have invariant surface conditions as follows:

$$\Omega_1 \equiv \xi(x,t,u,v)u_x + \tau(x,t,u,v)u_t - \eta = 0,$$
$$\Omega_2 \equiv \xi(x,t,u,v)v_x + \tau(x,t,u,v)v_t - \phi = 0.$$

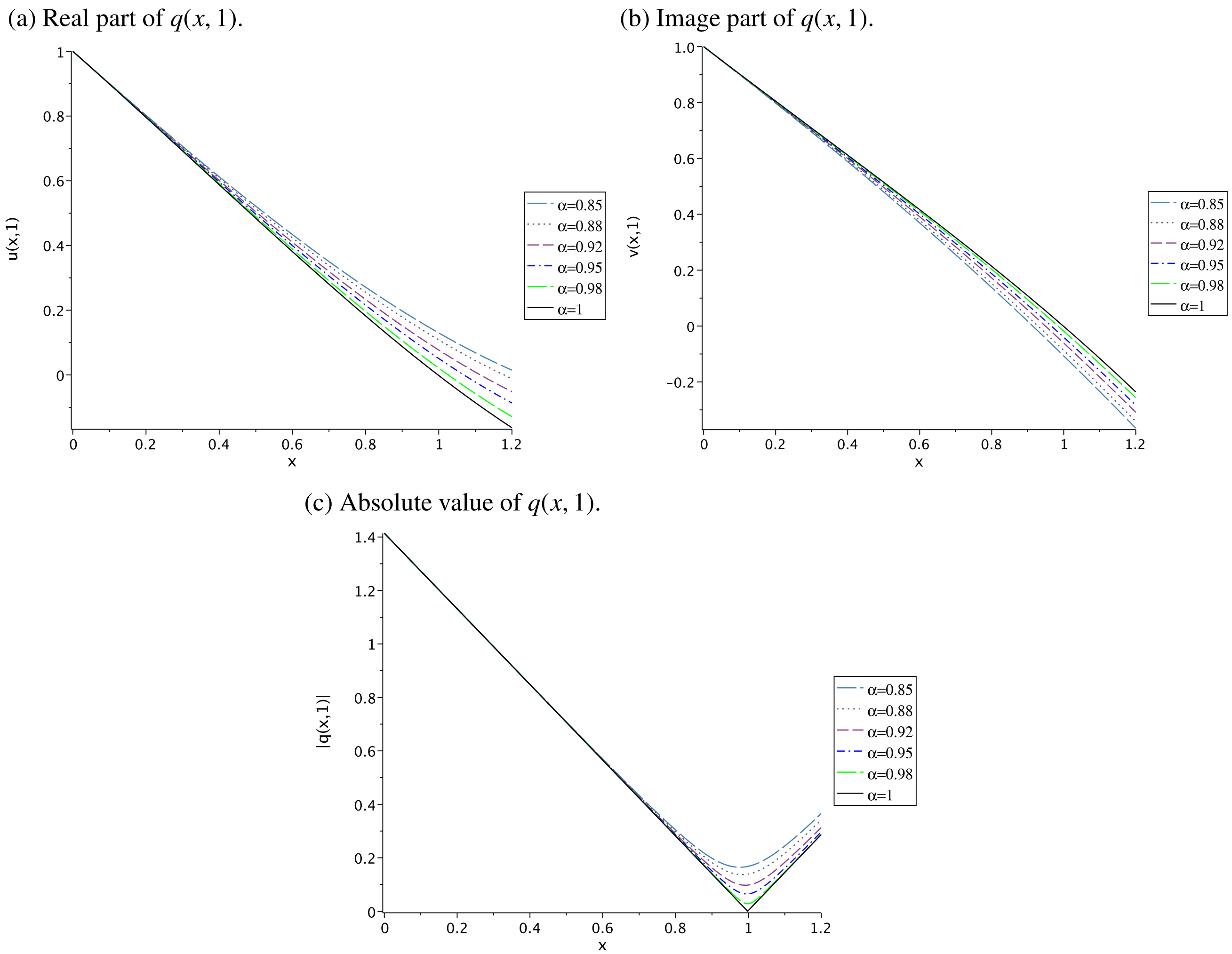

Figure 1. The related plots of the exact solution of (1) which are obtained by the classical generator of V_2 for different values of α at $t = 1$.

There is no loss of generality in assuming that $\xi = 1$, $\tau = 0$. Therefore, invariant surface conditions can be obtained as follows:

$$u_x = \eta, \quad v_x = \phi.$$

Then

$$u_{xx} = \eta_x + \eta\eta_u + \phi\eta_v,$$
$$v_{xx} = \phi_x + \eta\phi_u + \phi\phi_v.$$

Substituting the above relations into relation (3) gives $\phi = -u$ and $\eta = v$, and so we have the infinitesimal symmetry

$$V_3 = \frac{\partial}{\partial x} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}. \tag{21}$$

This vector field gives us the following invariant solutions:

$$u(x, t) = f(t)\sin(x) + g(t)\cos(x),$$
$$v(x, t) = f(t)\cos(x) - g(t)\sin(x). \tag{22}$$

So, the reduced system by using the above relations is

$$\begin{cases} \dfrac{\partial^\alpha f(t)}{\partial t^\alpha} + ag(t) - b(f(t)^2 + g(t)^2)^2 g(t) = 0, \\ \dfrac{\partial^\alpha g(t)}{\partial t^\alpha} - af(t) + b(f(t)^2 + g(t)^2)^2 f(t) = 0. \end{cases} \tag{23}$$

To find an exact solution for eq. (23), suppose that $g(t) = if(t)$. Therefore, it suffices to obtain the solution of the following equation:

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} + aif(t) = 0. \tag{24}$$

Applying the fractional Laplace transform [37,38] to solve eq. (24) yields

$$f(t) = t^{2\alpha-2}E_{\alpha,2\alpha-1}\left(-ait^\alpha\right), \tag{25}$$

and $g(t) = if(t)$. Substituting eq. (25) into eq. (22), exact solutions of eq. (3) are obtained.

$$u(x,t) = t^{2\alpha-2}E_{\alpha,2\alpha-1}\left(-ait^{\alpha}\right)\sin(x) + it^{2\alpha-2}E_{\alpha,2\alpha-1}\left(-ait^{\alpha}\right)\cos(x),$$
$$v(x,t) = t^{2\alpha-2}E_{\alpha,2\alpha-1}\left(-ait^{\alpha}\right)\cos(x) - it^{2\alpha-2}E_{\alpha,2\alpha-1}\left(-ait^{\alpha}\right)\sin(x).$$

Then

$$q(x,t) = 2t^{2\alpha-2}\Bigg[\sin(x)\left(\sum_{k=0}^{\infty}\frac{(-1)^k(at^{\alpha})^{2k}}{\Gamma(2k\alpha+2\alpha-1)}\right) - \cos(x)\left(\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(at^{\alpha})^{2k+1}}{\Gamma(2k\alpha+3\alpha-1)}\right)\Bigg] + i2t^{\alpha-2}\Bigg[\sin(x)\left(\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(at^{\alpha})^{2k+1}}{\Gamma(2k\alpha+3\alpha-1)}\right) + \cos(x)\left(\sum_{k=0}^{\infty}\frac{(-1)^k(at^{\alpha})^{2k}}{\Gamma(2k\alpha+2\alpha-1)}\right)\Bigg].$$

The plots of imaginary part, real part and $\mathcal{L}_2$ norm of the above-mentioned function $q(x,t)$ for $\alpha = 0.5$ and $a = 1$ are illustrated in figure 2.

For the other case when we have $\xi = 1, \tau \neq 0$, assume that $\tau_u = \tau_v = 0$. So the invariant surface conditions are as follows:

$$u_x = \eta - \tau u_t, \quad v_x = \phi - \tau v_t,$$

and the corresponding generator is

$$V_4 = \frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial t}.$$

With this vector field, the following order can be reduced:

$$u(x,t) = f(\zeta), \quad v(x,t) = g(\zeta), \qquad \zeta = t - \lambda x.$$

Hence, eq. (3) is reduced to the following system:

$$\begin{cases} \frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}f + a\lambda^2 g_{\zeta\zeta} + b(f^2+g^2)^2 g \\ -2c\lambda f g f_{\zeta} - 2c\lambda g^2 g_{\zeta} = 0, \\ -\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}g + a\lambda^2 f_{\zeta\zeta} + b(f^2+g^2)^2 f \\ -2c\lambda f g g_{\zeta} - 2c\lambda f^2 f_{\zeta} = 0. \end{cases} \tag{26}$$

If we assume that $g(\zeta) = if(\zeta)$, then eq. (26) can be rewritten as follows:

$$\frac{\partial^{\alpha} f}{\partial\zeta^{\alpha}} + aif_{\zeta\zeta} = 0. \tag{27}$$

Applying the fractional Laplace transform to solve eq. (27) yields

$$f(\zeta) = \frac{-k_1}{a\lambda^2 i}\zeta^{\alpha}E_{2-\alpha,\alpha+1}\left(\frac{1}{a\lambda^2 i}\zeta^{2-\alpha}\right) + k_2E_{2-\alpha,1}\left(\frac{1}{a\lambda^2 i}\zeta^{2-\alpha}\right) + k_3\zeta E_{2-\alpha,2}\left(\frac{1}{a\lambda^2 i}\zeta^{2-\alpha}\right),$$

where

$$k_1 = \frac{k_2}{\Gamma(1-\alpha)}$$

and k_2, k_3 are constants, and $g(\zeta) = if(\zeta)$.

Then,

$$q(x,t) = \frac{-k_1}{a\lambda^2 i}(t-\lambda x)^{\alpha}E_{2-\alpha,\alpha+1}\times\left(\frac{1}{a\lambda^2 i}(t-\lambda x)^{2-\alpha}\right) + k_2E_{2-\alpha,1}\left(\frac{1}{a\lambda^2 i}(t-\lambda x)^{2-\alpha}\right) + k_3(t-\lambda x)E_{2-\alpha,2}\left(\frac{1}{a\lambda^2 i}(t-\lambda x)^{2-\alpha}\right) + i\Bigg[\frac{-k_1}{a\lambda^2 i}(t-\lambda x)^{\alpha}E_{2-\alpha,\alpha+1}\times\left(\frac{1}{a\lambda^2 i}(t-\lambda x)^{2-\alpha}\right) + k_2E_{2-\alpha,1}\left(\frac{1}{a\lambda^2 i}(t-\lambda x)^{2-\alpha}\right) + k_3(t-\lambda x)E_{2-\alpha,2}\left(\frac{1}{a\lambda^2 i}(t-\lambda x)^{2-\alpha}\right)\Bigg].$$

Therefore, by separating the real and imaginary parts of $q(x,t)$, we have the following equivalence form:

$$q(x,t) = -\frac{k_1}{a\lambda^2}(t-\lambda x)^{\alpha}\times\sum_{k=0}^{\infty}\left[\frac{(-1)^{k+1}(t-\lambda x)^{(2k+1)(2-\alpha)}}{(a\lambda^2)^{2k+1}\Gamma(4k-2k\alpha+3)} + \frac{(-1)^k(t-\lambda x)^{2k(2-\alpha)}}{(a\lambda^2)^{2k}\Gamma(4k-2k\alpha+\alpha+1)}\right] + k_2\sum_{k=0}^{\infty}\left[\frac{(-1)^k(t-\lambda x)^{2k(2-\alpha)}}{(a\lambda^2)^{2k}\Gamma(4k-2k\alpha+1)} - \frac{(-1)^{k+1}(t-\lambda x)^{(2k+1)(2-\alpha)}}{(a\lambda^2)^{2k+1}\Gamma(4k-2k\alpha-\alpha+3)}\right]$$

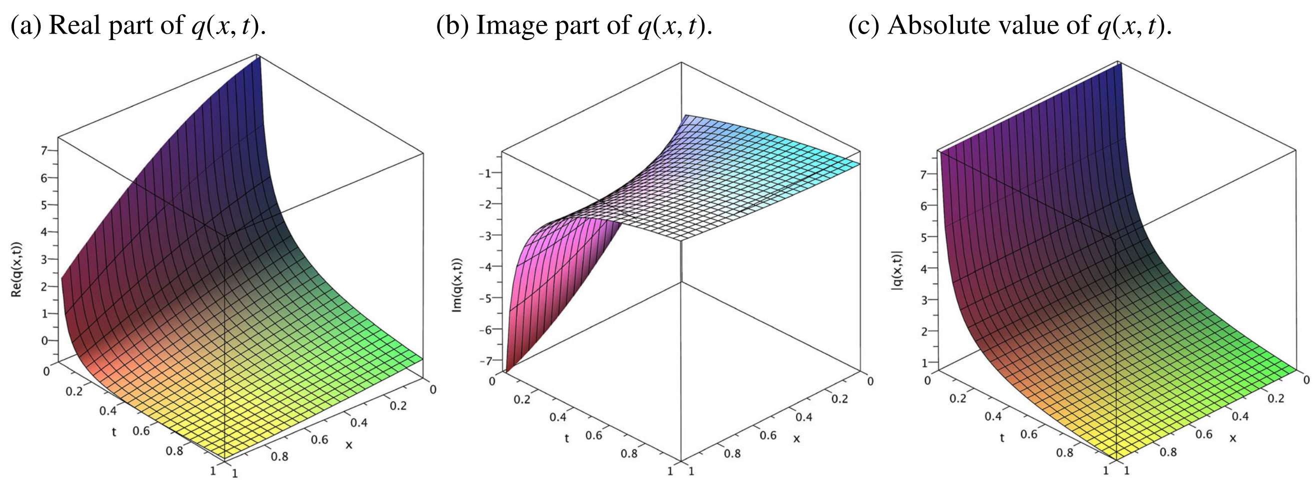

Figure 2. The related plots of the exact solution of (1) which are obtained by the non-classical generator of V_3 for $\alpha = 0.5$.

$$+k_3(t-\lambda x)\sum_{k=0}^{\infty}\left[\frac{(-1)^k(t-\lambda x)^{2k(2-\alpha)}}{(a\lambda^2)^{2k}\Gamma(4k-2k\alpha+2)} - \frac{(-1)^{k+1}(t-\lambda x)^{(2k+1)(2-\alpha)}}{(a\lambda^2)^{2k+1}\Gamma(4k-2k\alpha-\alpha+4)}\right]$$
$$+i\left(\frac{k_1}{a\lambda^2}(t-\lambda x)^{\alpha} \times \sum_{k=0}^{\infty}\left[\frac{(-1)^k(t-\lambda x)^{2k(2-\alpha)}}{(a\lambda^2)^{2k}\Gamma(4k-2k\alpha+\alpha+1)} - \frac{(-1)^{k+1}(t-\lambda x)^{(2k+1)(2-\alpha)}}{(a\lambda^2)^{2k+1}\Gamma(4k-2k\alpha+3)}\right]\right.$$
$$+k_2\sum_{k=0}^{\infty}\left[\frac{(-1)^{k+1}(t-\lambda x)^{(2k+1)(2-\alpha)}}{(a\lambda^2)^{2k+1}\Gamma(4k-2k\alpha-\alpha+3)} + \frac{(-1)^k(t-\lambda x)^{2k(2-\alpha)}}{(a\lambda^2)^{2k}\Gamma(4k-2k\alpha+1)}\right] + k_3(t-\lambda x)$$
$$\left.\times\sum_{k=0}^{\infty}\left[\frac{(-1)^{k+1}(t-\lambda x)^{(2k+1)(2-\alpha)}}{(a\lambda^2)^{2k+1}\Gamma(4k-2k\alpha-\alpha+4)} + \frac{(-1)^k(t-\lambda x)^{2k(2-\alpha)}}{(a\lambda^2)^{2k}\Gamma(4k-2k\alpha+2)}\right]\right).$$

In order to show the convergence behaviour of the obtained exact solution at $t = 0.5$ and $(a, k_2, k_3, \lambda) = (1, 0.2, 0.1, 0.25)$, we increase the value of α from 0.90 to 1 in figure 3. In this figure, we show the real part, imaginary part and $\mathcal{L}_2$ norm of $q(x, 0.5)$ for different values of α.

3. Conservation laws

A new conservation laws theorem is first presented by Ibragimov in [31]. This theorem uses Lie symmetries to find conserved vectors. Actually, this new theorem is applied to extract conservation laws for differential equations without Lagrangian. Then, this new theorem is developed into fractional differential equations [32–36]. To the best of the authors' knowledge, only classical symmetries are used to find conservation laws. Here, we use the vector fields obtained from the classical and non-classical types of the Lie symmetry group. First, we give an overview of this method. For the integer-order PDEs, the conserved vector $\mathcal{T} = (\mathcal{T}^t, \mathcal{T}^x)$ of (3) satisfies the following conservation equation:

$$D_t(\mathcal{T}^t) + D_x(\mathcal{T}^x)|_{(3)} = 0,$$

where $\mathcal{T}^t$ and $\mathcal{T}^x$ are time and space flow, respectively. The Lagrangian for (3) is given by

$$\mathcal{H} = \psi(x,t)\left(\frac{\partial^\alpha}{\partial t^\alpha}u + av_{xx} + b(u^2+v^2)^2 v + 2cuvu_x + 2cv^2v_x\right) + \phi(x,t)\left(\frac{\partial^\alpha}{\partial t^\alpha}v - au_{xx} - b(u^2+v^2)^2 u - 2cuvv_x - 2cu^2u_x\right). \tag{28}$$

The adjoint equations of the fractional KE equation can be specified as follows:

$$\begin{cases} \mathcal{F}_1^* \equiv \dfrac{\delta\mathcal{H}}{\delta u} = 0, \\ \mathcal{F}_2^* \equiv \dfrac{\delta\mathcal{H}}{\delta v} = 0, \end{cases} \tag{29}$$

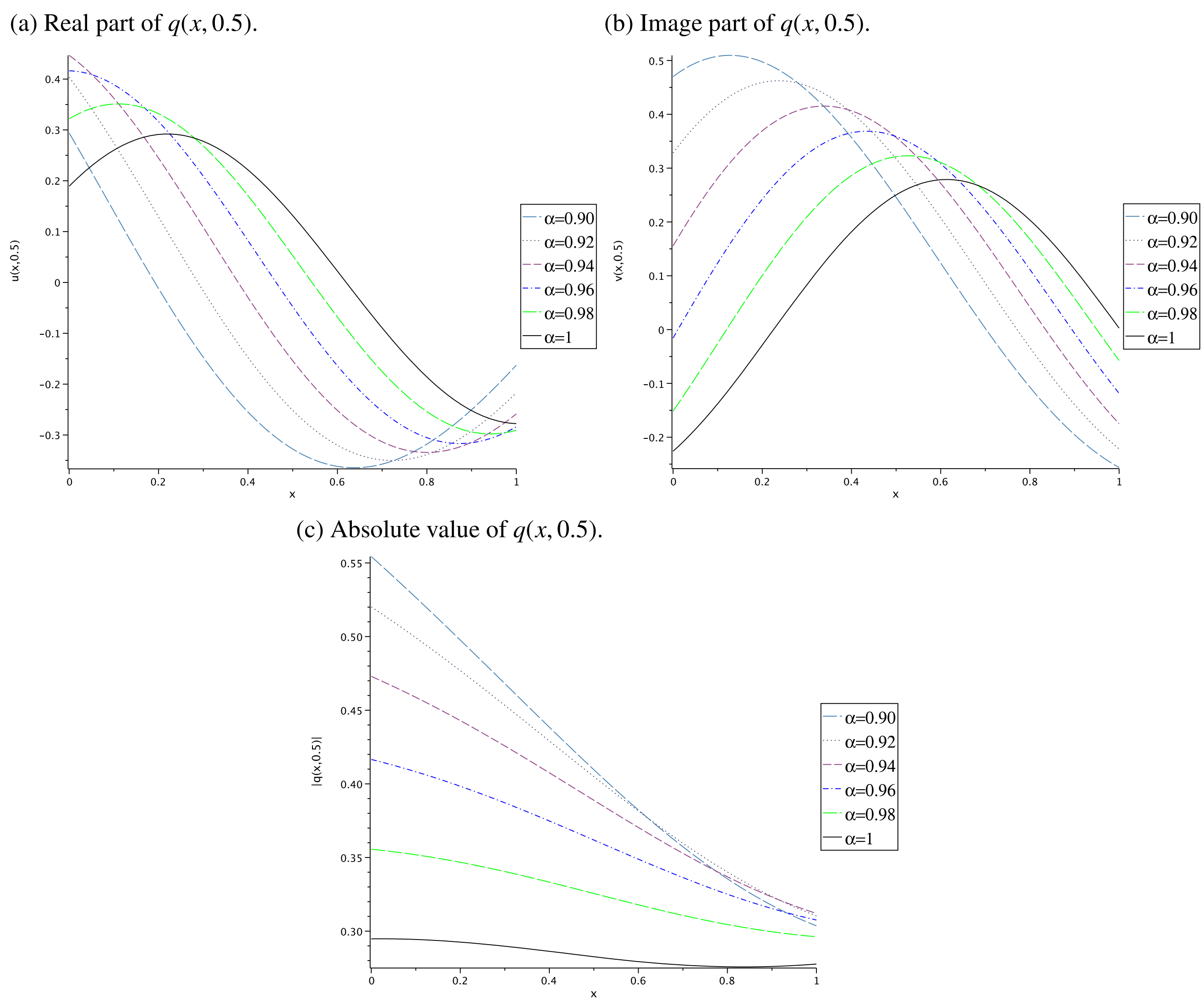

Figure 3. The related plots of the exact solution of (1) which are obtained by the classical generator of V_4 for different values of α at $t = 0.5$.

where the operators of Euler–Lagrange are given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (\partial_t^\alpha)^* \frac{\partial}{\partial(\partial_t^\alpha u)} + \sum_{k\geq 1}(-1)^k D_x \ldots D_x \frac{\partial}{\partial u_{kx}}$$

and

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} + (\partial_t^\alpha)^* \frac{\partial}{\partial(\partial_t^\alpha v)} + \sum_{k\geq 1}(-1)^k D_x \ldots D_x \frac{\partial}{\partial v_{kx}}.$$

$(\partial_t^\alpha)^*$ denotes the adjoint operator for ∂_t^α and D_x is the total derivative operator. According to the Riemann–Liouville fractional differential operator

$$(\partial_t^\alpha)^* = (-1)^n \mathcal{I}_T^{n-\alpha}(\partial_t^n) = (\partial_T^\alpha)_t^{\mathcal{C}},$$

$$\mathcal{I}_T^{n-\alpha} h(t, x) = \int_t^T \frac{h(\tau, x)(\tau - t)^{n-(1+\alpha)}}{\Gamma(n-\alpha)} \mathrm{d}\tau,$$

$$n = [\alpha] + 1,$$

where $(\partial_T^\alpha)_t^{\mathcal{C}}$ is the operator of the right-sided Caputo derivative. By introducing new variables $\psi(x, t) = \Psi(x, t, u, v)$ and $\phi(x, t) = \Theta(x, t, u, v)$ and substituting (28) into (29) we have the adjoint equations of (3) as

$$\begin{cases} \mathcal{F}_1^* = & (\partial_t^\alpha)^*\Psi + \Psi\big(4buv(u^2+v^2) \\ & -\Theta\left(b(u^2+v^2)^2+4bu^2(u^2+v^2)+2cvv_x\right) \\ & -2cuv_x\big) - 2cuv\Psi_x + 2cu^2\Theta_x - a\Theta_{xx}, \\ \mathcal{F}_2^* = & (\partial_t^\alpha)^*\Theta + \Psi\big(b(u^2+v^2)^2 + 4bv^2(u^2+v^2) \\ & +2cuu_x\big) - \Theta\left(4buv(u^2+v^2) - 2cvu_x\right) \\ & -2cv^2\Psi_x + 2cuv\Theta_x + a\Psi_{xx}. \end{cases}$$

The time-fractional system (3) is called nonlinearly self-adjoint if

$$\begin{cases} \mathcal{F}_1^* = \mu_1 F_1 + \mu_2 F_2, \\ \mathcal{F}_2^* = \mu_3 F_1 + \mu_4 F_2, \end{cases}$$

where μ_i for $i = 1, \ldots, 4$, are unknowns to be determined. In view of [31], for eq. (3) we have the following conservation law:

$$D_t \mathcal{T}_i^t + D_x \mathcal{T}_i^x = 0, \tag{30}$$

where the conserved vectors $\mathcal{T}_i = (\mathcal{T}_t^t, \mathcal{T}_i^x)$ have the following form:

$$\begin{aligned} \mathcal{T}_i^x &= \left({}^u\mathcal{W}_i \frac{\partial \mathcal{H}}{\partial u_x} + \sum_{k\geq 1} D_x \ldots D_x({}^u\mathcal{W}_i)\frac{\partial \mathcal{H}}{\partial u_{(k+1)x}}\right) \\ &\quad + \left({}^v\mathcal{W}_i \frac{\partial \mathcal{H}}{\partial v_x} + \sum_{k\geq 1} D_x \ldots D_x({}^v\mathcal{W}_i)\frac{\partial \mathcal{H}}{\partial v_{(k+1)x}}\right), \\ \mathcal{T}_i^t &= \sum_{k=0}^{n-1}(-1)^k\left[\partial_t^{\alpha-1-k}({}^u\mathcal{W}_i)D_t^k\left(\frac{\partial \mathcal{H}}{\partial(\partial_t^\alpha u)}\right)\right. \\ &\quad \left. +\partial_t^{\alpha-1-k}({}^v\mathcal{W}_i)D_t^k\left(\frac{\partial \mathcal{H}}{\partial(\partial_t^\alpha v)}\right)\right] \\ &\quad -(-1)^n\left[\mathcal{J}\left({}^u\mathcal{W}_i, D_t^n\left(\frac{\partial \mathcal{H}}{\partial(\partial_t^\alpha u)}\right)\right)\right. \\ &\quad \left. +\mathcal{J}\left({}^v\mathcal{W}_i, D_t^n(\frac{\partial \mathcal{H}}{\partial(\partial_t^\alpha v)})\right)\right], \\ n &= [\alpha]+1, \end{aligned} \tag{31}$$

in which

$$\begin{aligned} {}^u\mathcal{W}_i &= \eta_{1_i} - \xi_{1_i} u_x - \tau_{1_i} u_t, \\ {}^v\mathcal{W}_i &= \phi_{1_i} - \xi_{1_i} v_x - \tau_{1_i} v_t \end{aligned}$$

and the integral $\mathcal{J}$ is given by

$$\begin{aligned} &\mathcal{J}(h,g) \\ &= \int_0^t \int_t^T \frac{h(\lambda,x)g(\mu,x)(\mu-\lambda)^{n-(\alpha+1)}}{\Gamma(n-\alpha)} \mathrm{d}\lambda\, \mathrm{d}\mu. \end{aligned} \tag{32}$$

According to the above analysis and Lie symmetry generator, we consider the conserved vectors for classical and non-classical generators of fractional KE equation. We have the following modes for classical generators.

Case 1: For the generator $V_1 = \partial/\partial x$, the respective Lie characteristic function is

$$ {}^u\mathcal{W}_1 = -u_x, \qquad {}^v\mathcal{W}_1 = -v_x. \tag{33}$$

Substituting (37) into (31) yields the conserved vector as follows:

$$\begin{aligned} \mathcal{T}_1^x &= {}^u\mathcal{W}_1\left(\frac{\partial \mathcal{H}}{\partial u_x} - D_x \frac{\partial \mathcal{H}}{\partial u_{xx}}\right) \\ &\quad + D_x({}^u\mathcal{W}_1)\frac{\partial \mathcal{H}}{\partial u_{xx}} + {}^v\mathcal{W}_1\left(\frac{\partial \mathcal{H}}{\partial v_x} - D_x\frac{\partial \mathcal{H}}{\partial v_{xx}}\right) \\ &\quad + D_x({}^v\mathcal{W}_1)\frac{\partial \mathcal{H}}{\partial v_{xx}}, \\ \mathcal{T}_1^t &= -\mathcal{I}^{1-\alpha}({}^u\mathcal{W}_1)\Psi + \mathcal{J}({}^u\mathcal{W}_1, \Psi_t) \\ &\quad -\mathcal{I}^{1-\alpha}({}^v\mathcal{W}_1)\Theta + \mathcal{J}({}^v\mathcal{W}_1, \Theta_t). \end{aligned}$$

Then

$$\begin{aligned} \mathcal{T}_1^x &= (2cuvv_x + 2cu^2u_x + au_{xx})\Theta \\ &\quad - (2cuvu_x + 2cv^2v_x + av_{xx})\Psi \\ &\quad + av_x\Psi_x - au_x\Theta_x, \\ \mathcal{T}_1^t &= \mathcal{I}^{1-\alpha}u_x\Psi + \mathcal{J}\big(-u_x, \Psi_t\big) \\ &\quad + \mathcal{I}^{1-\alpha}v_x\Theta + \mathcal{J}\big(-v_x, \Theta_t\big). \end{aligned}$$

Case 2: For the generator

$$V_2 = 2\alpha x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u} - \alpha v\frac{\partial}{\partial v},$$

the respective Lie characteristic function is obtained by

$$\begin{aligned} {}^u\mathcal{W}_2 &= -\alpha u - 2\alpha x u_x - 4tu_t, \\ {}^v\mathcal{W}_2 &= -\alpha v - 2\alpha x v_x - 4tv_t. \end{aligned} \tag{34}$$

Substituting (38) into (31) yields the conserved vector

$$\begin{aligned} \mathcal{T}_2^x &= \Theta(2\alpha cu^3 + 2\alpha cxu^2u_x + 8ctu^2u_t \\ &\quad +2\alpha cuv^2 + 8ctuvv_t \\ &\quad +2\alpha(cxuv+a)v_x + \alpha(2x+a)v_{xx} \\ &\quad +4atv_{xt} + au_{xx}) \\ &\quad -\Psi(2\alpha cu^2v + 2\alpha(cxuv+a)u_x \\ &\quad +8ctuv2u_t + 2\alpha cv^3 \\ &\quad +2\alpha cxv^2v_x + 8ctv^2v_t + \alpha(2x+a)u_{xx} \\ &\quad +4atu_{xt} + av_{xx}) \\ &\quad +a\Psi_x\left(\alpha v + 2\alpha xv_x + 4tv_t\right) \\ &\quad -a\Theta_x\left(\alpha u + 2\alpha xu_x + 4tu_t\right), \\ \mathcal{T}_2^t &= -\mathcal{I}^{1-\alpha}(-\alpha u - 2\alpha xu_x - 4tu_t)\Psi \\ &\quad +\mathcal{J}(-\alpha u - 2\alpha xu_x - 4tu_t, \Psi_t) \\ &\quad -\mathcal{I}^{1-\alpha}(-\alpha v - 2\alpha xv_x - 4tv_t)\Theta \\ &\quad +\mathcal{J}(-\alpha v - 2\alpha xv_x - 4tv_t, \Theta_t). \end{aligned}$$

For non-classical generators:

Case 3: For $\xi = 1, \tau = 0$, we have

$$V_4 = \frac{\partial}{\partial x} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$$

and the respective Lie characteristic function is obtained by

$${}^u\mathcal{W}_4 = v - u_x, \qquad {}^v\mathcal{W}_4 = -u - v_x. \tag{35}$$

Hence, substituting (39) into (31) yields the conserved vector as follows:

$$\begin{aligned}\mathcal{T}_4^x &= \Theta(-2cvu^2 + 2cu^2u_x + 2cu^2v + 2cuvv_x \\ &\quad + av_x - au_{xx}) \\ &\quad + \Psi(2cuv^2 - 2cuvu_x - 2cv^2u - 2cv^2v_x \\ &\quad - au_x - av_{xx}) \\ &\quad + a\Psi_x\,(u + v_x) - a\Theta_x\,(u_x - v)\,, \\ \mathcal{T}_4^t &= -\mathcal{I}^{1-\alpha}(v - u_x)\Psi + \mathcal{J}(v - u_x, \Psi_t) \\ &\quad - \mathcal{I}^{1-\alpha}(-u - v_x)\Theta \\ &\quad + \mathcal{J}(-u - v_x, \Theta_t).\end{aligned}$$

Case 4: For $\xi = 1, \tau \neq 0$, we have

$$V_5 = \frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial t}$$

and the respective Lie characteristic function is obtained by

$${}^u\mathcal{W}_5 = -u_x - \lambda u_t, \quad {}^v\mathcal{W}_5 = -v_x - \lambda v_t. \tag{36}$$

The corresponding conserved vectors are

$$\begin{aligned}\mathcal{T}_5^x &= \Theta(2cu^2u_x + 2c\lambda u^2u_t + 2cuvv_x \\ &\quad + 2c\lambda uvv_t + au_{xx} + a\lambda u_{xt}) \\ &\quad - \Psi(2cuvu_x + 2c\lambda uvu_t + 2cv^2v_x \\ &\quad + 2c\lambda v^2v_t + av_{xx} + a\lambda v_{xt}) \\ &\quad + a\Psi_x\,(v_x + \lambda v_t) - a\Theta_x\,(u_x + \lambda u_t)\,, \\ \mathcal{T}_5^t &= -\mathcal{I}^{1-\alpha}(-u_x - \lambda u_t)\Psi + \mathcal{J}(-u_x - \lambda u_t, \Psi_t) \\ &\quad - \mathcal{I}^{1-\alpha}(-v_x - \lambda v_t)\Theta + \mathcal{J}(-v_x - \lambda v_t, \Theta_t).\end{aligned}$$

In the special case, if it is assumed that $v = iu$, then we can find $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$ and $\Psi = 1, \Theta = i$. Thus, conservation laws can be written as follows:

Case 1: For the generator $V_1 = \partial/\partial x$, the respective Lie characteristic function is

$${}^u\mathcal{W}_1 = -u_x, \quad {}^v\mathcal{W}_1 = -v_x. \tag{37}$$

Substituting (37) into (31) yields the conserved vector as follows:

$$\begin{aligned}\mathcal{T}_1^x &= aiu_{xx} - av_{xx}, \\ \mathcal{T}_1^t &= \mathcal{I}^{1-\alpha}(iu_x + v_x).\end{aligned}$$

Case 2: For the generator

$$V_2 = 2\alpha x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u} - \alpha v\frac{\partial}{\partial v}.$$

and the respective Lie characteristic function is obtained by

$$\begin{aligned}{}^u\mathcal{W}_2 &= -\alpha u - 2\alpha xu_x - 4tu_t, \\ {}^v\mathcal{W}_2 &= -\alpha v - 2\alpha xv_x - 4tv_t.\end{aligned} \tag{38}$$

Then

$$\begin{aligned}\mathcal{T}_2^x &= ai\,(3\alpha u_x \\ &\quad + 2\alpha xu_{xx} + 4tu_{xt}) \\ &\quad - a\,(3\alpha v_x + 2\alpha xv_{xx} + 4tv_{xt})\,, \\ \mathcal{T}_2^t &= -\mathcal{I}^{1-\alpha}(-\alpha u \\ &\quad - 2\alpha xu_x - 4tu_t) \\ &\quad - i\mathcal{I}^{1-\alpha}(-\alpha v - 2\alpha xv_x - 4tv_t).\end{aligned}$$

For non-classical generators:

Case 3: When $\xi = 1, \tau = 0$, we have

$$V_4 = \frac{\partial}{\partial x} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}.$$

The corresponding Lie characteristic function is obtained by

$${}^u\mathcal{W}_4 = v - u_x, \quad {}^v\mathcal{W}_4 = -u - v_x. \tag{39}$$

Hence

$$\begin{aligned}\mathcal{T}_4^x &= a\,(-u_x - v_{xx}) - ai\,(v_x - au_{xx})\,, \\ \mathcal{T}_4^t &= -\mathcal{I}^{1-\alpha}(v - u_x) - i\mathcal{I}^{1-\alpha}(-u - v_x).\end{aligned}$$

Case 4: For state ($\xi = 1, \tau \neq 0$) we have

$$V_5 = \frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial t},$$

and the respective Lie characteristic function is obtained by

$${}^u\mathcal{W}_5 = -u_x - \lambda u_t, \quad {}^v\mathcal{W}_5 = -v_x - \lambda v_t. \tag{40}$$

The corresponding conserved vectors are

$$\begin{aligned}\mathcal{T}_5^x &= ai\,(2u_{xx} + \lambda u_{xt}) - a\,(v_{xx} + \lambda v_{xt})\,, \\ \mathcal{T}_5^t &= -\mathcal{I}^{1-\alpha}(-u_x - \lambda u_t) - i\mathcal{I}^{1-\alpha}(-v_x - \lambda v_t).\end{aligned}$$

4. Conclusion

Here, non-classical and classical Lie symmetry group analyses are performed on the time-fractional KE equation. By using the obtained vector fields, the governing time-fractional KE equation is reduced to a system of time-fractional ODE. The corresponding exact solutions of the reduced systems are discussed. In summary, based on the advantages of the obtained exact solutions, one can find this method a promising one to solve different nonlinear time-fractional equations. According to Ibragimov's method, the conservation laws are evaluated for the time-fractional KE equation for both classical and non-classical vector fields.

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