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Anisotropic expansion, second-order hydrodynamics and holographic dual

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Abstract. We consider Kasner space–time describing anisotropic three-dimensional expansion of RHIC and LHC fireball and study the generalisation of Bjorken's one-dimensional expansion by taking into account second-order relativistic viscous hydrodynamics. Using time-dependent AdS/CFT correspondence, we study the late-time behaviour of the Bjorken flow. From the conditions of conformal invariance and energy–momentum conservation, we obtain the explicit expression for the energy density as a function of proper time in terms of Kasner parameters. The proper time dependence of the temperature and entropy have also been obtained in terms of Kasner parameters. We consider Eddington–Finkelstein-type coordinates and discuss the gravity dual of the anisotropically expanding fluid in the late-time regime.

Keywords. Quark gluon plasma; anisotropic expansion; second-order hydrodynamics; Kasner Universe.

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1. Introduction

AdS/CFT correspondence has provided a very important tool for studying the strongly coupled dynamics in a class of superconformal field theories, in particular, $\mathcal{N} = 4$ super Yang–Mills theory and the corresponding gravity dual description in AdS space–time [1,2]. One of the fundamental questions in the field of high-energy physics is to understand the properties of matter at extreme density and temperature in the first few microseconds after the big bang. Such a state of matter is known as quark–gluon–plasma (QGP) state where the quarks and the gluons are in a deconfined state. A lot of progress has been made in understanding the properties and various aspects of the evolution of strongly coupled QGP through the heavy-ion collision experiments at RHIC [3,4] (see also [5]) as well as in LHC (see [6] for a comprehensive review). Hydrodynamics plays an important role after the system undergoes a rapid thermalisation and local thermal equilibrium is reached. Since it is difficult to solve the strongly coupled QCD, the qualitative features of the hydrodynamics regime in the evolution of QGP has been studied by using the AdS/CFT duality. In the context of heavy-ion collisions, AdS/CFT correspondence has led to very interesting results like computation of shear viscosity of finite temperature $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [7], viscosity from gravity dual description involving black holes in AdS space [8] etc.

As we know, perfect local equilibrium system is described by ideal fluid dynamics. For small departures from equilibrium, the system is described by dissipative fluid dynamics. The Navier–Stokes equation is the fundamental equation in the non-relativistic viscous hydrodynamics. However, in the case of relativistic Navier–Stokes equation, due to the lack of an initial value formulation in first-order hydrodynamics, signals can be transmitted with arbitrarily high speed thereby violating causality. The system in this case is described by parabolic equations. The first-order theory has been extended by Müller [9]; Israel and Stewart [10] by including the second-order gradient terms thereby preserving causality in the resulting relativistic hydrodynamics equations. The sets of transport coefficients are extended in the second-order hydrodynamics and the resulting equations become hyperbolic.

The study of time-dependent AdS/CFT correspondence in the context of expansion of the RHIC, LHC fireball has been an important area of research. In the context of heavy-ion collisions, the basic features of

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the boost-invariant evolution of the plasma have been discussed by studying the one-dimensional expansion under the assumption that the system remains invariant under longitudinal boosts in the central rapidity region of the expanding plasma [11]. The gravity dual desription of the expansion of the strongly coupled QGP has been very useful in understanding various aspects of both the boundary as well as bulk theory. The earlier work has also suggested that the gravity dual of RHIC fireball is a black hole [12,13].

In a pioneering work, Janik and Peschanski have used the AdS/CFT correspondence and Bjorken's boost invariance symmetry to study the dynamics of the strongly coupled QGP in non-viscous case [14]. Using conformal invariance of the gauge theory, energy–momentum conservation condition and hologhaphic renormalisation method [15], they have constructed the dual geometry from the given boundary data on the gauge theory side. Using Fefferman–Graham (FG) coordinates and solving the nonlinear Einstein's equations, the non-singular bulk geometry in large proper time has been obtained [14]. The regularity of the bulk geometry has been an important aspect in determining the hydrodynamic quantities in the corresponding boundary theory. The boost-invariant dual geometry has also been constructed by including the conserved R-charge [16].

In view of the results from RHIC as well as LHC, it is important to include shear viscosity for the expanding plasma and analyse the corresponding dynamics. By including the shear viscosity in the context of relativistic first-order hydrodynamics, the dual geometry has been constructed in ref. [17]. Then the above study corresponding to Bjorken's one-dimensional expansion has been extended by Sin *et al* to anisotropic three-dimensional expansion of the plasma by considering Kasner space–time as the local rest frame of the fluid [18]. Though Kasner space–time is a curved space–time, Sin *et al* [18] have shown that under a well-controlled approximation, it can be considered as the local rest frame of the anisotropically expanding fluid on Minkowski space–time. The corresponding gravity dual in large proper time regime has been obtained in first-order gradient expansion using Fefferman–Graham coordinates [18]. Subsequently, the form of the stress tensor within the framework of second-order viscous hydrodynamics has been obtained and the associated transport coefficients have been explicitly determined in [19,20]. The relaxation time in second-order viscous hydrodynamics has also been computed from the analysis of the regularity of the dual geometry [21].

Though the Fefferman–Graham coordinates have been very useful for the gravity dual description, the regularity of the dual geometry in late-time regime has been an important issue. This problem has been addressed by working with Eddington–Finkelstein-type coordinates. In the framework of second-order viscous hydrodynamics, it has been shown explicitly that the dual geometry is regular in the late-time approximation and the regularity of the dual geometry determines the transport coefficients uniquely [22–24]. The dual geometry is regular except for the physical singularity at the origin which is covered by the event horizon of the bulk geometry. However, the concept of event horizon in a time-dependent dual geometry is not entirely clear. Instead of an event horizon whose position is time-dependent, it is more appropriate to consider a locally defined apparent horizon which is crucial in establishing that the dual geometry corresponds to a dynamical black hole [23,25].

In this paper, we consider second-order relativistic viscous hydrodynamics and study the anisotropic three-dimensional expansion of the relativistic plasma in a time-dependent background. The local rest frame (LRF) of the anisotropically expanding fluid is described by the time-dependent Kasner space–time. We compute the dissipative part of the energy–momentum tensor in this set-up and obtain explicit expressions for the components of the energy–momentum tensor in terms of Kasner parameters, pressure, energy density, shear viscosity, relaxation time and other transport coefficients in second-order relativistic hydrodynamics. Then, we solve the corresponding equations of state and hydrodynamic conservation equation for the conformal fluid and obtain the expressions for the time evolution of the energy density, pressure, temperature and entropy per unit co-moving volume in terms of the transport coefficients in second-order gradient expansion and the Kasner parameters. These expressions have not been obtained before. Subsequently, we have made an attempt to find the holographic dual of the anisotropically expanding plasma in the late-time approximation in the context of time-dependent AdS/CFT correspondence. We have tried to obtain the zeroth-order solution for the corresponding dual five-dimensional bulk space–time in Eddington–Finkelstein coordinates. We find that the zeroth-order solution is an exact solution in the large proper time limit with constraints on the values of the Kasner parameters required for consistency.

The paper is organised as follows: In §2, we have considered Kasner space–time as the local rest frame of the fluid and have studied the anisotropically expanding fluid in three dimensions thereby generalising Bjorken's one-dimensional expansion. Section 3 deals with the study of the evolution of the QGP in the context of second-order viscous hydrodynamics and time-dependent AdS/CFT correspondence. Using the conditions of conformal invariance and energy–momentum conservation, we have obtained explicit expression for the energy density as a function of proper time in terms of Kasner parameters with second-order gradient expansion. The proper time dependence of pressure, temperature and entropy per unit co-moving volume has also been obtained in terms of Kasner parameters. Our results reduce to that of the earlier discussions in the framework of one-dimensional Bjorken expansion in appropriate limit of the Kasner parameters where the local rest frame is described by Minkowski space–time in terms of proper time and rapidity. In §4, we have made a proposal for the gravity dual of the anisotropically expanding fluid using Eddington–Finkelstein-type coordinates and have shown that the zeroth-order solution is an exact solution of the five-dimensional Einstein's equation in the large proper time limit with constraints on the Kasner parameters. We summarise and discuss the future perspective in §5.

2. Bjorken's hydrodynamics and Kasner space–time

In this section, we consider the viscous hydrodynamics including second-order gradient expansion terms and study the anisotropic three-dimensional expansion of the fluid with Kasner space–time as the local rest frame. For the one-dimensional Bjorken expansion case, the local rest frame of the fluid is described by proper time τ and rapidity y and they are related to the Cartesian coordinates in the following way:

$$(X_0, X_1, X_2, X_3) = (\tau \cosh y, \tau \sinh y, X_2, X_3). \quad (1)$$

Here the collision axis is taken along the X_1 direction. The Minkowski metric in these coordinate is given by

$$\mathrm{d}s^2 = -(\mathrm{d}\tau)^2 + \tau^2\,\mathrm{d}y^2 + (\mathrm{d}X_2)^2 + (\mathrm{d}X_3)^2. \quad (2)$$

We consider the generalisation of Bjorken's one-dimensional expansion to three-dimensional expansion of the plasma in order to connect to the realistic description of the RHIC and LHC fireball and for this, we consider Kasner space–time as the local rest frame of the fluid [18]. The metric is given by

$$\mathrm{d}s^2 = -(\mathrm{d}\tau)^2 + \tau^{2a}(\mathrm{d}x_1)^2 + \tau^{2b}(\mathrm{d}x_2)^2 + \tau^{2c}(\mathrm{d}x_3)^2. \quad (3)$$

Here x_1, x_2, x_3 are the co-moving coordinates. a, b, c are constants and are known as Kasner parameters. The Kasner parameters satisfy the conditions

$$a + b + c = 1, \quad a^2 + b^2 + c^2 = 1. \quad (4)$$

The above Kasner metric is an exact solution of vacuum Einstein's equation and it describes a homogeneous and anisotropic expansion of the Universe. The physical quantities are assumed to depend only on proper time τ. The non-zero components of the affine connection for the Kasner metric are given by

$$\Gamma^{\tau}_{x_1x_1} = a\tau^{2a-1},\ \Gamma^{\tau}_{x_2x_2} = b\tau^{2b-1},\ \Gamma^{\tau}_{x_3x_3} = c\tau^{2c-1},$$
$$\Gamma^{x_1}_{x_1\tau} = \frac{a}{\tau},\quad \Gamma^{x_2}_{x_2\tau} = \frac{b}{\tau},\quad \Gamma^{x_3}_{x_3\tau} = \frac{c}{\tau}. \quad (5)$$

In ref. [26], Kasner space–time has also been studied to relate the anisotropic expansion with anisotropic hydrodynamics (see ref. [27] for a review on anisotropic hydrodynamics).

Since the gauge theory is conformal, from the tracelessness condition of the energy–momentum tensor, we obtain

$$-T_{\tau\tau} + \frac{1}{\tau^{2a}}T_{x_1x_1} + \frac{1}{\tau^{2b}}T_{x_2x_2} + \frac{1}{\tau^{2c}}T_{x_3x_3} = 0. \quad (6)$$

From the conservation of energy–momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ (where $\mu, \nu = \tau, x_1, x_2, x_3$), one gets further relation among the components of $T_{\mu\nu}$:

$$\partial_\tau T_{\tau\tau} + \frac{(a+b+c)}{\tau}T_{\tau\tau} + \frac{a}{\tau}\tau^{-2a}T_{x_1x_1}$$
$$+\frac{b}{\tau}\tau^{-2b}T_{x_2x_2} + \frac{c}{\tau}\tau^{-2c}T_{x_3x_3} = 0. \quad (7)$$

When $a = 1, b = 0$ and $c = 0$, these equations reduce to that of the one-dimensional expansion case with Minkowski space–time as the local rest frame of the fluid and the components of the energy–momentum tensor can be written in terms of a single function which is interpreted as the energy density as a function of time.

In the relativistic viscous hydrodynamics, the energy–momentum tensor is given by

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P\Delta^{\mu\nu} + \Pi^{\mu\nu}, \quad (8)$$

where u^μ, ϵ and P are 4-velocity, the energy density and pressure respectively. $\Pi^{\mu\nu}$ represents the dissipative part. The dissipative part including second-order gradient expansion terms is given by [19,20] (we use the notations of [19]),

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\pi\left[{}^{\langle}D\sigma^{\mu\nu\rangle} + \frac{1}{3}\sigma^{\mu\nu}(\nabla\cdot u)\right]$$
$$+\kappa\left[R^{\langle\mu\nu\rangle} - 2u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta\right]$$
$$+\lambda_1\sigma^{<\mu}{}_\lambda\sigma^{\nu>\lambda} + \lambda_2\sigma^{<\mu}{}_\lambda\Omega^{\nu>\lambda}$$
$$+\lambda_3\Omega^{<\mu}{}_\lambda\Omega^{\nu>\lambda}, \quad (9)$$

where η is the shear viscosity, τ_π is the relaxation time and $\kappa, \lambda_1, \lambda_2, \lambda_3$ are the other second-order transport coefficients. For flat space, κ term vanishes. In the first-order hydrodynamics, only the first term in $\Pi^{\mu\nu}$ involving shear viscosity η is relevant. The bulk viscosity is zero as we are considering a conformal fluid.

Various terms appearing in $\Pi^{\mu\nu}$ are given by

$$
\begin{aligned}
D &\equiv u^\mu \nabla_\mu \\
\Delta^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu \\
{}^{\langle}\nabla^\mu u^{\nu\rangle} &= \left(\Delta^{\mu\lambda}\nabla_\lambda u^\nu + \Delta^{\nu\lambda}\nabla_\lambda u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\lambda u^\lambda\right) \\
\sigma^{\mu\nu} &= 2^{\langle}\nabla^\mu u^{\nu\rangle} \\
{}^{\langle}R^{\mu\nu\rangle} &= \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(R_{\alpha\beta} + R_{\beta\alpha}) \\
&\quad -\frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}R_{\alpha\beta} = R^{\langle\mu\nu\rangle} \\
R^{\alpha\langle\mu\nu\rangle\beta} &= \frac{1}{2}\Delta^{\mu\sigma}\Delta^{\nu\rho}(R^{\alpha\beta}_{\sigma\rho} + R^{\alpha\beta}_{\rho\sigma}) \\
&\quad -\frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\sigma\rho}R^{\alpha\beta}_{\sigma\rho} \\
\Omega^{\mu\nu} &= \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha),
\end{aligned} \tag{10}
$$

where $u^\mu = (1, 0, 0, 0)$, $\Delta^{\mu\nu}$ is the projector on the spatial subspace and Ω is the vorticity term. In general, the bracket for a second rank tensor implies,

$$
\begin{aligned}
{}^{\langle}A^{\mu\nu\rangle} &\equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(A_{\alpha\beta} + A_{\beta\alpha}) \\
&\quad -\frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta} \equiv A^{\langle\mu\nu\rangle}.
\end{aligned} \tag{11}
$$

3. Energy–momentum tensor in second-order hydrodynamics

In the case of first-order gradient expansion, the energy–momentum tensor in Kasner space–time is obtained as

$$
\begin{aligned}
T^{00} &= \epsilon(\tau) \\
T^{11} &= \tau^{-2a}\left(P - \frac{2\eta}{3\tau}(3a-1)\right) \\
T^{22} &= \tau^{-2b}\left(P - \frac{2\eta}{3\tau}(3b-1)\right) \\
T^{33} &= \tau^{-2c}\left(P - \frac{2\eta}{3\tau}(3c-1)\right).
\end{aligned} \tag{12}
$$

The corresponding equation of state and conservation condition are respectively given as

$$
P = \frac{\epsilon}{3} \tag{13}
$$

$$
\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} + \frac{4\epsilon}{3\tau} - \frac{4\eta}{3\tau^2} = 0. \tag{14}
$$

As one can see, these equations are independent of the Kasner parameters.

For the second-order hydrodynamics, we obtain explicit expressions for the components of the dissipative part ($\Pi^{\mu\nu}$) of the energy–momentum tensor in terms of Kasner parameters, which are given by

$$
\begin{aligned}
\Pi^{00} &= 0 \\
\Pi^{11} &= -\frac{2\eta(3a-1)}{3\tau^{2a+1}} - \frac{4\eta\tau_\pi(3a-1)}{9\tau^{2a+2}} \\
&\quad -\frac{2\kappa(-1+a)a}{\tau^{2a+2}} + \frac{4\lambda_1(9a^2-6a-1)}{9\tau^{2a+2}} \\
\Pi^{22} &= -\frac{2\eta(3b-1)}{3\tau^{2b+1}} - \frac{4\eta\tau_\pi(3b-1)}{9\tau^{2b+2}} \\
&\quad -\frac{2\kappa(-1+b)b}{\tau^{2b+2}} + \frac{4\lambda_1(9b^2-6b-1)}{9\tau^{2b+2}} \\
\Pi^{33} &= -\frac{2\eta(3c-1)}{3\tau^{2c+1}} - \frac{4\eta\tau_\pi(3c-1)}{9\tau^{2c+2}} \\
&\quad -\frac{2\kappa(-1+c)c}{\tau^{2c+2}} + \frac{4\lambda_1(9c^2-6c-1)}{9\tau^{2c+2}}.
\end{aligned} \tag{15}
$$

One can check, for the one-dimensional expansion of the fluid, corresponding to $a = 1, b = 0, c = 0$, the above expressions become (written in a matrix form) [23]

$$
\begin{aligned}
\Pi^{\mu\nu} &= -\eta \begin{pmatrix} 0 & & & \\ & \frac{4}{3}\tau^{-3} & & \\ & & -\frac{2}{3}\tau^{-1} & \\ & & & -\frac{2}{3}\tau^{-1} \end{pmatrix} \\
&\quad +(\eta\tau_\pi - \lambda_1)\begin{pmatrix} 0 & & & \\ & -\frac{8}{9}\tau^{-4} & & \\ & & \frac{4}{9}\tau^{-2} & \\ & & & \frac{4}{9}\tau^{-2} \end{pmatrix}.
\end{aligned} \tag{16}
$$

Next, for the three-dimensional expansion of the fluid, we get the components of the energy–momentum tensor as

$$
\begin{aligned}
T^{00} &= \epsilon(\tau) \\
T^{11} &= \frac{P(\tau)}{\tau^{2a}} - \frac{2\eta(3a-1)}{3\tau^{2a+1}} \\
&\quad -\frac{4\eta\tau_\pi(3a-1)}{9\tau^{2a+2}} + \frac{2\kappa(1-a)a}{\tau^{2a+2}} \\
&\quad +\frac{4\lambda_1(9a^2-6a-1)}{9\tau^{2a+2}} \\
T^{22} &= \frac{P(\tau)}{\tau^{2b}} - \frac{2\eta(3b-1)}{3\tau^{2b+1}}
\end{aligned}
$$

$$-\frac{4\eta\tau_\pi(3b-1)}{9\tau^{2b+2}}+\frac{2\kappa(1-b)b}{\tau^{2b+2}}+\frac{4\lambda_1(9b^2-6b-1)}{9\tau^{2b+2}}$$

$$T^{33}=\frac{P(\tau)}{\tau^{2c}}-\frac{2\eta(3c-1)}{3\tau^{2c+1}}-\frac{4\eta\tau_\pi(3c-1)}{9\tau^{2c+2}}+\frac{2\kappa(1-c)c}{\tau^{2c+2}}+\frac{4\lambda_1(9c^2-6c-1)}{9\tau^{2c+2}}. \quad (17)$$

We have obtained these expressions after putting the Kasner conditions

$$\sum_i a_i=1,\quad \sum_i a_i^2=1 \quad (a_i=a,b,c). \quad (18)$$

The hydrodynamic conservation equation $\nabla_\mu T^{\mu\nu}=0$ becomes,

$$\begin{aligned}\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}=&-\frac{(a+b+c)\epsilon}{\tau}-\frac{(a+b+c)P}{\tau}\\&+\frac{2\eta}{\tau^2}\left((a^2+b^2+c^2)-\frac{1}{3}(a+b+c)^2\right)\\&-\frac{\eta\tau_\pi}{\tau^3}\Big\{-2(a^2+b^2+c^2)+\frac{2}{3}(a+b+c)^2\\&+\frac{2}{3}(a^2+b^2+c^2)(a+b+c)-\frac{2}{9}(a+b+c)^3\Big\}\\&-\frac{\lambda_1}{3\tau^3}\left\{\left(a-\frac{1}{3}(a+b+c)\right)^2(8a-4b-4c)\right.\\&+\left(b-\frac{1}{3}(a+b+c)\right)^2(8b-4a-4c)\\&\left.+\left(c-\frac{1}{3}(a+b+c)\right)^2(8c-4a-4b)\right\}\\&-\frac{\kappa}{\tau^3}\Big\{(a+b+c)(a^2+b^2+c^2)\\&-(a^2+b^2+c^2)-\frac{1}{3}(a+b+c)^3+\frac{1}{3}(a+b+c)\\&-2\left((a^3+b^3+c^3)-2(a^2+b^2+c^2)\right)\\&-2(a+b+c)\Big\}\end{aligned} \quad (19)$$

The equation of state and the conservation law become (using Kasner conditions),

$$P=\frac{\epsilon}{3} \quad (20)$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}+\frac{4\epsilon}{3\tau}=\frac{4\eta}{3\tau^2}+\frac{8\eta\tau_\pi}{9\tau^3}+\frac{2\kappa(-1+a^3+b^3+c^3)}{\tau^3}-\frac{\lambda_1(-7+9(a^3+b^3+c^3))}{9\tau^3}. \quad (21)$$

As one can see, the equation of state is independent of the Kasner parameters, but the energy–momentum conservation law does depend on the Kasner parameters.

From the conformal invariance of the fluid, the proper time dependence of the transport coefficients are given by

$$\eta=\epsilon_0\,\eta_0\left(\frac{\epsilon}{\epsilon_0}\right)^{3/4},\quad \tau_\pi=\tau_\pi^0\left(\frac{\epsilon}{\epsilon_0}\right)^{-1/4},$$
$$\lambda_1=\epsilon_0\,\lambda_1^0\left(\frac{\epsilon}{\epsilon_0}\right)^{1/2},\quad \kappa=\epsilon_0\,\kappa_0\left(\frac{\epsilon}{\epsilon_0}\right)^{1/2}, \quad (22)$$

where ϵ_0, η_0, τ_π^0, λ_1^0, κ_0 are constants.

The solution of the equation for energy density $\epsilon(\tau)$ is obtained as

$$\frac{\epsilon(\tau)}{\epsilon_0}=\tau^{-4/3}-2\frac{\eta_0}{\tau^2}+\left[\frac{3\eta_0^2}{2}+\frac{\lambda_1^0}{3}(-7+9a^3+9b^3+9c^3)-\frac{2\eta_0\tau_\pi^0}{3}-\frac{3\kappa_0}{2}(-1+a^3+b^3+c^3)\right]\tau^{-8/3}+\cdots. \quad (23)$$

Denoting the term in the above square bracket as

$$\left[\frac{3\eta_0^2}{2}+\frac{\lambda_1^0}{3}(-7+9a^3+9b^3+9c^3)-\frac{2\eta_0\tau_\pi^0}{3}-\frac{3\kappa_0}{2}(-1+a^3+b^3+c^3)\right]=\tilde{\epsilon}_0^{(2)}, \quad (24)$$

the solution for $\epsilon(\tau)$ can be written as

$$\frac{\epsilon(\tau)}{\epsilon_0}=\tau^{-4/3}-2\eta_0\tau^{-2}+\tilde{\epsilon}_0^{(2)}\tau^{-8/3}+\cdots. \quad (25)$$

For $a=1, b=0, c=0$, the above solution reduces to that of the one-dimensional expansion case (where $\kappa=0$ in flat space) in second-order hydrodynamics [19]. Now we write the components of the energy–momentum tensor in terms of energy density and the expressions are given by

$$T_{00}/\epsilon_0=\tau^{-4/3}-2\eta_0\tau^{-2}+\tilde{\epsilon}_0^{(2)}\tau^{-8/3}+\cdots$$

$$\begin{aligned}\frac{T_{11}}{\epsilon_0\tau^{2a}}=&\frac{1}{3}\tau^{-4/3}-2\eta_0 a\tau^{-2}+\eta_0^2\left(3a-\frac{1}{2}\right)\tau^{-8/3}\\&-\frac{\kappa_0}{2}((a^3+b^3+c^3)+(4a^2-4a-1))\tau^{-8/3}\\&-\frac{2}{9}\eta_0\tau_\pi^0(6a-1)\tau^{-8/3}+\frac{\lambda_1^0}{9}(9(a^3+b^3+c^3)\\&+(36a^2-24a-11))\tau^{-8/3}+\cdots\end{aligned}$$

$$\frac{T_{22}}{\epsilon_0\tau^{2b}} = \frac{1}{3}\tau^{-4/3} - 2\eta_0 b\tau^{-2} + \eta_0^2\left(3b - \frac{1}{2}\right)\tau^{-8/3} - \frac{\kappa_0}{2}((a^3+b^3+c^3)+(4b^2-4b-1))\tau^{-8/3} - \frac{2}{9}\eta_0\tau_\pi^0(6b-1)\tau^{-8/3} + \frac{\lambda_1^0}{9}(9(a^3+b^3+c^3) + (36b^2-24b-11))\tau^{-8/3} + \cdots$$

$$\frac{T_{33}}{\epsilon_0\tau^{2c}} = \frac{1}{3}\tau^{-4/3} - 2\eta_0 c\tau^{-2} + \eta_0^2\left(3c - \frac{1}{2}\right)\tau^{-8/3} - \frac{\kappa_0}{2}((a^3+b^3+c^3)+(4c^2-4c-1))\tau^{-8/3} - \frac{2}{9}\eta_0\tau_\pi^0(6c-1)\tau^{-8/3} + \frac{\lambda_1^0}{9}(9(a^3+b^3+c^3) + (36c^2-24c-11))\tau^{-8/3} + \cdots . \tag{26}$$

From Stefan–Boltzmann's law, where $\epsilon \propto T^4$, we obtain the proper time dependence of temperature T as

$$T(\tau) = \epsilon_0^{1/4}\left(\frac{1}{\tau^{1/3}} - \frac{\eta_0}{2\tau} + \frac{3\kappa_0(-1+a^3+b^3+c^3)}{8\tau^{5/3}} + \frac{\lambda_1^0(-7+9a^3+9b^3+9c^3)}{12\tau^{5/3}} - \frac{\eta_0\tau_\pi^0}{6\tau^{5/3}} + \cdots\right). \tag{27}$$

It is useful to re-express the conservation law of energy–momentum tensor as

$$\frac{\mathrm{d}(\sqrt{g}\epsilon)}{\mathrm{d}\tau} + \frac{\mathrm{d}\sqrt{g}}{\mathrm{d}\tau}P = \frac{4}{3}\frac{\sqrt{g}\eta}{\tau^2} + \frac{8}{9}\frac{\sqrt{g}\eta\tau_\pi}{\tau^3} + \frac{2\sqrt{g}\kappa(-1+a^3+b^3+c^3)}{\tau^3} - \frac{4\sqrt{g}\lambda_1(-7+9a^3+9b^3+9c^3)}{9\tau^3}, \tag{28}$$

where $\sqrt{g} = \tau$ is the volume element in the co-moving coordinate. Using the thermodynamic relation $\mathrm{d}E + P\mathrm{d}V = T\,\mathrm{d}S$, the above equation can be expressed as

$$T\frac{\mathrm{d}(\sqrt{g}s)}{\mathrm{d}\tau} = \frac{4}{3}\frac{\sqrt{g}\eta}{\tau^2} + \frac{2\sqrt{g}\kappa(-1+a^3+b^3+c^3)}{\tau^3} + \frac{8}{9}\frac{\sqrt{g}\eta\tau_\pi}{\tau^3} - \frac{4\sqrt{g}\lambda_1(-7+9a^3+9b^3+9c^3)}{9\tau^3}, \tag{29}$$

where s is the entropy density and $\tau s = \sqrt{g}s = S$ is the entropy per unit co-moving volume. Integrating the above equation and using the proper time dependence of the temperature, the entropy per unit co-moving volume (as a function of proper time τ) is obtained as

$$S(\tau) = \frac{4}{3}\int_0^\tau \mathrm{d}\tau\frac{\sqrt{g}\eta}{\tau^2 T} + \frac{8}{9}\int_0^\tau \mathrm{d}\tau\frac{\sqrt{g}\eta\tau_\pi}{\tau^3 T} + 2\int_0^\tau \mathrm{d}\tau\frac{\sqrt{g}\kappa(-1+a^3+b^3+c^3)}{\tau^3 T} - \frac{4}{9}\int_0^\tau \mathrm{d}\tau\frac{\sqrt{g}\lambda_1(-7+9a^3+9b^3+9c^3)}{9\tau^3 T} \tag{30}$$

$$= \epsilon_0^{3/4}\left\{1 - \frac{3\eta_0}{2}\tau^{-2/3} + \frac{3\eta_0^2}{4}\tau^{-4/3} - \frac{\eta_0\tau_\pi^0}{2}\tau^{-4/3} + \frac{\lambda_1^0(-7+9a^3+9b^3+9c^3)}{4}\tau^{-4/3} - \frac{9\kappa_0(-1+a^3+b^3+c^3)}{8}\tau^{-4/3} + 0(\tau^{-2})\right\}. \tag{31}$$

As one can see, our expressions for energy density, temperature and entropy as a function of proper time τ in second-order hydrodynamics in Kasner space–time depend on the Kasner parameters a, b and c.

4. A proposal for the gravity dual of anisotropic expansion

In this section, we shall discuss the holographic dual of the anisotropically expanding fluid in the late-time approximation using Eddington–Finkelstein (EF) coordinates. For the one-dimensional expansion case, the gravity dual geometry has been obtained in [18] by using the Fefferman–Graham (FG) coordinates. The five-dimensional asymptotically AdS metric in FG coordinates is given by

$$\mathrm{d}s^2 = \frac{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}z^2}{z^2}, \tag{32}$$

where $x^\mu = (\tau, x_1, x_2, x_3)$. $g_{\mu\nu}$ is the four-dimensional metric which depends on both τ and z and it is expanded with respect to z as [15]

$$g_{\mu\nu}(\tau, z) = g_{\mu\nu}^{(0)}(\tau) + z^2 g_{\mu\nu}^{(2)}(\tau) + z^4 g_{\mu\nu}^{(4)}(\tau) + z^6 g_{\mu\nu}^{(6)}(\tau) + \cdots . \tag{33}$$

Here, $g_{\mu\nu}^{(0)}$ is the gauge theory metric on the boundary. In our case, $g_{\mu\nu}^{(0)}$ corresponds to the Kasner metric. $g_{\mu\nu}^{(2)} = 0$ as the four-dimensional metric is Ricci flat. $g_{\mu\nu}^{(4)}$ is proportional to the boundary energy–momentum tensor, that is,

$$g_{\mu\nu}^{(4)} = \mathrm{const}\langle T_{\mu\nu}\rangle. \tag{34}$$

One obtains the higher-order terms in the expansion of $g_{\mu\nu}(\tau, z)$ by solving the five-dimensional bulk Einstein's equation with a negative cosmological constant recursively:

$$R_{MN} - \frac{1}{2}G_{MN}R - 6G_{MN} = 0, \tag{35}$$

where G_{MN}, R_{MN} and R correspond to the metric, Ricci tensor and Ricci scalar respectively in the five-dimensional theory. After solving the Einstein's equation recursively, the late-time 5D bulk geometry can be obtained in a compact form. The dual metric has been obtained by Sin, Nakamura and Kim with first-order gradient expansion terms and is given by [18]

$$\mathrm{d}s^2 = \frac{1}{z^2}\left\{ -\frac{\left(1 - \frac{\epsilon z^4}{3}\right)^2}{1 + \frac{\epsilon z^4}{3}}\mathrm{d}\tau^2 + \left(1 + \frac{\epsilon z^4}{3}\right) \times \sum_{i=1}^{3} \left(\frac{1 + \frac{\epsilon z^4}{3}}{1 - \frac{\epsilon z^4}{3}}\right)^{(1-3a_i)\gamma} \tau^{2a_i}(\mathrm{d}x^i)^2 \right\} + \frac{\mathrm{d}z^2}{z^2}, \tag{36}$$

where

$$\epsilon(\tau) = \frac{\epsilon_0}{\tau^{4/3}} - \frac{2\eta_0}{\tau^2}, \qquad \gamma = \frac{\eta_0}{\epsilon_0 \tau^{2/3}},$$
$$a_i (i = 1, 2, 3) \equiv a, b, c. \tag{37}$$

The above 5D metric in FG coordinate is a solution in the late-time regime and is correct only upto order γ ($\gamma \propto \tau^{-2/3}$) [18].

Though the holographic dual of Bjorken flow has been described very well by using the Fefferman–Graham (FG) coordinate, it is difficult to define the location of the event horizon in a time-dependent geometry in this coordinate. In a very interesting paper, Kinoshita *et al* [23] (see also [24]) have proposed the dual geometry in late-time expansion by using Eddington–Finkelstein (EF) coordinates in the one-dimensional expansion case and have also computed the location of the apparent horizon (boundary between the trapped and untrapped regions). The regularity of the dual geometry has been shown to all orders and that in turn, determines the transport coefficients uniquely [23,24]. Eddington–Finkelstein coordinates have also been used before to construct the dual geometry which is regular except at the origin [20,22].

In analogy with the one-dimensional expansion of the fluid [23,24], we propose the following parametrisation for the dual geometry in the late-time regime corresponding to the three-dimensional expansion case with Kasner space–time as the local rest frame of the fluid:

$$\mathrm{d}s^2 = -r^2 P\,\mathrm{d}\tau^2 + 2\mathrm{d}\tau\,\mathrm{d}r + r^2\tau^{2a}\mathrm{e}^{2Q-2R}\left(1 + \frac{1}{u\tau^{2/3}}\right)^2 \mathrm{d}x_1^2 + r^2\tau^{2b}\mathrm{e}^R\,\mathrm{d}x_2^2 + r^2\tau^{2c}\mathrm{e}^R\,\mathrm{d}x_3^2, \tag{38}$$

where r is the fifth dimension. The variable u is defined as $u = r\tau^{1/3}$ and the late-time approximation is taken in an expansion in $\tau^{-2/3}$ keeping u fixed. a, b, c are Kasner parameters and P, Q, R are functions of u and τ. The boundary conditions correspond to $P \to 1$, $Q \to 0$ and $R \to 0$ as $r \to \infty$ (corresponds to the spatial boundary with r as the fifth dimension).

With these boundary conditions, the 5D bulk metric in the limit as $r \to \infty$ becomes

$$\mathrm{d}s^2|_{r\to\infty} = r^2\Big[-(\mathrm{d}\tau)^2 + \tau^{2a}(\mathrm{d}x_1)^2 + \tau^{2b}(\mathrm{d}x_2)^2 + \tau^{2c}(\mathrm{d}x_3)^2\Big] + 2\mathrm{d}\tau\mathrm{d}r, \tag{39}$$

where the quantity inside the square bracket is the boundary four-dimensional Kasner metric on the local rest frame of the fluid. This is in analogy with the one-dimensional expansion with the Minkowski metric on the LRF of the fluid [24]. The 4D part of the proposed dual bulk metric has been taken to be diagonal because the 4D boundary energy–momentum tensor is diagonal. The parameters P, Q, R are expanded in powers of $\tau^{-2/3}$ as [23,24],

$$P(\tau, u) = P_0(u) + P_1(u)\tau^{-2/3} + P_2(u)\tau^{-4/3} + \cdots$$
$$Q(\tau, u) = Q_0(u) + Q_1(u)\tau^{-2/3} + Q_2(u)\tau^{-4/3} + \cdots$$
$$R(\tau, u) = R_0(u) + R_1(u)\tau^{-2/3} + R_2(u)\tau^{-4/3} + \cdots, \tag{40}$$

where P_n, Q_n and R_n are obtained by solving the 5D Einstein's equation order by order in late-time regime with the boundary conditions as mentioned above. The zeroth-order solution is given by (we have set the integration constant to zero),

$$P_0(u) = 1 - \frac{w^4}{u^4}, \quad Q_0 = 0, \quad R_0 = 0, \tag{41}$$

where w is a constant. We would like to clarify that the corresponding zeroth-order metric given by

$$\mathrm{d}s^2 = -r^2\left(1 - \frac{w^4}{u^4}\right)\mathrm{d}\tau^2 + 2\mathrm{d}\tau\,\mathrm{d}r + r^2\tau^{2a}\left(1 + \frac{1}{u\tau^{2/3}}\right)^2 \mathrm{d}x_1^2 + r^2\tau^{2b}\mathrm{d}x_2^2 + r^2\tau^{2c}\mathrm{d}x_3^2 \tag{42}$$

is an exact solution of the 5D Einstein's equation in the large τ limit. Consistency of the above solution puts a constraint on the values of the Kasner parameters, namely $a = 1$, $b = 0$ and $c = 0$ so that the zeroth-order metric becomes an exact solution in the large τ limit. There can be other choices for a, b and c, but they do not satisfy the Kasner conditions, $a + b + c = 1$ and $a^2 + b^2 + c^2 = 1$. At zeroth order, the term $1/u\tau^{2/3}$ in the bracket in the coefficient of $\mathrm{d}x_1^2$ is ignored [23]. This term was introduced so that the zeroth-order metric in the one-dimensional expansion case reduces to an exact AdS metric (through a coordinate transformation) in the $w \to 0$ limit [23,24].

The corresponding Kretschmann scalar for our zeroth-order metric is obtained as

$$R_{MNKL}\, R^{MNKL} = \left[40 + \frac{72w^8}{u^8} + \frac{32(a+b+c)}{u\tau^{2/3}} + \frac{8(ab+bc+ca)}{u^2\tau^{4/3}} + \frac{16(a^2+b^2+c^2)}{u^2\tau^{4/3}}\right], \tag{43}$$

where M, N, K, L are the indices corresponding to the 5D metric. We have checked that the above expression reduces exactly to the zeroth-order computation of the Kretschmann scalar in the one-dimensional expansion case [23] in appropriate limit of the Kasner parameters. After putting the Kasner conditions ($a + b + c = 1$, $a^2 + b^2 + c^2 = 1$ and $ab + bc + ca = 0$ by virtue of the Kasner conditions), the above expression for the Kretschmann scalar becomes independent of the Kasner parameters and is given by

$$R_{KLMN}\, R^{KLMN} = 8\left(5 + \frac{9w^8}{u^8}\right) + O(\tau^{-2/3}), \tag{44}$$

where the physical singularity is at $u = 0$. This matches with the result of ref. [23] in zeroth order. The Ricci scalar of the zeroth-order metric is given by

$$R = -20 - \frac{8(a+b+c)}{u\tau^{2/3}} = -20 - \frac{8}{u\tau^{2/3}}. \tag{45}$$

A detailed analysis regarding the location of the apparent horizon of the dual geometry in an expansion in $\tau^{-2/3}$ has been done in ref. [23], where the position of the apparent horizon has been expanded as

$$u_H = u_0 + u_1\tau^{-2/3} + u_2\tau^{-4/3} + \cdots. \tag{46}$$

The zeroth-order result gives $u_0 = w$. The higher-order results have also been obtained by using the regularity condition of the dual geometry [23,24], showing that there is indeed an apparent horizon. The existence of the apparent horizon also means that there is an event horizon so that the physical singularity at the origin is covered and it is not a naked singularity (see ref. [25] for a rigorous discussion on the location of the apparent horizon and event horizon). We have not done a detailed analysis for higher orders in expansion in $\tau^{-2/3}$ in the three-dimensional expansion case. The volume element of the apparent horizon of the dual geometry in our three-dimensional case (after putting Kasner condition) is given by

$$\text{vol.} = r^3\tau e^{Q}\left(1 + \frac{1}{r\tau}\right) \tag{47}$$

which can be computed at $u = u_H$ in an expansion in $\tau^{-2/3}$. We have checked that our expression for the zeroth-order and first-order terms in entropy density computed from hydrodynamics in the limit of $a = 1$, $b = 0$, $c = 0$ match with the results computed from the volume element of the apparent horizon of the dual geometry in late-time regime with appropriate normalisation factor [23]. It has been noticed that at second order, there is a mismatch between the geometrical and hydrodynamical entropy densities in the one-dimensional expansion case. This does not imply any physical inconsistency and there can be several intricate reasons for this [23].

5. Summary and discussion

In this paper, we have studied the three-dimensional anisotropic expansion of a conformal fluid by using Kasner space–time as the local rest frame of the fluid as an example of time-dependent AdS/CFT correspondence. We have considered relativistic viscous hydrodynamics to second order in gradient expansion and have obtained the expressions for energy density, temperature and the components of the energy–momentum tensor in terms of Kasner parameters and the transport coefficients in the late-time regime. We have also obtained the entropy density per unity rapidity from the hydrodynamics side. In analogy with the one-dimensional expansion case [23,24], we have made a proposal for the five-dimensional dual geometry in the large proper time approximation using Eddington–Finkelstein coordinates in the three-dimensional expansion case with the boundary metric as the 4D Kasner space–time. We find that the zeroth-order metric is an exact solution in the large proper time limit with constraints on the Kasner parameters. The zeroth-order computation agrees with the one-dimensional case. The corresponding Kretshmann scalar has been computed and is found to be regular except for the physical singularity at the origin $u = 0$. We plan to make a detailed analysis of the regularity of the dual geometry as well as the location of the apparent horizon at higher orders in an expansion in $\tau^{-2/3}$ in future.

Though the focus has been on the applications of fluid dynamics near the local equilibrium of the system by using the gradient expansion, it is important to explore whether the applicability can also be extended to fluid dynamics far from local equilibrium [28]. This issue has opened up a new direction called 'resurgence' giving rise to hydrodynamic attractor solutions. Resurgence theory suggests that the gradient series becomes divergent but is Borel summable giving rise to hydrodynamic attractor solutions [29]. It is also applicable to large gradients. It will be interesting to study the attractor solution in the present case of three-dimensional anisotropic expansion of the fluid with second- and higher-order viscous hydrodynamics (see ref. [30] for related discussion). To conclude, the present study of QGP dynamics with second-order relativistic viscous hydrodynamics and anisotropic expansion of the fluid using Kasner space–time is expected to provide a better understanding of the physics of early Universe as well as strongly coupled theories.

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