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Phase equivalent Coulomb-like potential

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Abstract. An equivalent energy-dependent local potential corresponding to Coulomb plus Graz separable potential is constructed through simple rearrangement of the Schrödinger equation. It is conjectured that local Coulomb-like potential is equally applicable for the traditional phase function method. The merit of our constructed potential is thus judged by studying nucleon–nucleon and alpha–nucleon systems through the phase function method. Good agreement in phase shift values with standard data is achieved.

Keywords. Energy-dependent local potential; phase function method; Coulomb-like potential; application.

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1. Introduction

The equivalent potential is an aid to understand the properties of the non-local potential in terms of those concepts which are familiar to the physicists. Since the localisation depends on the solutions of non-local equation, equivalent potential is energy-dependent. In the framework of all partial waves, it is also angular momentum-dependent. This energy-dependent local potential numerically reproduces observables like phase shifts and T-matrices of the parent non-local potential as a function of energy. It also may be useful in determining to what degree the angular momentum barrier exists originally in non-local interaction. Previously, some groups made attempt to construct energy-dependent potential to the non-local potential [1–8]. Coz *et al* [1] have shown a more precise way to obtain an equivalent potential from the two independent solutions of the non-local interaction. They have applied this method to several non-local Hatree–Fock nucleon–nucleus potentials.

Separable interactions have been successfully applied in different areas of a physics such as particle, nuclear and atomic physics. In general, non-local potential is a function of two coordinate variables. In the separable model

$$V_\ell(r, r') = \sum_{i=1}^{N} \lambda_\ell^i \left| g_\ell^i(r) g_\ell^i(r') \right|$$

where λ_ℓ^i and $g_\ell^i(r)$ represent the state-dependent strength parameter and form factor of the potential. Separable potentials have been frequently used in different areas of physics because of its simplicity involved in analytical calculations [9–15]. Among the separable potentials, the Graz model [11–15] provides a reasonable description of the nucleon–nucleon data. The scattering of the particles under the combined influence of electromagnetic plus nuclear force is often studied by the interaction consisting of the sum of a short-range finite rank separable potential and an electromagnetic one. A simple approach for localisation is presented by rearranging the Schrödinger equation for the electromagnetic plus separable interaction. The Coulomb potential is known as the best example of an electromagnetic interaction. But the Coulomb potential is not a well-behaved potential as per the ordinary scattering theory due to logarithmic divergence of the phase shifts. The pure Coulomb potential is an infinitely long-range interaction but in reality, it becomes screened at a certain distance. Thus, people prefer to use screened or cut-off Coulomb interaction to deal with such situations within the framework of ordinary scattering theory. It is argued that the traditional phase function method (PFM) [16] does not hold good for pure Coulomb and Coulomb-like interactions and needs modification [17–19]. As the Coulomb potential turns out to be insignificant after a finite distance, the use of pure Coulomb interaction may be justified in many situations where the scattering takes place under the combined influence of Coulomb-like potentials. For

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brevity, we use the traditional PFM [16] for our energy-dependent local potential to compute scattering phase shifts for some nuclear systems to judge the validity of our conjecture. This is the main motivation of this text. In §2 we address the localisation process for all the partial waves to construct energy-dependent local potentials in terms of the regular solution. Section 3 gives results and discussion while §4 is devoted to conclusion.

2. Localisation process

The radial Schrödinger wave equation for the Coulomb plus Graz separable potential is written as [9–15,20,21]

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-\frac{\ell\,(\ell+1)}{r^2}-\frac{2k\eta}{r}\right\}\psi_\ell(k,r) = \lambda 2^{-2l}\,(\ell!)^{-2}\,r^\ell \mathrm{e}^{-\alpha r} d_\ell(k) \tag{1}$$

with

$$d_\ell(k) = \int_0^\infty r'^\ell \mathrm{e}^{-\beta r'}\psi_\ell(k,r')\mathrm{d}r'. \tag{2}$$

In eqs (1) and (2) the quantities η, α, β and λ stand for the Sommerfeld parameter, inverse range and strength parameters of the Graz separable potential respectively. Rearranging the above equation, one has

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-\frac{\ell\,(\ell+1)}{r^2}\right\}\psi_\ell(k,r) = \left\{\frac{2k\eta}{r}+\frac{1}{\psi_\ell(k,r)}\lambda 2^{-2\ell} \times (\ell!)^{-2}\,r^\ell \mathrm{e}^{-\alpha r} d_\ell(k)\right\}\psi_\ell(k,r). \tag{3}$$

Comparing the above equation to the original eq. (1), one can determine the equivalent local potential present in eq. (3) as

$$V_{\mathrm{EQ}}(k,r) = \frac{2k\eta}{r}+\frac{1}{\psi_\ell(k,r)}\lambda 2^{-2\ell}\,(\ell!)^{-2}\,r^\ell \mathrm{e}^{-\alpha r} d_\ell(k). \tag{4}$$

Within the formalism under regular boundary condition, the regular solution to eq. (1) reads as

$$\phi_\ell(k,r) = \phi_\ell^C(k,r) + \lambda 2^{-2\ell}\,(\ell!)^{-2}\,d_\ell(k) \times \int_0^r \mathrm{e}^{-\alpha r'}(r')^\ell G_\ell^{C(R)}(r,r')\mathrm{d}r'. \tag{5}$$

Multiplying eq. (5) with $r^\ell \mathrm{e}^{-\beta r}$ on both sides and integrating from 0 to ∞ one can find

$$d_\ell(k) = \frac{1}{D_\ell(k)}\int_0^\infty \mathrm{e}^{-\beta r}(r)^\ell \phi_\ell^C(k,r)\mathrm{d}r \tag{6}$$

with

$$D_\ell(k) = 1-\lambda 2^{-2\ell}\,(\ell!)^{-2} \times \int_0^\infty\int_0^r \mathrm{e}^{-\alpha r'}\mathrm{e}^{-\beta r}(r')^\ell(r)^\ell G_\ell^{C(R)}(r,r')\mathrm{d}r'\mathrm{d}r. \tag{7}$$

Regular Coulomb's Green function [22,23] can be evaluated from the relation

$$G_\ell^{C(R)}(r,r') = \frac{1}{\mathfrak{J}_\ell^C(k)} \times \left[\phi_\ell^C(k,r)\,f_\ell^C(k,r') - \phi_\ell^C(k,r')\,f_\ell^C(k,r)\right], \tag{8}$$

where $\phi_\ell^C(k,r)$ and $f_\ell^C(k,r)$ are the regular and the irregular solutions [20–23] for the pure Coulomb potential respectively and are given by

$$\phi_\ell^C(k,r) = r^{\ell+1}\mathrm{e}^{ikr}\Phi(\ell+1+i\eta,2\ell+2;-2ikr), \tag{9}$$

$$f_\ell^C(k,r) = -i(2kr)^{\ell+1}\mathrm{e}^{i(kr-\ell\pi/2)} \times \mathrm{e}^{\pi\eta/2}\Psi(\ell+1+i\eta,2\ell+2;-2ikr) \tag{10}$$

and the Jost function [20–23]

$$\mathfrak{J}_\ell^C(k) = \frac{(2\ell+1)!!}{k^{-\ell}}\mathrm{e}^{i\ell\pi/2}f_\ell^C(k) \tag{11}$$

with

$$f_\ell^C(k) = (2k)^{-\ell}\,\mathrm{e}^{i\ell\pi/2}\mathrm{e}^{\pi\eta/2}\frac{\Gamma(2\ell+2)}{\Gamma(\ell+1+i\eta)}. \tag{12}$$

Combining eqs (9)–(12) and using the following transformation for confluent hypergeometric function [24,25]

$$\Psi(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a,c;x) + \frac{\Gamma(c-1)}{\Gamma(a)}\overline{\Phi}(a,c;x) \tag{13}$$

eq. (8) changes to

$$G_\ell^{C(R)}(r,r') = -\frac{(2ik)^{2\ell+1}}{(2\ell+1)}\left(rr'\right)^{2\ell+1} \times \mathrm{e}^{ikr}\mathrm{e}^{ikr'}\{\Phi(\ell+1+i\eta,2\ell+2;-2ikr) \times \overline{\Phi}(\ell+1+i\eta,2\ell+2;-2ikr') - \overline{\Phi}(\ell+1+i\eta,2\ell+2;-2ikr) \times \Phi(\ell+1+i\eta,2\ell+2;-2ikr')\}. \tag{14}$$

Now the single transformation of Coulomb's Green function with form factor $(r')^\ell \mathrm{e}^{-\alpha r'}$ reads as

$$G_\ell^{C(R)}(\alpha,r) = \int_0^r \mathrm{e}^{-\alpha r'}(r')^\ell G_\ell^{C(R)}(r,r')\mathrm{d}r'. \tag{15}$$

Substitution of eq. (14) along with the help of the expansion of exponential function, eq. (15) converts to

$$\begin{aligned} G_\ell^{C(R)}(\alpha, r) &= \frac{1}{2\ell+1} e^{ikr} r^{\ell+1} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha+ik}{2ik}\right)^n \\ &\times \Big\{ \Phi(\ell+1+i\eta, 2\ell+2; -2ikr) \\ &\times \int_0^r \left(-2ikr'\right)^{n+2\ell+1} e^{2ikr'} \\ &\times \overline{\Phi}(\ell+1+i\eta, 2\ell+2; -2ikr')dr' \\ &- \overline{\Phi}(\ell+1+i\eta, 2\ell+2; -2ikr) \\ &\times \int_0^r \left(-2ikr'\right)^{n+2\ell+1} \\ &\times e^{2ikr'} \Phi(\ell+1+i\eta, 2\ell+2; -2ikr')dr' \Big\}. \end{aligned} \tag{16}$$

Equation (16) is exactly similar to the integral representation of the inhomogeneous confluent hypergeometric function [26] written as

$$\begin{aligned} \Theta_\sigma(a, c; z) &= \frac{1}{c-1} \Big\{ \Phi(a, c; z) \int_0^z (z')^{\sigma+c-2} e^{-z'} \overline{\Phi}(a, c; z')dz' \\ &- \overline{\Phi}(a, c; z) \int_0^z (z')^{\sigma+c-2} e^{-z'} \Phi(a, c; z')dz' \Big\}. \end{aligned} \tag{17}$$

Utilising eq. (17), eq. (16) yields

$$\begin{aligned} G_\ell^{C(R)}(\alpha, r) = -\frac{1}{2ik} e^{ikr} r^{\ell+1} \\ \times \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha+ik}{2ik}\right)^n \\ \times \Theta_\sigma(\ell+1+i\eta, 2\ell+2; -2ikr). \end{aligned} \tag{18}$$

From eq. (18), the double Laplace transformation of the Coulomb's Green function with form factor $(r)^\ell e^{-\beta r}$ reads as

$$G_\ell^{C(R)}(\alpha, \beta) = \int_0^\infty r^\ell e^{-\beta r} G_\ell^{C(R)}(\alpha, r)\, dr. \tag{19}$$

Substitution of eq. (18) in eq. (19), application of the following standard integral [27–30]

$$\begin{aligned} \int_0^\infty e^{-bz} z^{c-1} \Theta_\sigma(a, c; \rho z)dz \\ = \frac{\Gamma(\sigma+c-1)\, \rho^\sigma}{\sigma\, b^{\sigma+c}} {}_2F_1(1, \sigma+a; \sigma+1; \rho/b), \end{aligned} \tag{20}$$

leads to

$$\begin{aligned} G_\ell^{C(R)}(\alpha, \beta) \\ = -\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} \frac{(\alpha+ik)^n\, \Gamma(n+2\ell+2)}{(\beta-ik)^{n+2\ell+3}} \\ \times {}_2F_1\left(1, n+\ell+2+i\eta; n+2; \frac{-2ik}{\beta-ik}\right). \end{aligned} \tag{21}$$

To remove the infinite sum series present in eq. (21) we proceed further by using the following analytic continuation relation of the Gaussian hypergeometric function [27–29]

$$\begin{aligned} {}_2F_1(a, b; c; z) = \frac{\Gamma(c)\, \Gamma(c-a-b)}{\Gamma(c-a)\, \Gamma(c-b)} \\ \times {}_2F_1(a, b; a+b-c+1; 1-z) \\ + (1-z)^{c-a-b} \frac{\Gamma(c)\, \Gamma(a+b-c)}{\Gamma(a)\, \Gamma(b)} \\ \times {}_2F_1(c-a, c-b; c-a-b+1; 1-z). \end{aligned} \tag{22}$$

Using eq. (22) followed by the implication of some general properties for Gaussian hypergeometric function [27–29]

$${}_2F_1(a, b; b; z) = (1-z)^{c-a-b} \tag{23}$$

and

$$\begin{aligned} {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\, \Gamma(b)} \\ \times \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\, \Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \end{aligned} \tag{24}$$

eq. (21) converts to

$$\begin{aligned} G_\ell^{C(R)}(\alpha, \beta) \\ = -\frac{1}{(\ell+1+i\eta)\, (\beta-ik)^{2\ell+3}} \\ \times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(n+2\ell+2) \left(\frac{\alpha+ik}{\beta-ik}\right)^n \\ \times {}_2F_1\left(1, n+\ell+2+i\eta; \ell+2+i\eta; \frac{\beta+ik}{\beta-ik}\right) \\ - \frac{\Gamma(2\ell+2)}{2ik\, (\ell+1+i\eta)\, \left(\beta^2+k^2\right)^{\ell+1}} \left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta} \\ \times {}_2F_1\left(1, 2\ell+2; \ell+2+i\eta; \frac{\alpha+ik}{2ik}\right). \end{aligned} \tag{25}$$

Application of the following relation [27–29] to the ${}_2F_1(*)$ function present in eq. (25)

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b; c; z) \tag{26}$$

and

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1(a,c-b;c;z/z-1) \tag{27}$$

$G_\ell^{C(R)}(\alpha,\beta)$ simplifies to

$$\begin{aligned} G_\ell^{C(R)}(\alpha,\beta) &= \frac{1}{2ik\,(\ell+1+i\eta)\,(\beta-ik)^{2\ell+2}} \\ &\times \sum_{n=0}^{\infty} \frac{\Gamma(n+2\ell+2)}{n!}\left(\frac{\alpha+ik}{2ik}\right)^n \\ &\times {}_2F_1\left(-n,\ell+1+i\eta;\ell+2+i\eta;\frac{\beta+ik}{\beta-ik}\right) \\ &+ \frac{\Gamma(2\ell+2)}{(\alpha-ik)\,(\ell+1+i\eta)\,\left(\beta^2+k^2\right)^{\ell+1}}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta} \\ &\times {}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\frac{\alpha+ik}{\alpha-ik}\right). \end{aligned} \tag{28}$$

The integral representation of the Gaussian hypergeometric function ${}_2F_1(*)$ is written as [27–29]

$$\begin{aligned} {}_2F_1\,(a,b;c;z) &= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \\ &\times \int_0^1 \mathrm{d}t\, t^{b-1}\,(1-t)^{c-b-1}\,(1-tz)^{-a}. \end{aligned} \tag{29}$$

Utilisation of eq. (29) to the first term of eq. (28) along with some algebraic works, one gets

$$\begin{aligned} G_\ell^{C(R)}(\alpha,\beta) &= \frac{\Gamma(2\ell+2)}{2ik\,(\beta-ik)^{2\ell+2}}\left(\frac{-2ik}{\alpha-ik}\right)^{2l+2} \\ &\times \int_0^1 \mathrm{d}t\, t^{\ell+i\eta}\left\{1-t\frac{(\alpha+ik)\,(\beta+ik)}{(\alpha-ik)\,(\beta-ik)}\right\}^{-2\ell-2} \\ &+ \frac{\Gamma(2\ell+2)}{(\alpha-ik)\,(\ell+1+i\eta)\,\left(\beta^2+k^2\right)^{\ell+1}}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta} \\ &\times {}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\frac{\alpha+ik}{\alpha-ik}\right). \end{aligned} \tag{30}$$

Equation (30), in conjunction with eqs (26) and (29) yields

$$\begin{aligned} G_\ell^{C(R)}(\alpha,\beta) &= \frac{\Gamma(2\ell+2)}{(\alpha-ik)\,(\ell+1+i\eta)} \\ &\times \left\{\frac{1}{\left(\beta^2+k^2\right)^{\ell+1}}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta}\right. \\ &\times {}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\frac{\alpha+ik}{\alpha-ik}\right) \\ &- \frac{1}{(\beta-ik)\,(\alpha+\beta)^{2\ell+1}} \\ &\left.\times\, {}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\frac{(\alpha+ik)\,(\beta+ik)}{(\alpha-ik)\,(\beta-ik)}\right)\right\}. \end{aligned} \tag{31}$$

From eqs (7) and (31), the expression for the Fredhom determinant associated with regular solution is obtained as

$$\begin{aligned} D_\ell(k) &= 1-\lambda\frac{2^{-2\ell}\,(\ell!)^{-2}\,\Gamma(2\ell+2)}{(\alpha-ik)\,(\ell+1+i\eta)} \\ &\times \left\{\frac{1}{\left(\beta^2+k^2\right)^{\ell+1}}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta}\right. \\ &\times {}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\frac{\alpha+ik}{\alpha-ik}\right) \\ &- \frac{1}{(\beta-ik)\,(\alpha+\beta)^{2\ell+1}} \\ &\left.\times {}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\frac{(\alpha+ik)\,(\beta+ik)}{(\alpha-ik)\,(\beta-ik)}\right)\right\}. \end{aligned} \tag{32}$$

To find expression for $d_\ell(k)$ one has to substitute eqs (9) and (32) in eq. (6). Application of the following definite integral of confluent hypergeometric function [24,25,30]:

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-\lambda z} z^{\upsilon}\,\Phi(a,c;\rho z)\mathrm{d}z \\ &= \frac{\Gamma(\upsilon+1)\rho^\sigma}{\lambda^{\upsilon+1}}\,{}_2F_1(1,\upsilon+1;c;\rho/\lambda) \end{aligned} \tag{33}$$

along with (23) the resultant expression for $d_\ell(k)$ is obtained as

$$d_\ell(k) = \frac{\Gamma(2\ell+2)}{D_\ell(k)\,(\beta-ik)^{2\ell+2}}\left(\frac{\beta-ik}{\beta+ik}\right)^{\ell+1+i\eta}. \tag{34}$$

Substituting eqs (9), (18) and (34) in eq. (5) one gets the required regular solution to eq. (1) as

$$\begin{aligned} \phi_\ell(k,r) &= \phi_\ell^C(k,r) \\ &- \lambda\frac{2^{-2l}\,(\ell!)^{-2}\,\Gamma(2\ell+2)}{(2ik)\,D_\ell(k)\,(\beta-ik)^{2\ell+2}}\left(\frac{\beta-ik}{\beta+ik}\right)^{\ell+1+i\eta} \\ &\times \mathrm{e}^{ikr} r^{\ell+1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha+ik}{2ik}\right)^n \\ &\times \Theta_\sigma(\ell+1+i\eta,2\ell+2;-2ikr). \end{aligned} \tag{35}$$

Finally, eqs (9), (32), (34) and (35) together with eq. (4) produce the desired expression for the equivalent local-potential $V_{\mathrm{EQ}}(k,r)$ for the Coulomb-modified Graz separable non-local interaction.

3. Results and discussion

For scattering phase shift computation, we find it more convenient to work with phase function method (PFM). To compute scattering phase shifts for quantum mechanical problems involving local and non-local interactions one can easily rely on this efficient and simple approach. In this method, one can calculate phase shifts for various nuclear systems under consideration by applying the phase equation given by Calogero [16]

$$\delta'_\ell(k,r) = -k^{-1}V_\ell(k,r) \times\left[\hat{J}_\ell(kr)\cos\delta_\ell(k,r) - \hat{\eta}_\ell(kr)\sin\delta_\ell(k,r)\right]^2. \quad (36)$$

Exploiting eqs (4) and (36) for $\ell = 0$, 1 and 2, we have computed scattering phase shifts for p–p and α–p systems using the parameters given in tables 1 and 2 and compared our results with the standard data. Considering $\alpha = \beta$ all numerical calculations are carried out by using $\frac{\hbar^2}{m_p} = 41.47$ MeV fm^2 and $2k\eta = 0.0347\,\text{fm}^{-1}$ and $0.1117\,\text{fm}^{-1}$ for p–p and α–p systems respectively. We have also studied n–p and α–n systems by turning off the Coulomb part in our numerical routine along with the respective parameters listed in tables. Phase shifts for different states along with standard values [32,33] are plotted in figures 1–8. Scattering phase shifts for n–p and p–p systems are depicted in figures 1–4 respectively with energies up to $E_{\text{Lab}} = 100\,\text{MeV}$ and for α–n and α–p systems we portray phase shifts with energies up to $E_{\text{Lab}} = 16\,\text{MeV}$ in figures 5–8. In figure 1

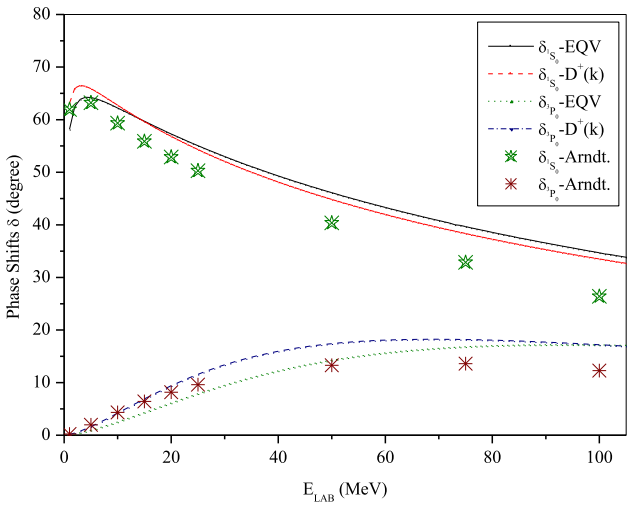

Figure 1. n–p phase shifts for $^1S_0$ and $^3P_0$ states as a function of energy.

our computed n–p phase shifts for $^1S_0$ state are in reasonable agreement with Arndt *et al* [32] up to 20 MeV and beyond that slightly larger values are reproduced. Between the two sets of phase-shift values, the values with equivalent local potential are in better agreement, particularly at the peak of the phase shifts, with those of ref. [32]. Similar results are also observed for $^3P_0$ n–p state. For $^3P_2$ and $^1D_2$ n–p states, as shown in figure 2, it is noticed that energy-dependent local potentials are superior to their energy-independent counterparts. The phase shifts for the p–p system, portrayed in figures 3 and 4 follow the same trend as that of the n–p system.

Looking closely into figures 5–8, it is seen that better results are achieved for equivalent local potentials than the Coulomb plus non-local one for all the partial wave states with those of Satchler *et al* [33] at all energies under consideration except for $1/2^-$ α–p states. The $1/2^-$ α–p state phase shifts have slightly lower values up to 7 MeV. Although small differences in phase values between equivalent local and Coulomb-modified non-local treatments are observed for $1/2^-$ α–p state, they follow the correct trends of the phase shifts [33]. Also,

Table 1. Parameters for n–p/p–p system [2].

State	λ (fm$^{-2\ell-3}$)	β (fm^{-1})
$^1S_0$	−5.237	1.4054
$^3P_0$	−18.5	1.36
$^3P_2$	−66.5	1.89
$^1D_2$	−110	1.45

Table 2. Parameters for α–n and α–p systems [31].

States	Alpha–neutron system		Alpha–proton system	
	λ (MeV fm$^{-2\ell-3}$)	β (fm^{-1})	λ (MeV fm$^{-2\ell-3}$)	β (fm^{-1})
$1/2^+$	−9.995	1.2	−21.56	1.3
$1/2^-$	−25.28	1.2	−37.28	1.3
$3/2^-$	−36.5	1.2	−76.2	1.4
$3/2^+$	−27.2	1.2	−42.7	1.3
$5/2^+$	−40.2	1.2	−58.7	1.3

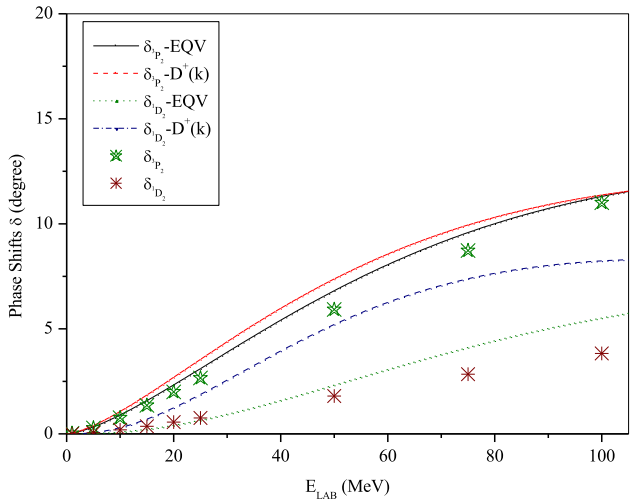

Figure 2. n–p phase shifts for $^3P_2$ and $^1D_2$ states as a function of energy.

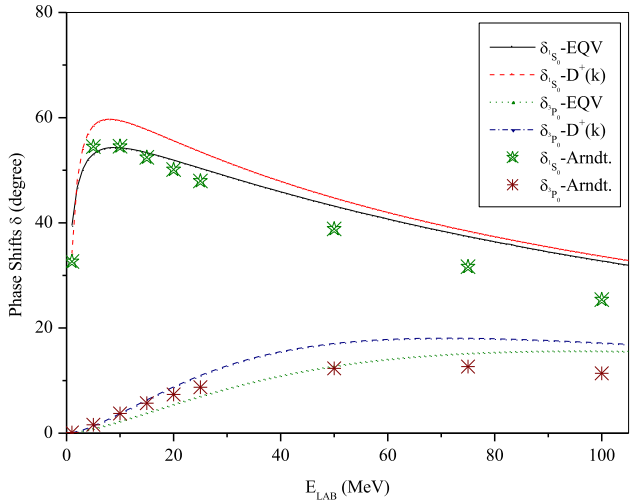

Figure 3. p–p phase shifts for $^1S_0$ and $^3P_0$ states as a function of energy.

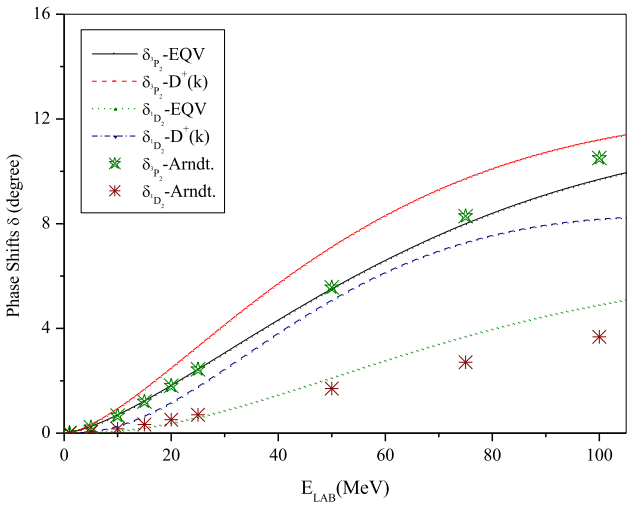

Figure 4. p–p phase shifts for $^3P_2$ and $^1D_2$ states as a function of energy.

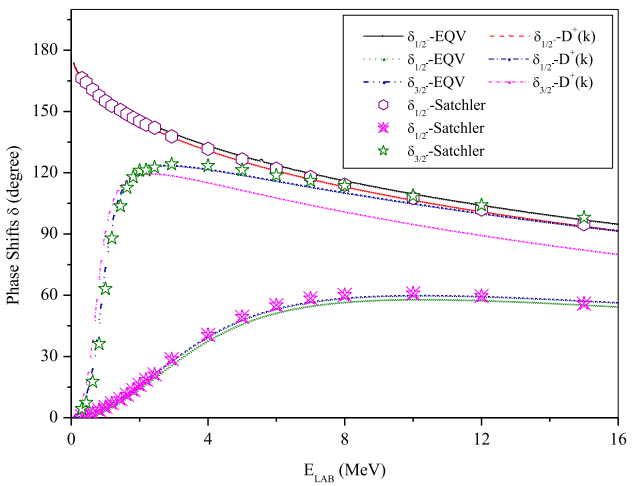

Figure 5. Phase shifts for $1/2^+$, $1/2^-$ and $3/2^-$ states of α–n system as a function of E_{LAB}.

it is worthwhile to mention that our results for the α–nucleon systems are at par with the theoretical analysis of the previous works [6,34–46].

We have also plotted the nature of energy-dependent potentials for the p–p and α–p systems at different energies in figures 9–12. The generated energy-dependent potentials exhibit finite discontinuities at certain points within their ranges. The sharp peaks vary from −15000 to 400 MeV for the p–p system and from −4000 to 7000 MeV for the α–p system. However, the depths of the finite sharp peaks are found to be smaller at higher momentum states. These finite discontinuities are observed in potentials due to the behaviour of regular solution with distance. As energy increases, sharp peaks in the potential occur at smaller values of r. It is also observed that for higher angular momentum states, peaks arise comparatively at larger distances. However, for sufficiently large ℓ these abrupt changes are not observed within the ranges of the respective potentials. Hence, for the states $^1D_2$, $3/2^+$ and $5/2^+$ potentials vary smoothly within their range of interactions. We have also verified that these forgoing discussions are equally applicable to local potentials generated for n–p and α–n systems which are not shown in the figures.

4. Conclusion

In this paper, we have localised the Coulomb plus Graz separable non-local interaction by rearranging the

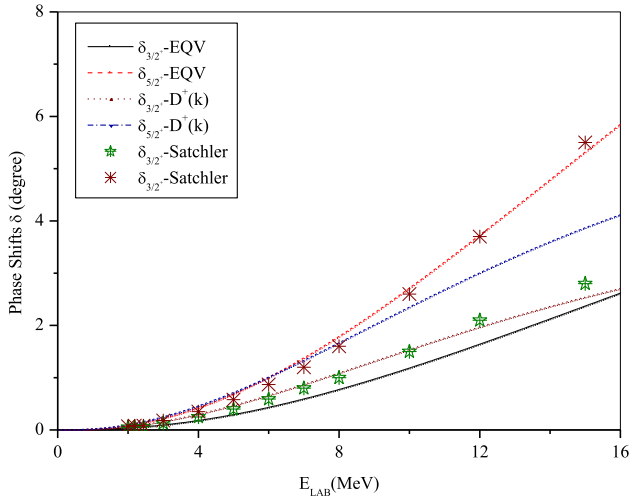

Figure 6. Phase shifts for $3/2^+$ and $5/2^+$ states of α–n system as a function of E_{LAB}.

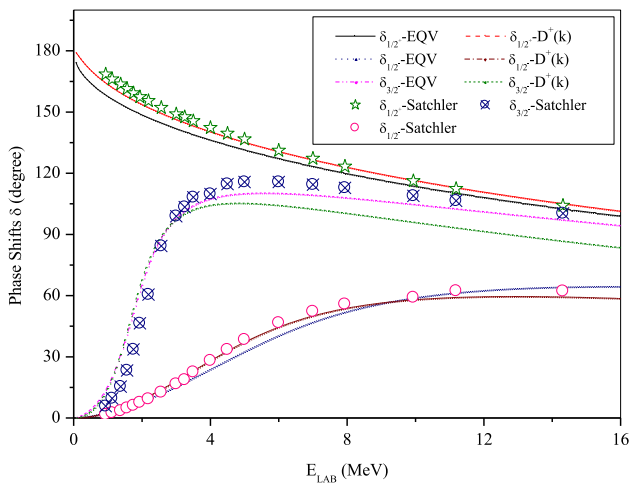

Figure 7. Phase shifts for $1/2^+$, $1/2^-$ and $3/2^-$ states of α–p system as a function of E_{LAB}.

Schrödinger wave equation using regular boundary condition. One may use the irregular boundary condition to get a complex equivalent potential as the irregular solution is a complex quantity [6]. The real phase shifts are obtained from the real part of the potential as discussed in ref. [6]. The generated equivalent local potential produces better results than the Coulomb-modified nonlocal interaction for different nuclear systems under considerations which are in reasonable agreement with those of refs [6,31–46]. The Graz separable potential also has the ability to reproduce nucleus–nucleus scattering data quite efficiently. We have studied α–carbon and α–α systems by replacing the nuclear part of the interaction by the Graz separable one and obtained good fit to the experimental data [47,48]. Thus, the results of the present work and those of ref. [47] establish the

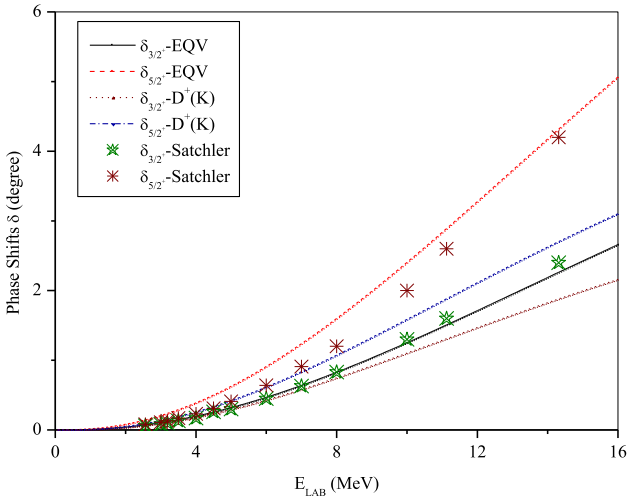

Figure 8. Phase shifts for $3/2^+$ and $5/2^+$ states of α–p system as a function of E_{LAB}.

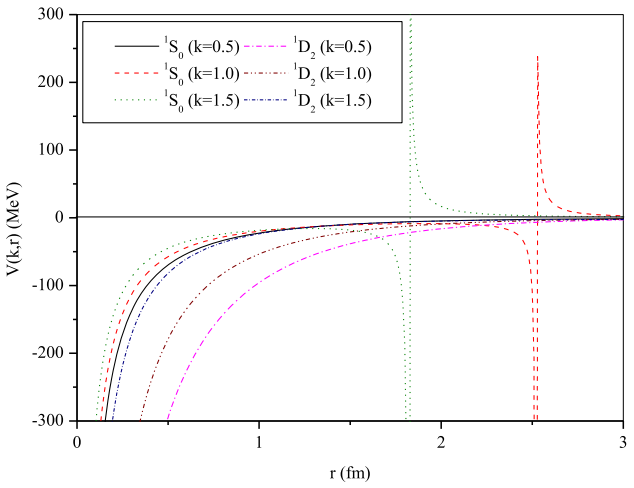

Figure 9. p–p potential for $^1S_0$ and $^1D_2$ states as a function of r.

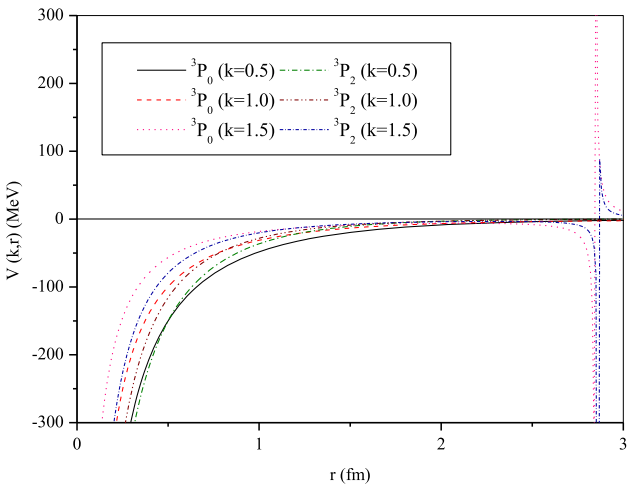

Figure 10. p–p potential for $^3P_0$ and $^3P_2$ states as a function of r.

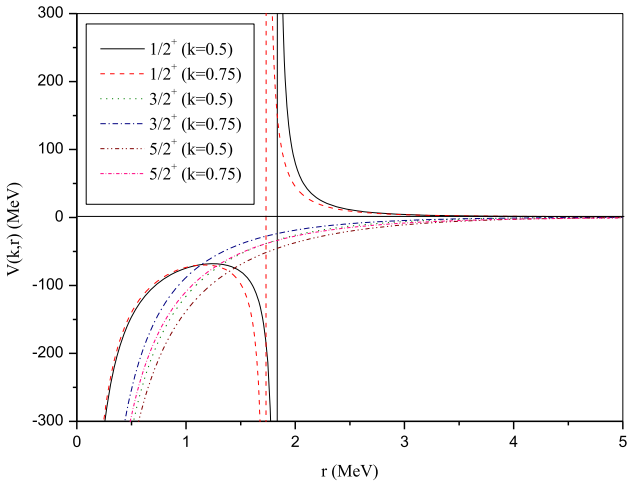

Figure 11. α–p potential for $1/2^+$, $3/2^+$ and $5/2^+$ states as a function of r.

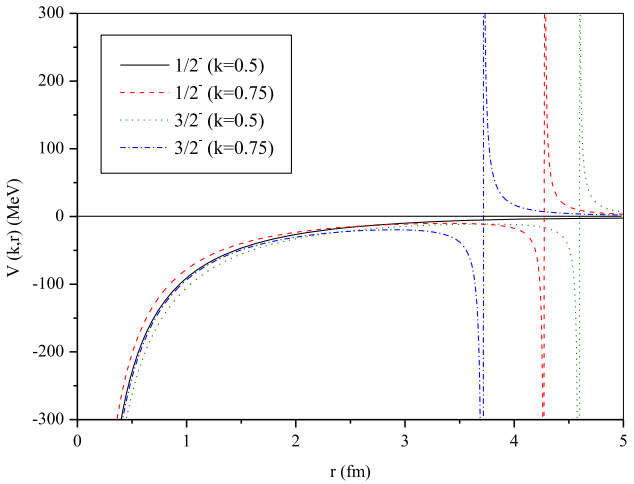

Figure 12. α–p potential for $1/2^-$ and $3/2^-$ states as a function of r.

usefulness of separable representations of the nuclear part of interaction without any doubt. From this simple model calculation, one may infer that the non-central potential dominates over the central one in nuclear systems. Although some finite discontinuities appear in the potentials, the computed phase shifts are found in order. This is due to the fact that the resultant contribution to scattering phase shifts from either side of the point of finite discontinuities in the related potentials is of definite values. It is found that at higher partial waves, those finite discontinuities arising in the generated local potentials with regular solution are likely to appear less within the range of interactions. From our observations in this paper, it is conclusively proved that our initial conjecture of using Coulomb-like local potential in the traditional phase function method is correct. Thus, it is concluded that the use of pure Coulomb or Coulomb-like potential in the traditional phase function method is justified in many situations where the Coulomb potential is in fact always screened.

References

[1] M Coz, L G Arnold and A D MacKellar, *Ann. Phys. (NY)* **59**, 219 (1970)
[2] L G Arnold and A D MacKellar, *Phys. Rev. C* **3**,1095 (1971)
[3] J P McTavish, *J. Phys. G* **8**, 1037 (1982)
[4] B Talukdar, G C Sett and S R Bhattaru, *J. Phys. G* **11**, 591 (1985)
[5] G C Sett, U Laha and B Talukdar, *Pramana – J. Phys.* **28**, 325 (1987)
[6] U Laha, S K Das and J Bhoi, *Turk. J. Phys.* **41**, 447 (2017)
[7] U Laha, *J. Kor. Phys. Soc.* **75**, 935 (2019)
[8] A K Behera, U Laha and J Bhoi, *Turk. J. Phys.* **44**, 229 (2020)
[9] B Talukdar, D K Ghosh and T Sasakawa, *J. Math. Phys.* **23**, 1700 (1982)
[10] H van Haeringen and R van Wageningen, *J. Math. Phys.* **16**, 1441 (1975)
[11] L Cripinsek, C B Lang, H Oberhummer, W Plessas and H F K Zingl, *Acta Phys. Austriaca* **42**, 139 (1975)
[12] W Schweiger, W Plessas, L P Kok and H van Haeringen, *Phys. Rev. C* **27**, 515 (1983)
[13] J Haidenbauer and W Plessas, *Phys. Rev. C* **30**, 1822 (1984)
[14] B Talukdar, U Laha and T Sasakawa, *J. Math. Phys.* **27**, 2080 (1986)
[15] U Laha, B J Roy and B Talukdar, *J. Phys. A* **22**, 3597 (1989)
[16] F Calogero, *Variable phase approach to potential scattering* (Academic, New York, 1967)
[17] G C Sett, U Laha and B Talukdar, *J. Phys. A* **21**, 3643 (1988)
[18] U Laha, N Haque, T Nandi and G C Sett, *Z. Phys. A At. Nucl.* **332**, 305 (1989)
[19] U Laha, A K Jana and T Nandi, *Pramana – J. Phys.* **37**, 387 (1991)
[20] U Laha and J Bhoi, *J. Math. Phys.* **54**, 013514 (2013)
[21] U Laha and J Bhoi, *Phys. Rev. C* **88**, 064001 (2013)
[22] R G Newton, *Scattering theory of waves and particles* (McGraw-Hill, New York, 1982)
[23] J R Taylor, *Scattering theory: The quantum theory of non-relativistic collisions* (John Wiley & Sons, Inc, New York, 1972)
[24] L J Slater, *Confluent hypergeometric functions* (Cambridge University Press, New York, 1960)

[25] H Buchholz, *The confluent hypergeometric function* (Springer, New York, 1969)
[26] A W Babister, *Transcendental functions satisfying non-homogeneous linear differential equations* (The MacMillan Company, New York, 1967)
[27] A Erdeyli, *Higher transcendental functions* (McGraw-Hill, New York, 1953) Vol. 1
[28] W Magnus and F Oberhettinger, *Formulas and theorems for the special functions of mathematical physics* (Chelsea, New York, 1949)
[29] L J Slater, *Generalized hypergeometric functions* (Cambridge University Press, Cambridge, 1966)
[30] I S Gradshteyn and I M Ryzhik, *Tables of integrals, series and products* (Academic Press, London, 2000)
[31] J Bhoi and U Laha, *Pramana – J. Phys.* **91**: 77 (2018)
[32] R A Arndt, L D Roper, R A Bryan, R B Clark, B J Ver West and P Signell, *Phys. Rev. D* **28**, 97 (1983)
[33] G R Satchler, L W Owen, A J Elwin, G L Morgan and R L Walter, *Nucl. Phys. A* **112**, 1 (1968)
[34] P Swan, *Proc. Roy. Soc.* **228**, 10 (1955)
[35] E Van der Spuy, *Nucl. Phys.* **1**, 381 (1956)
[36] J Gammel and R Thaler, *Phys. Rev.* **109**, 2041 (1958)
[37] A M Mitra, V S Bhasin and B S Bhakar, *Nucl. Phys.* **38**, 316 (1962)
[38] J Pigeon, J Barguil, C Fayard, G H Lamot and E El Baz, *Nucl. Phys. A* **145**, 319 (1970)
[39] S Ali, M Rahaman and D Hussain, *Phys. Rev. D* **6**, 1178 (1972)
[40] S Ali, M Rahaman and D Hussain, *Phys. Rev. C* **9**, 1657 (1974)
[41] A A Z Ahmad, S Ali, N Ferdous and M Ahmed, *Nuovo Cimento A* **30**, 385 (1975)
[42] G Cattapan, G Pisent and V Vanzani, *Nucl. Phys. A* **241**, 204 (1975)
[43] A K Rafiquallah, S Hossain, N Chowdhury, A Begum and N Ferdous, *Daccan Univ. Stud. B* **23**, 5 (1975)
[44] C L Lee and D Robson, *Nucl. Phys. A* **379**, 11 (1982)
[45] J Dohet-Eraly and D Baye, *Phys. Rev. C* **84**, 014604 (2011)
[46] U Laha and J Bhoi, *Phys. Rev. C* **91**, 034614 (2015)
[47] A K Behera, U Laha, M Majumder and J Bhoi, *J. Kor. Phys. Soc.* **74**, 428 (2019)
[48] A K Behera, B Khirali, U Laha and J Bhoi, *Theor. Math. Phys.* **205**, 1353 (2020)