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Fredholm determinants for the Hulthén-distorted separable potential

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Abstract. By exploiting higher partial wave solutions for the Hulthén potential, constructed via the factorisation method, closed form analytical expressions of the Fredholm determinants for motion in Hulthén plus modified Graz separable potential are constructed to study on-shell scattering up to partial wave $\ell = 2$. Phase shifts for different states of α-^{3}H and α-^{3}He are obtained by exploiting the expression of the Fredholm determinant. The results are found in reasonable agreement with the standard data (Spiger and Tombrello 1967).

Keywords. Hulthén plus Graz potential; Fredholm determinants; scattering phase shifts; nucleus–nucleus systems.

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1. Introduction

The asymptotic condition for the well-behaved potential does not hold good and as a consequence the concept of phase shift is ill defined for Coulomb scattering [1–3]. Therefore, in many situations the electromagnetic part of the interaction is described by a relatively short-range screened Coulomb potential. The Hulthén potential [4] is a famous example of the exponentially screened Coulomb potential. We have applied the Hulthén potential successfully to compute scattering phase shifts for various nuclear systems [5–10]. The Hulthén potential is exactly solvable for s-wave only. However, the Hulthén potential with $\ell > 0$ has been solved with some approximation techniques and published in a number of publications [11–16]. Considering the s-wave solutions as the basic inputs we shall derive scattering state solutions and the Fredholm determinants for Hulthén plus Graz separable [17] potential up to partial wave $\ell = 2$ by exploiting the technique of supersymmetric algebra [18–23]. The expressions for the Fredholm determinants will be used to compute scattering phase shifts for nucleus–nucleus systems to judge the merit of our approach. The present paper is an effort in this direction. In §2 we construct analytical expressions for the Fredholm determinants by exploiting higher partial wave solutions of the pure Hulthén potential obtained from supersymmetry algebra. Section 3 is related to the results and §4 gives summary and conclusion.

2. Supersymmetry and Fredholm determinants

One of us (UL) derived a simple factorisation method [24] for differential equations satisfied by hypergeometric functions. This has been achieved by constructing a pair of first-order differential operators obtained from near the origin behaviour of the associated functions. These operators are seen to be very similar to those obtained in supersymmetric algebra [18–23]. Using this formalism higher partial wave solutions, in the positive energy state, for the Hulthén potential have been constructed by Laha *et al* [21]. These approximate higher partial wave solutions will be exploited to derive analytical expressions for the Fredholm determinants for the Hulthén plus Graz [17] separable potential. The Graz separable potential is expressed as

$$V(r, r') = \lambda_\ell g_\ell(r)\, g_\ell(r') \tag{1}$$

with

$$g_\ell(r') = 2^{-\ell}(\ell\,!)^{-1} r'^{\ell}\, \mathrm{e}^{-\beta_\ell r'} \tag{2a}$$

and

$$g_\ell(r) = 2^{-\ell}(\ell\,!)^{-1} r^{\ell}\, \mathrm{e}^{-\alpha_\ell r}. \tag{2b}$$

For $\ell > 0$, however, we approximate the form factor of the Graz potential as

$$g_\ell(r') = 2^{-\ell}(\ell\,!)^{-1} a^{\ell}\left(1 - \mathrm{e}^{-r'/a}\right)^{\ell} \mathrm{e}^{-\beta_\ell r'} \tag{3a}$$

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and

$$g_\ell(r) = 2^{-\ell}(\ell\,!)^{-1}a^\ell\left(1-e^{-r/a}\right)^\ell \mathrm{e}^{-\alpha_\ell r} \tag{3b}$$

to derive closed form expressions for the Fredholm determinants up to partial waves $\ell = 2$.

The s-wave regular and irregular solutions for the Hulthén potential [21] are written as

$$\varphi_{\mathrm{H}0}(k,r) = a\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/a}\right){}_2F_1(A+1,B+1;\,2;\,1-\mathrm{e}^{-r/a}) \tag{4}$$

and

$$f_{\mathrm{H}0}(k,r) = \mathrm{e}^{ikr}{}_2F_1(A,B;C;\mathrm{e}^{-r/a}) \tag{5}$$

respectively and the Jost function reads as

$$f_{\mathrm{H}0}(k) = \frac{\Gamma(C)}{\Gamma(1+A)\Gamma(1+B)} \tag{6}$$

with

$$A = -iak + ia\left(k^2+V_0^2\right)^{1/2}, \tag{7}$$

$$B = -iak - ia\left(k^2+V_0^2\right)^{1/2} \tag{8}$$

and

$$C = 1-2iak. \tag{9}$$

With the knowledge of the regular and the irregular solutions as well as the Jost function, one can construct the physical Green's function as [1,25–28]

$$G_{\mathrm{H}}^{(+)}(r,r') = -\frac{\varphi_{\mathrm{H}}(k,r_<)f_{\mathrm{H}}(k,r_>)}{\Im_{\mathrm{H}}(k)}. \tag{10}$$

Now utilising eqs (4)–(6), eq. (10) can be expressed in terms of regular Green's function $G_{\mathrm{H}0}^{(+)}(r,r')$, as [29]

$$G_{\mathrm{H}0}^{(+)}(r,r') = G_{\mathrm{H}0}^{(R)}(r,r') - \frac{1}{f_{\mathrm{H}0}(k)}a\mathrm{e}^{ik(r+r')} \times\left(1-\mathrm{e}^{-r/a}\right){}_2F_1(A+1,B+1;2;1-\mathrm{e}^{-r/a}) \times{}_2F_1(A,B;C;\mathrm{e}^{-r'/a}), \tag{11}$$

where

$$G_{\mathrm{H}0}^{(R)}(r,r') = \frac{1}{f_{\mathrm{H}0}(k)}a\mathrm{e}^{ik(r+r')} \times\Big\{\left(1-\mathrm{e}^{-r/a}\right){}_2F_1(A+1,B+1;2;1-\mathrm{e}^{-r/a}) \times{}_2F_1(A,B;C;\mathrm{e}^{-r'/a}) - {}_2F_1(A,B;C;\mathrm{e}^{-r/a}) \times(1-\mathrm{e}^{-r'/a}){}_2F_1(A+1,B+1;2;1-\mathrm{e}^{-r'/a})\Big\}. \tag{12}$$

With the help of the regular boundary condition, the single Laplace transformation of eq. (11) with the form factor of the s-wave Graz separable potential $g(r') = \mathrm{e}^{-\beta r'}$ can be expressed as

$$\begin{aligned} G_{\mathrm{H}0}^{(+)}(r,\beta) &= \int_0^\infty G_{\mathrm{H}0}^{(+)}(r,r')\mathrm{e}^{-\beta r'}\mathrm{d}r' \\ &= \int_0^r G_{\mathrm{H}0}^{(R)}(r,r')\mathrm{e}^{-\beta r'}\mathrm{d}r' \\ &\quad -a\left(1-\mathrm{e}^{-r/a}\right)\mathrm{e}^{ikr}\frac{1}{f_{\mathrm{H}0}(k)} \\ &\quad \times{}_2F_1(A+1,B+1;2;1-\mathrm{e}^{-r/a}) \\ &\quad \times\int_0^\infty \mathrm{e}^{-(\beta-ik)r'}{}_2F_1(A,B;C;\mathrm{e}^{-r'/a})\mathrm{d}r'. \end{aligned} \tag{13}$$

To simplify eq. (13), one has to solve the integrations present in the above equation. The integration over the whole space can be evaluated easily by utilising the following standard integral relation [30–32]:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}{}_2F_1(\alpha,\beta;\gamma;x)\mathrm{d}x = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;1). \tag{14}$$

However, to solve the integration involving regular boundary condition, one has to apply the following analytic continuation formula [30–32] in eq. (12):

$$\begin{aligned} {}_2F_1(A,B;C;Z) &= \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} \\ &\quad\times{}_2F_1(A,B;A+B-C+1;1-Z) \\ &\quad+(1-Z)^{C-A-B}\frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} \\ &\quad\times{}_2F_1(C-A,C-B;C-A-B+1;1-Z) \end{aligned} \tag{15}$$

to get

$$\begin{aligned} G_{\mathrm{H}0}^{(R)}(r,r') &= \lim_{\epsilon\to 0} a\mathrm{e}^{ik(r+r')}\Big\{\left(1-\mathrm{e}^{-r/a}\right) \\ &\quad\times{}_2F_1(A+1,B+1;2;1-\mathrm{e}^{-r/a}) \\ &\quad\times{}_2F_1(A,B;\epsilon;1-\mathrm{e}^{-r'/a}) \\ &\quad-\left(1-\mathrm{e}^{-r'/a}\right){}_2F_1(A+1,B+1;2;1-\mathrm{e}^{-r'/a}) \\ &\quad\times{}_2F_1(A,B;\epsilon;1-\mathrm{e}^{-r/a})\Big\}. \end{aligned} \tag{16}$$

Substituting eq. (16) in eq. (13) and using the following standard integral relation [33,34]

$$f_\sigma(a,b;c;z) = \frac{1}{c-1}\Big\{{}_2F_1(a,b;c;z) \times \int_0^z s^{\sigma-1}(1-s)^{a+b-c} \times {}_2F_1(a-c+1,b-c+1;2-c;s)\mathrm{d}s - z^{1-c}{}_2F_1(a-c+1,b-c+1;2-c;z) \times \int_0^z s^{\sigma+c-2}(1-s)^{a+b-c}{}_2F_1(a,b;c;s)\mathrm{d}s\Big\} \quad (17)$$

along with the help of eq. (14), eq. (13) simplifies to

$$G_{\mathrm{H0}}^{(+)}(r,\beta) = a\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/a}\right) \times \Big\{a\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta+ik)a)}{\Gamma(1-(\beta+ik)a)}\frac{1}{n!} \times f_{n+1}(A+1,B+1;2;1-\mathrm{e}^{-r/a}) - \frac{1}{f_{\mathrm{H0}}(k)(\beta-ik)}\,{}_2F_1(A+1,B+1;2;1-\mathrm{e}^{-r/a}) \times {}_3F_2(A,B,(\beta-ik)a;C,1+(\beta-ik)a;1)\Big\}. \quad (18)$$

From eq. (18) the double transformation of the physical Green's function with form factor of the Yamaguchi potential $g(r) = \mathrm{e}^{-\alpha r}$ reads as

$$G_{\mathrm{H0}}^{(+)}(\alpha,\beta) = a^2\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta+ik)a)}{\Gamma(1-(\beta+ik)a)}\frac{1}{n!} \times \int_0^\infty \mathrm{e}^{-(\alpha-ik)r} f_{n+1}(A+1,B+1;2;1-\mathrm{e}^{-r/a}) \times \left(1-\mathrm{e}^{-r/a}\right)\mathrm{d}r - a\frac{1}{f_{\mathrm{H0}}(k)(\beta-ik)} \times {}_3F_2(A,B,(\beta-ik)a; C,1+(\beta-ik)a;1)\int_0^\infty \mathrm{e}^{-(\alpha-ik)r}\left(1-\mathrm{e}^{-r/a}\right) \times {}_2F_1(A+1,B+1;2;1-\mathrm{e}^{-r/a})\mathrm{d}r. \quad (19)$$

Here eq. (19) involves integral expression containing both homogeneous and non-homogeneous Gaussian hypergeometric functions. The integration containing homogeneous hypergeometric function can be evaluated by using eq. (14) whereas for the non-homogeneous function the following standard integral formula [33] is used:

$$\int_0^1 z^{c-1}(1-z)^{\upsilon-1}f_\sigma(a,b;c;pz)\mathrm{d}z = \frac{\Gamma(\sigma+c-1)\Gamma(\upsilon)}{\Gamma(\sigma+c+\upsilon-1)}f_\sigma(a,b;c+\upsilon;p). \quad (20)$$

In addition to eqs (14) and (20), use of the following hypergeometric transformations [30–33]

$$f_\sigma(a,b;c;z) = \frac{z^\sigma}{\sigma(\sigma+c-1)} \times {}_3F_2(1,\sigma+a,\sigma+b;\sigma+1,\sigma+c;z) \quad (21)$$

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (22)$$

change eq. (19) to

$$G_{\mathrm{H0}}^{(+)}(\alpha,\beta) = a^3\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta+ik)a)\Gamma((\alpha-ik)a)}{\Gamma(1-(\beta+ik)a)\Gamma(n+3+(\alpha-ik)a)} \times {}_3F_2(1,n+A+2,n+B+2;n+2,n+3+(\alpha-ik)a;1) - a^2\frac{1}{f_{\mathrm{H0}}(k)(\beta-ik)} \times \frac{\Gamma((\alpha-ik)a)\Gamma((\alpha+ik)a)}{\Gamma(1+(\alpha-ik)a-A)\Gamma(1+(\alpha-ik)a-B)} \times {}_3F_2(A,B,(\beta-ik)a;C,1+(\beta-ik)a;1). \quad (23)$$

The Fredholm determinant $D^{(+)}(k)$ for the physical solution is generally expressed by $G_{\mathrm{H}}^{(+)}(\alpha,\beta)$ which has very important application to nuclear scattering theory. Therefore, to make eq. (23) helpful for the numerical treatment, it is necessary to remove the infinite summation series present in the above equation.

For this, we proceed by applying the following analytic continuation [34]:

$${}_3F_2(a,b,c;e,f;1) = \frac{\Gamma(e)\Gamma(e-a-b)}{\Gamma(e-a)\Gamma(e-b)} \times {}_3F_2(a,b,f-c;a+b\text{-}e+1,f;1) + \frac{\Gamma(e)\Gamma(f)\Gamma(a+b-e)\Gamma(e+f-a-b-c)}{\Gamma(a)\Gamma(b)\Gamma(f-c)\Gamma(e+f-a-b)} \times {}_3F_2(e-a,e-b,e+f-a-b-c; e-a-b+1,\mathrm{e}+f-a-b;1) \quad (24)$$

to the hypergeometric series present in the first term of eq. (23). Using eq. (24) along with the help of eq. (22)

and the general series expansion of a ${}_3F_2(*)$ hypergeometric function [33]

$$
{}_3F_2(a,b,c;e,f;z) = \frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(b)\Gamma(c)} \times \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+n)\Gamma(f+n)} \frac{z^n}{n!}, \tag{25}
$$

eq. (23) simplifies to

$$
\begin{aligned}
G_{\mathrm{H0}}^{(+)}(\alpha,\beta) &= -a^3 \frac{\Gamma((\alpha-ik)a)}{(A+1)\Gamma(1-(\beta+ik)a)} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-(\beta+ik)a)(n+1)}{\Gamma(n+3+(\alpha-ik)a)} \\
&\times {}_3F_2(1, n+A+2, 1+(\alpha-ik)a-B; A+2, n+3+(\alpha-ik)a; 1) \\
&+a^3 \frac{\Gamma((\alpha-ik)a)\Gamma((\alpha+ik)a)}{(A+1)(B+1)\Gamma(1+(\alpha-ik)a-A)\Gamma(1+(\alpha-ik)a-B)} \\
&\times {}_3F_2(1,2,1-(\beta+ik)a; A+2, 2-A-2iak; 1) - a^2 \frac{1}{f_{\mathrm{H0}}(k)(\beta-ik)} \\
&\times \frac{\Gamma((\alpha-ik)a)\Gamma((\alpha+ik)a)}{\Gamma(1+(\alpha-ik)a-A)\Gamma(1+(\alpha-ik)a-B)} {}_3F_2(A,B,(\beta-ik)a; C, 1+(\beta-ik)a; 1).
\end{aligned} \tag{26}
$$

Now, by applying the following transformation [33] to the ${}_3F_2$ function present in the second term of eq. (26)

$$
\begin{aligned}
{}_3F_2(a,b,c;e,f;1) &= \frac{\Gamma(s)\Gamma(f)\Gamma(e)}{\Gamma(a)\Gamma(s+c)\Gamma(s+b)} \\
&\times {}_3F_2(s, e-a, f-a; s+b, s+c; 1); \\
s &= e+f-a-b-c
\end{aligned} \tag{27}
$$

$G_{\mathrm{H0}}^{(+)}(\alpha,\beta)$ becomes

$$
\begin{aligned}
G_{\mathrm{H0}}^{(+)}(\alpha,\beta) &= -a^3 \frac{\Gamma((\alpha-ik)a)}{(A+1)\Gamma(1-(\beta+ik)a)} \\
&\times \sum_{n=0}^{\infty} \frac{\Gamma(n+1-(\beta+ik)a)(n+1)}{\Gamma(n+3+(\alpha-ik)a)} \\
&\times {}_3F_2(1, n+A+2, 1+(\alpha-ik)a-B; \\
&A+2, n+3+(\alpha-ik)a; 1).
\end{aligned} \tag{28}
$$

Then proceeding with the following hypergeometric transformation [33]

$$
\begin{aligned}
&{}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;1) \\
&= \frac{\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)} \\
&\times {}_3F_2(\beta_1-\alpha_1, \beta_2-\alpha_2, \alpha_3; \\
&\beta_1, \beta_1+\beta_2-\alpha_1-\alpha_2; 1),
\end{aligned} \tag{29}
$$

eq. (28) converts to

$$
\begin{aligned}
G_{\mathrm{H0}}^{(+)}(\alpha,\beta) &= -a^3 \frac{\Gamma((\alpha-ik)a)}{(A+1)(B+1)\Gamma(1-(\beta+ik)a)} \\
&\times \sum_{n=0}^{\infty} \frac{\Gamma(n+1-(\beta+ik)a)(n+1)}{\Gamma(n+2+(\alpha-ik)a)} \\
&\times {}_3F_2(-n, 1-(\alpha+ik)a, 1; A+2, B+2; 1).
\end{aligned} \tag{30}
$$

Expanding eq. (30) and rearranging the terms suitably along with the help of the following general properties of Gaussian hypergeometric series [30–34]

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}, \tag{31}
$$

$$
{}_4F_3(a,b,c,d;e,f,g;z) = \frac{\Gamma(e)\Gamma(f)\Gamma(g)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} \times \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)\Gamma(d+n)}{\Gamma(e+n)\Gamma(f+n)\Gamma(g+n)} \frac{z^n}{n!} \tag{32}
$$

and eq. (22), eq. (30) changes to

$$
\begin{aligned}
G_{\mathrm{H0}}^{(+)}(\alpha,\beta) &= -a^3 \frac{\Gamma((\alpha+\beta)a-1)}{(A+1)(B+1)\Gamma((\alpha+\beta)a+1)} \\
&\times {}_4F_3(1, 2, 1-(\alpha+ik)a, 1-(\beta+ik)a; \\
&A+2, B+2, 2-(\alpha+\beta)a; 1).
\end{aligned} \tag{33}
$$

From the supersymmetry algebra [21], the ‘p’ partial wave regular and irregular solutions read as

$$
\begin{aligned}
\varphi_{\mathrm{H1}}(k,r) &= a^2 \mathrm{e}^{ikr} \left(1-\mathrm{e}^{-r/a}\right)^2 \\
&\times {}_{2F_1}(A+2, B+2; 4; 1-\mathrm{e}^{-r/a})
\end{aligned} \tag{34}
$$

and

$$f_{\mathrm{H1}}(k,r)=\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/a}\right)^{-1}\times{}_2F_1(A-1,B-1;C;\mathrm{e}^{-r/a}),\quad(35)$$

where the Jost function is given by

$$\Im_{\mathrm{H1}}(k)=3!!k^{-1}\mathrm{e}^{i\pi/2}f_{\mathrm{H1}}(k)\quad(36)$$

with

$$f_{\mathrm{H1}}(k)=-\frac{2iak\Gamma(C)}{\Gamma(A+2)\Gamma(B+2)}.\quad(37)$$

Considering eqs (34)–(37) in conjunction with eq. (10) one can write an expression for the p-wave physical Green's function as

$$G^{(+)}_{\mathrm{H1}}(r,r')=-\frac{2iak}{\Gamma(4)f_{\mathrm{H1}}(k)}a\mathrm{e}^{ikr_<}\times\left(1-\mathrm{e}^{-r_</a}\right)^2{}_2F_1(A+2,B+2;4;1-\mathrm{e}^{-r_</a})\times\mathrm{e}^{ikr_>}\left(1-\mathrm{e}^{-r_>/a}\right)^{-1}\times{}_2F_1(A-1,B-1;C;\mathrm{e}^{-r_>/a}).\quad(38)$$

The transformation of eq. (38) with the form factor of p-wave Graz separable potential from eq. (3a) reads as

$$G^{(+)}_{\mathrm{H1}}(r,\beta)=\frac{1}{2}a\int_0^\infty G^{(+)}_{\mathrm{H1}}(r,r')\left(1-\mathrm{e}^{-r'/a}\right)\mathrm{e}^{-\beta r'}\mathrm{d}r'.\quad(39)$$

Taking care of regular and irregular solutions, substitution of eq. (38) in eq. (39) results in

$$G^{(+)}_{\mathrm{H1}}(r,\beta)=-\frac{2iak}{2\Gamma(4)f_{\mathrm{H1}}(k)}a^2\Big\{\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/a}\right)^{-1}\times{}_2F_1(A-1,B-1;C;\mathrm{e}^{-r/a})\times\int_0^r\mathrm{e}^{-(\beta-ik)r'}(1-\mathrm{e}^{-r'/a})^3\times{}_2F_1(A+2,B+2;4;1-\mathrm{e}^{-r'/a})\mathrm{d}r'+\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/a}\right)^2\times{}_2F_1(A+2,B+2;4;1-\mathrm{e}^{-r/a})\int_r^\infty\mathrm{e}^{-(\beta-ik)r'}\times{}_2F_1(A-1,B-1;C;\mathrm{e}^{-r'/a})\mathrm{d}r'\Big\}.\quad(40)$$

In the above equation, an indefinite integral is involved. To evaluate it eq. (40) is rewritten as

$$G^{(+)}_{\mathrm{H1}}(r,\beta)=-\frac{iak}{\Gamma(4)f_{\mathrm{H1}}(k)}a^2\Big[\mathrm{e}^{ikr}\Big\{\left(1-\mathrm{e}^{-r/a}\right)^{-1}\times{}_2F_1(A-1,B-1;C;\mathrm{e}^{-r/a})\times\int_0^r\mathrm{e}^{-(\beta-ik)r'}\left(1-\mathrm{e}^{-r'/a}\right)^3\times{}_2F_1(A+2,B+2;4;1-\mathrm{e}^{-r'/a})\mathrm{d}r'-\left(1-\mathrm{e}^{-r/a}\right)^2\times{}_2F_1(A+2,B+2;4;1-\mathrm{e}^{-r/a})\int_0^r\mathrm{e}^{-(\beta-ik)r'}\times{}_2F_1(A-1,B-1;C;\mathrm{e}^{-r'/a})\mathrm{d}r'\Big\}+\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/a}\right)^2\times{}_2F_1(A+2,B+2;4;1-\mathrm{e}^{-r/a})\times\int_0^\infty\mathrm{e}^{-(\beta-ik)r'}{}_2F_1(A-1,B-1;C;\mathrm{e}^{-r'/a})\mathrm{d}r'\Big].\quad(41)$$

Integration with limit 0 to ∞ can be directly evaluated by using eq. (14) whereas for the integration with 0 to r one has to apply the analytic continuation to Gaussian hypergeometric function given in eq. (15). Thus, the resulting equation leads to

$$G^{(+)}_{\mathrm{H1}}(r,\beta)=\mathrm{e}^{ikr}\frac{f_{\mathrm{H1}}(\mathrm{k})\Gamma(3)}{2iak}\left(1-\mathrm{e}^{-r/a}\right)^{-1}\times\Big[\Big\{{}_2F_1(A-1,B-1;-2;1-\mathrm{e}^{-r/a})\int_0^r\mathrm{e}^{-(\beta-ik)r'}\times(1-\mathrm{e}^{-r'/a})^3{}_2F_1(A+2,B+2;4;1-\mathrm{e}^{-r'/a})\mathrm{d}r'-\left(1-\mathrm{e}^{-r/a}\right)^3{}_2F_1(A+2,B+2;4;1-\mathrm{e}^{-r/a})\times\int_0^r\mathrm{e}^{-(\beta-ik)r'}{}_2F_1(A-1,B-1;-2;1-\mathrm{e}^{-r'/a})\mathrm{d}r'\Big\}+\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/a}\right)^2{}_2F_1(A+2,B+2;4;1-\mathrm{e}^{-r/a})\times\int_0^\infty\mathrm{e}^{-(\beta-ik)r'}{}_2F_1(A-1,B-1;C;\mathrm{e}^{-r'/a})\mathrm{d}r'\Big].\quad(42)$$

Utilising eqs (14) and (17), $G^{(+)}_{\mathrm{H1}}(r,\beta)$ changes to

$$G^{(+)}_{\mathrm{H1}}(r,\beta)=\frac{1}{2}a^2\mathrm{e}^{ikr}\left(1-\mathrm{e}^{-r/a}\right)^2\times\Big\{a\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta+ik)a)}{\Gamma(1-(\beta+ik)a)}\frac{1}{n!}\times f_{n+1}(A+2,B+2;4;1-\mathrm{e}^{-r/a})$$

$$-\frac{\Gamma(A+2)\Gamma(B+2)}{\Gamma(4)\Gamma(C)(\beta-ik)} \times {}_2F_1(A+2, B+2; 4; 1-\mathrm{e}^{-r/a}) \times {}_3F_2(A-1, B-1, (\beta-ik)a; C, 1+(\beta-ik)a; 1). \quad (43)$$

The double transformation of eq. (43) with the p-wave form factor from eq. (3b) can be written as

$$G_{\mathrm{H1}}^{(+)}(\alpha,\beta) = \frac{1}{4}\Bigg[a^3\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta+ik)a)}{\Gamma(1-(\beta+ik)a)}\frac{1}{n!} \times \int_0^{\infty} r\left(1-\mathrm{e}^{-r/a}\right)^2 f_{n+1}(A+2, B+2; 4; 1-\mathrm{e}^{-r/a}) \times \mathrm{e}^{-(\alpha-ik)r}\,\mathrm{d}r - \frac{2iak}{\Gamma(4) f_{\mathrm{H1}}(k)(\beta-ik)}a^2 \times {}_3F_2(A-1, B-1, (\beta-ik)a; C, 1+(\beta-ik)a; 1) \times \int_0^{\infty} r\left(1-\mathrm{e}^{-r/a}\right)^2 \mathrm{e}^{-(\alpha-ik)r} \times {}_2F_1(A+2, B+2; 4; 1-\mathrm{e}^{-r/a})\mathrm{d}r\Bigg]. \quad (44)$$

Solving eq. (44) by using eqs (14), (20)–(22), one obtains

$$G_{\mathrm{H1}}^{(+)}(\alpha,\beta) = \frac{1}{4}\Bigg[a^5\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta+ik)a)\Gamma((\alpha-ik)a)\Gamma(n+4)}{\Gamma(1-(\beta+ik)a)\Gamma(n+5+(\alpha-ik)a)\Gamma(n+2)} \times {}_3F_2(1, n+A+3, n+B+3; n+2, n+5+(\alpha-ik)a; 1) - a^4\frac{2iak}{f_{\mathrm{H1}}(\mathrm{k})(\beta-ik)}\frac{\Gamma((\alpha-ik)a)\Gamma((\alpha+ik)a)}{\Gamma(2+(\alpha-ik)a-A)\Gamma(2+(\alpha-ik)a-B)} \times {}_3F_2(A-1, B-1, (\beta-ik)a; C, 1+(\beta-ik)a; 1)\Bigg]. \quad (45)$$

Simplifying eq. (45) by applying the transformations given in eqs (22), (24) and (25) one gets

$$G_{\mathrm{H1}}^{(+)}(\alpha,\beta) = \frac{1}{4}\Bigg[-a^5\frac{\Gamma((\alpha-ik)a)}{(A+2)\Gamma(1-(\beta+ik)a)}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta+ik)a)\Gamma(n+4)}{\Gamma(n+5+(\alpha-ik)a)\Gamma(n+1)} \times {}_3F_2(1, n+A+3, 2+(\alpha-ik)a-B; A+3, n+5+(\alpha-ik)a; 1) + a^5\frac{\Gamma(4)\Gamma((\alpha-ik)a)\Gamma((\alpha+ik)a)}{(A+2)(B+2)\Gamma(2+(\alpha-ik)a-A)\Gamma(2+(\alpha-ik)a-B)} \times {}_3F_2(1, 4, 1-(\beta+ik)a; A+3, 3-A-2iak; 1) - a^4\frac{2iak}{f_{\mathrm{H1}}(\mathrm{k})(\beta-ik)}\frac{\Gamma((\alpha-ik)a)\Gamma((\alpha+ik)a)}{\Gamma(2+(\alpha-ik)a-A)\Gamma(2+(\alpha-ik)a-B)} \times {}_3F_2(A-1, B-1, (\beta-ik)a; C, 1+(\beta-ik)a; 1)\Bigg]. \quad (46)$$

Using eqs (27) and (29) in eq. (46), $G_{\mathrm{H1}}^{(+)}(\alpha,\beta)$ changes to

$$G_{\mathrm{H1}}^{(+)}(\alpha,\beta) = -\frac{1}{4}a^5\frac{\Gamma((\alpha-ik)a)}{(A+2)(B+2)\Gamma(1-(\beta+ik)a)} \times \sum_{n=0}^{\infty}\frac{\Gamma(n+1-(\beta+ik)a)\Gamma(n+4)}{\Gamma(n+4+(\alpha-ik)a)\Gamma(n+1)} \times {}_3F_2(-n, 1-(\alpha+ik)a, 1; A+3, B+3; 1). \quad (47)$$

Expanding the infinite sum in eq. (47) and rearranging the terms suitably along with the help of general properties of Gaussian hypergeometric series, one obtains the maximal reduced form of eq. (47) as

$$G_{\mathrm{H1}}^{(+)}(\alpha,\beta) = -\frac{1}{4}a^5\frac{\Gamma(4)\Gamma((\alpha+\beta)a-1)}{(A+2)(B+2)\Gamma((\alpha+\beta)a+3)} \times {}_4F_3(1, 4, 1-(\alpha+ik)a, 1-(\beta+ik)a; A+3, B+3, 2-(\alpha+\beta)a; 1). \quad (48)$$

The regular and the irregular solutions for d-partial wave analysis are given by [21]

$$\varphi_{\mathrm{H2}}(k,r) = a^3 e^{ikr}\left(1-e^{-r/a}\right)^3 \times {}_2F_1(A+3, B+3; 6; 1-e^{-r/a}) \quad (49)$$

and

$$f_{\mathrm{H2}}(k,r) = e^{ikr}\left(1-e^{-r/a}\right)^{-2} \times {}_2F_1(A-2, B-2; C; e^{-r/a}), \quad (50)$$

where the Jost function

$$\Im_{\mathrm{H2}}(k) = 5!!k^{-2}e^{i\pi} f_{\mathrm{H2}}(k) \quad (51)$$

with

$$f_{\mathrm{H2}}(k) = -\frac{8a^2k^2\Gamma(C)}{\Gamma(A+3)\Gamma(B+3)}. \quad (52)$$

With the help of eqs (10) and (49)–(52), the d-wave physical Green's function is expressed as

$$G^{(+)}_{\mathrm{H2}}(r,r') = -\frac{8a^2k^2}{\Gamma(6) f_{\mathrm{H2}}(k)} a e^{ikr_<}\left(1-e^{-r_</a}\right)^3 \times {}_2F_1(A+3, B+3; 6; 1-e^{-r_</a}) \times e^{ikr_>}\left(1-e^{-r_>/a}\right)^{-2} \times {}_2F_1(A-2, B-2; C; e^{-r_>/a}). \quad (53)$$

The single Laplace transformation of eq. (53) with the form factor of d-wave Graz separable potential is written as

$$G^{(+)}_{\mathrm{H2}}(r,\beta) = \frac{1}{8}a^2 \int_0^\infty G^{(+)}_{\mathrm{H2}}(r,r') \times (1-e^{-r'/a})^2 e^{-\beta r'} \mathrm{d}r'. \quad (54)$$

Following the same procedure as discussed earlier, one can get an expression for the single transform

$$G^{(+)}_{\mathrm{H2}}(r,\beta) = \frac{1}{8}a^3 e^{ikr}\left(1-e^{-r/a}\right)^3 \times \Bigg\{ a\sum_{n=0}^{\infty} \frac{\Gamma(n+1-(\beta+ik)a)}{\Gamma(1-(\beta+ik)a)} \times \frac{1}{n!} f_{n+1}(A+3, B+3; 6; 1-e^{-r/a}) - \frac{8a^2k^2}{\Gamma(6) f_{\mathrm{H2}}(k)(\beta-ik)} \times {}_2F_1(A+3, B+3; 6; 1-e^{-r/a}) \times {}_3F_2(A-2, B-2, (\beta-ik)a; C, 1+(\beta-ik)a; 1)\Bigg\}. \quad (55)$$

Similarly, the double transformation of eq. (55) with the form factors given in eq. (3b) is obtained as

$$G^{(+)}_{\mathrm{H2}}(\alpha,\beta) = -\frac{1}{64}a^7 \times \frac{\Gamma(6)\Gamma((\alpha+\beta)a-1)}{(A+3)(B+3)\Gamma((\alpha+\beta)a+5)} \times {}_4F_3(1, 6, 1-(\alpha+ik)a, 1-(\beta+ik)a; A+4, B+4, 2-(\alpha+\beta)a; 1). \quad (56)$$

Having the expressions for the double transforms of the physical Green's functions by the form factors of the separable potential, one is able to write the desired expressions for the Fredholm determinants for motion in Hulthén plus Graz separable potential up to partial wave $\ell = 2$ by exploiting the relation

$$D^{(+)}_{\mathrm{H}n}(k) = 1 - \lambda_n G^{(+)}_{\mathrm{H}n}(\alpha,\beta), \quad n = 0, 1, 2. \quad (57)$$

It well known that the phase of the Fredholm determinant associated with physical boundary condition is equal to the negative of the scattering phase shift for motion in a local plus non-local separable potential [35–38]. Thus, utilising the expressions for $D^{(+)}_{\mathrm{H}n}(k)$; $n = 0, 1, 2$ one can extract the scattering phase shifts for charged hadron systems. In the next section we shall compute the phase shifts for α-^{3}He and α-^{3}H systems.

3. Calculations and results

As the nuclear potentials are highly state dependent, we use different strength and inverse range parameters for various angular momentum states. These parameters for the α-^{3}He and α-^{3}H systems are given in table 1.

Considering $\alpha = \beta$, we have computed the phase shifts for α-^{3}He and α-^{3}H systems by exploiting eqs (33), (48), (56) and (57) for different states using the parameters presented in table 1 and portrayed our results along with standard data [39] in figures 1–4. For the computation purposes we have worked with $\hbar^2/m_p = 41.47$ MeV fm^2; $V_0 a = 0.2384$ fm^{-1} and 0.4762 fm^{-1} for α-^{3}H and α-^{3}He systems respectively. The computed phase shifts for various states of s and p waves for α-^{3}H and α-^{3}He systems are plotted in figures 1 and 3 respectively. The phase shifts $\delta_{1/2^+}$ and $\delta_{3/2^-}$ are in good agreement with ref. [39] for both the systems under consideration (see figures 1 and 3). As the standard results [39] for $\delta_{1/2^-}$ phase shifts have fluctuations in certain energy ranges, our computed data do not follow such variations. Rather, they reproduce smooth variation in the phase shift values. But our results for $1/2^-$ state have correct trends for α-^{3}H and α-^{3}He systems. As the d-wave phase shifts have very small values, they are depicted separately in figures 2 and 4 for α-^{3}H

Table 1. Parameters for the α-^{3}H and α-^{3}He systems.

States	α-^{3}H			α-^{3}He		
	λ (MeV fm^{-2l-1})	β (fm^{-1})	a (au)	λ (MeV fm^{-2l-1})	β (fm^{-1})	a (au)
$1/2^+$	– 13.5	1.19	100	– 7.47	0.95	100
$1/2^-$	– 86.2221	1.22	50	– 60.5074	1.2	100
$3/2^-$	– 325.6596	1.65	50	– 73.9957	1.245	100
$3/2^+$	– 45	1.18	50	– 299980	2.73	25
$5/2^+$	– 789980	2.9	25	– 279980	2.4	25

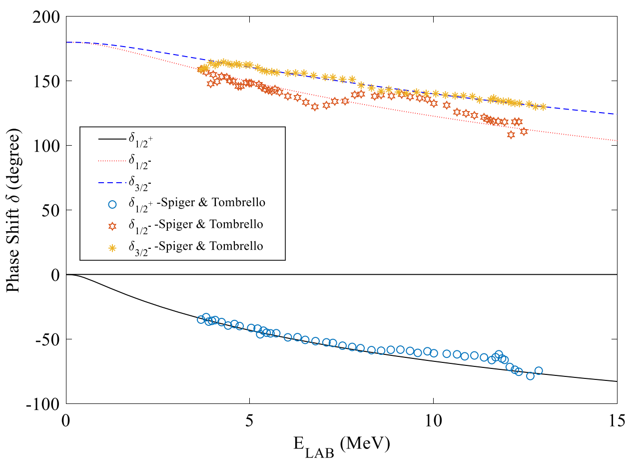

Figure 1. α-^{3}H phase shift for s and p waves as a function of laboratory energy.

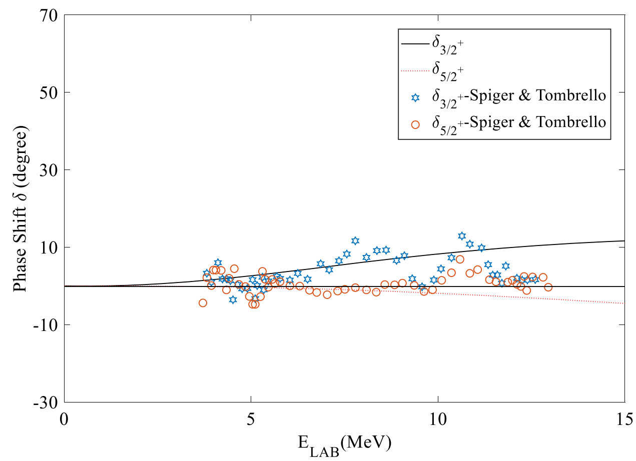

Figure 2. α-^{3}H phase shift for d waves as a function of laboratory energy.

and α-^{3}He systems respectively. For both the systems it is noticed that our computed $\delta_{3/2^+}$ and $\delta_{5/2^+}$ phase shifts are also in reasonable agreement with the scattered phase shift values of Spiger and Tombrello [39]. Previously, one of us [40] proposed Hulthén potential model and computed elastic scattering phase shifts for the α-^{3}H and α-^{3}He systems via the phase function method. In ref. [40] a screened centrifugal barrier term is added to the s-wave nuclear Hulthén potential to describe higher partial wave effective potentials. The present results are in better agreement with those of ref. [40].

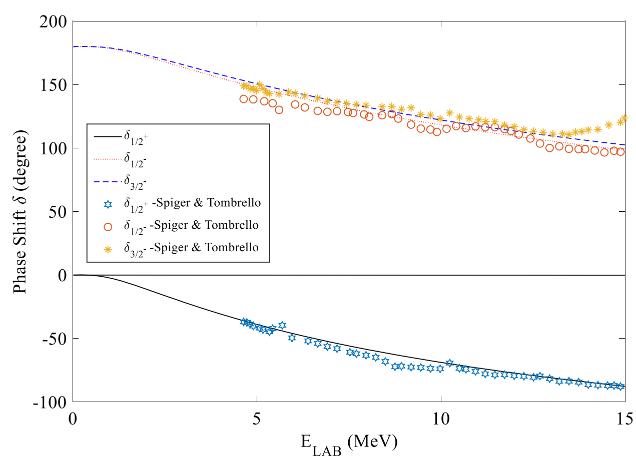

Figure 3. α-^{3}He phase shift for s and p waves as a function of laboratory energy.

4. Summary and conclusion

It is well known that fundamental studies on α-^{3}H and α-^{3}He interactions provide useful basis for understanding interactions between complex nuclei. In this context α-^{3}H and α-^{3}He scattering have been studied by several groups [39–47]. Mohr *et al* [41] have measured differential cross-sections for elastic scattering of α-^{3}H and α-^{3}He systems. They have analysed scattering phase shifts up to 10 MeV within the framework of optical model using double folded potentials. In the recent past, Neff [47] has calculated astrophysical S-factor for ^{3}H$(\alpha,\gamma)^7$Be and ^{3}He$(\alpha,\gamma)^7$Li in fermionic molecular dynamic approach using a realistic two-body effective interaction. In this paper, higher partial wave solutions, generated from supersymmetry formalism, are considered to calculate Green's functions and their integral transforms for atomic Hulthén plus a separable

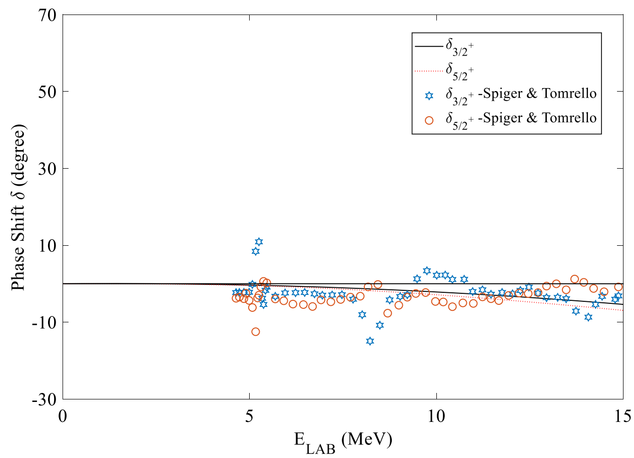

Figure 4. α-^{3}He phase shift for d waves as a function of laboratory energy.

nuclear potential. The associated Fredholm determinants are applied to study low-energy scattering phase shifts of α-^{3}H and α-^{3}He systems. All partial wave treatments for both on- and off-shell scattering are in our active consideration and will be addressed in future. For charged hadronic systems, generally, the screened/cut-off Coulomb interaction is applicable [48–50]. Therefore, the present approach may constitute a convenient starting point for treating complex nuclear systems with screened electromagnetic potential. Our approach is equally applicable for asymmetric form factors of the separable potential.

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