



# Thermal properties of anharmonic Eckart potential model using Euler–MacLaurin formula

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**Abstract.** By employing the asymptotic iteration method (AIM), we solved the three-dimensional time-independent Schrödinger equation with the anharmonic Eckart potential model. The expression for the eigensolution of the anharmonic Eckart potential was obtained. With the help of the ro-vibrational energy spectra obtained, we derived the expressions for the ro-vibrational partition function and other thermodynamic functions, via the Euler MacLaurin formula. Effects of temperature and upper bound vibration quantum number on the thermodynamic functions of anharmonic Eckart potential were discussed for some diatomic molecular systems. It has been established that unique critical temperatures of ro-vibrational entropy and ro-vibrational specific heat capacity exist for the selected diatomic molecules.

**Keywords.** Anharmonic Eckart potential; asymptotic iteration method; Euler MacLaurin formula; Schrödinger equation; thermal properties.

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## 1. Introduction

Since the advent of quantum mechanics, researchers have been engaged with the development of potential models to fit various complex physical systems like quantum information theory, mass spectra of quarkonium system, thermodynamic and thermochemical studies of diatomic and polyatomic molecule systems, and others. Examples of such potential models include the Morse potential [1], the Hellmann potential [2], the Kratzer potential [3], Rosen–Morse potential [4], Manning–Rosen potential [5], Poschl–Teller potential [6], screened Kratzer potential [7], anharmonic Eckart potential [8], etc.

The anharmonic Eckart potential, among other exponential-type potential models and their various modified forms, has gained popularity in research circle as a powerful and very useful potential model widely used in physics and chemical physics to study diatomic and polyatomic systems [9–22]. The Eckart potential is

given as [8]

$$V(r) = D_e \left( 1 - \frac{be^{-\alpha r}}{1 - e^{-\alpha r}} \right)^2; \quad 0 \leq r \leq \infty;$$
$$b = e^{\alpha r_e} - 1, \quad (1)$$

where  $D_e$ ,  $r_e$  and  $\alpha$  represent the dissociation energy, the equilibrium bond length and the screening parameter, respectively.

The anharmonic Eckart potential has been applied successfully in the study of Schrödinger equation, Klein–Gordon and Dirac equations [23–25], using different methods [26–28]. For instance, Zang *et al* [29] investigated the three-dimensional Schrödinger equation including centrifugal term using the NU method and obtained eigenvalues and eigenfunctions for the Eckart potential. Diao *et al* [30] obtained approximately the analytical solutions to the Schrödinger equation in the Eckart potential for the arbitrary l-wave bound states, using modified approximation scheme to deal with the

centrifugal term. Gao and Zhang [31] investigated the analytical solution of the Schrödinger equation in D-dimension with the Eckart potential using the modified approximation scheme. Chen *et al* [32] studied the characteristics of the scattering states for Eckart potential.

Earlier, different researchers have employed various potential functions to study the thermodynamic properties of different systems, using numerous techniques [33–41]. Recently, Boumali [42] presented the closed-form expressions of the vibrational partition function and other related thermodynamic functions for the one-dimensional  $q$ -deformed Morse potential energy model, using the Euler–MacLaurin method [43]. Also, the thermal properties of three-dimensional Morse potential for some selected diatomic molecules were obtained [44] using Euler–MacLaurin method. Similarly, the thermodynamic properties of some potential models have been studied for some diatomic molecular systems recently [45–48]. In addition, Valencia-Ortega and Arias-Hernandez [49] considered a one-dimensional microscopic system for the generalised Morse potential (GMP), describing local anharmonic effects. The authors obtained some thermodynamic functions for a heteronuclear diatomic system (HCl molecule) and compared these functions with that of the Morse potential, graphically. In their studies also, a connection between the GMP and the Eckart potential was established using SO (2, 1) algebras.

The aim of this research is to solve the three-dimensional Schrödinger equation with the anharmonic Eckart potential using the asymptotic iteration method (AIM). The eigensolutions obtained will be used to study the effects of temperature and upper bound vibration quantum number on the thermodynamic functions of the anharmonic Eckart potential for some diatomic molecules, using the Euler–MacLaurin formula.

This study is organised as follows. In §2, we solve the three-dimensional Schrödinger equation with the anharmonic Eckart potential and obtained its corresponding eigensolutions via the asymptotic iteration method (AIM), as presented in Appendix. The thermodynamic properties of the anharmonic Eckart potential are evaluated in §3. Section 4 gives results and discussion. Finally, a brief conclusion is presented in §5.

## 2. Eigensolutions of the Schrödinger equation with anharmonic Eckart potential model

The general form of the three-dimensional time-independent Schrödinger equation is given as [50]

$$\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \right] \psi(r, \phi, \varphi) = E_{nl} \psi(r, \phi, \varphi). \quad (2)$$

Defining

$$\psi(r, \phi, \varphi) = \frac{1}{r} R_{nl}(r) Y_{jm}(\phi, \varphi)$$

and considering the potential given in eq. (1), we obtain the radial part of the Schrödinger equation as

$$\frac{d^2 R_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2} [E_{nl} - V_{\text{eff}}(r)] R_{nl}(r) = 0, \quad (3)$$

where

$$V_{\text{eff}}(r) = D_e \left( 1 - \frac{be^{-\alpha r}}{1 - e^{-\alpha r}} \right)^2 + \frac{\hbar^2 l(l+1)}{2\mu r^2}. \quad (4)$$

This equation cannot be solved analytically for  $l \neq 0$  due to the centrifugal term. Hence, an approximation scheme is employed to handle the centrifugal term. This is given as [51]

$$\frac{1}{r^2} \approx \alpha^2 \left[ K_0 + \frac{e^{-\alpha r}}{(1 - e^{-\alpha r})^2} \right]; \quad K_0 = \frac{1}{12}. \quad (5)$$

Inserting eqs (5) and (5) into eq. (3) and introducing a new variable  $z = e^{-\alpha r}$ , we obtain the differential equation of the form

$$z(1-z) \frac{d^2 R_{nl}(z)}{dz^2} + (1-z) \frac{dR_{nl}(z)}{dz} + \frac{1}{z(1-z)} \times [-Az^2 + Bz - C] R_{nl}(z) = 0, \quad (6)$$

where

$$\begin{aligned} A &= \varepsilon_{nl}^2 + \frac{2\mu D_e}{\alpha^2 \hbar^2} + K_0 l(l+1) + \frac{4\mu D_e b}{\alpha^2 \hbar^2} + \frac{2\mu D_e b^2}{\alpha^2 \hbar^2}; \\ B &= \varepsilon_{nl}^2 + \frac{4\mu D_e}{\alpha^2 \hbar^2} + 2K_0 l(l+1) + \frac{4\mu D_e b}{\alpha^2 \hbar^2} - l(l+1); \\ C &= \varepsilon_{nl}^2 + \frac{2\mu D_e}{\alpha^2 \hbar^2} + K_0 l(l+1); \\ \varepsilon_{nl}^2 &= -\frac{2\mu E_{nl}}{\alpha^2 \hbar^2}. \end{aligned} \quad (7)$$

Equation (6) has two singularities at  $z = 0$  and  $z = 1$ . Therefore, a trial solution of the form is proposed:

$$R_{nl}(z) = z^u (1-z)^\delta f_{nl}(z), \quad (8)$$

where

$$\begin{aligned} u &= \pm \sqrt{-\frac{2\mu E_{nl}}{\alpha^2 \hbar^2} + \frac{2\mu D_e}{\alpha^2 \hbar^2} + K_0 l(l+1)}; \\ \delta &= \frac{1}{2} \left( 1 + \sqrt{(1+2l)^2 + \frac{8\mu D_e b^2}{\alpha^2 \hbar^2}} \right). \end{aligned} \quad (9)$$

Substituting eq. (8) into eq. (6) gives

$$\frac{d^2 f_{nl}(z)}{dz^2} = \frac{[(2u + 2\delta + 1)z - (1 + 2u)]}{z(1 - z)} \frac{df_{nl}(z)}{dz} + \frac{[(u + \delta) - \sqrt{A}][(u + \delta) + \sqrt{A}]}{z(1 - z)} f_{nl}(z). \quad (10)$$

Comparing eq. (10) with the AIM procedure (see Appendix A), we have

$$\lambda_0(z) = \frac{[(2u + 2\delta + 1)z - (1 + 2u)]}{z(1 - z)} ;$$

$$s_0(z) = \frac{[(\delta^2 + 2u\delta + C - A)]}{z(1 - z)}. \quad (11)$$

With the aid of the quantisation condition given in Appendix A, we obtain

$$\frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} \Rightarrow u_0 = - \frac{[\delta^2 + A - C]}{2\delta},$$

$$\frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2} \Rightarrow u_1 = - \frac{[\delta^2 + 2\delta + 1 + A - C]}{2(\delta + 1)},$$

$$\frac{s_2}{\lambda_2} = \frac{s_3}{\lambda_3} \Rightarrow u_2 = - \frac{[\delta^2 + 4\delta + 4 + A - C]}{2(\delta + 2)}. \quad (12)$$

Generalising eq. (12) for arbitrary  $n$  gives

$$u_n = - \left[ \frac{(n + \delta)^2 + C - A}{2(n + \delta)} \right]. \quad (13)$$

By substituting eqs (7) and (9) into eq. (13), we obtain the ro-vibrational energy spectra of the anharmonic Eckart potential model as

$$E_{nl} = D_e + \frac{\alpha^2 \hbar^2}{2\mu} K_0 l(l + 1) - \frac{\alpha^2 \hbar^2}{2\mu} \left[ \frac{\left( n + \frac{1}{2} \left( 1 + \sqrt{(1 + 2l)^2 + \frac{8\mu D_e b^2}{\alpha^2 \hbar^2}} \right) \right)^2 - \left( \frac{4\mu D_e b}{\alpha^2 \hbar^2} + \frac{2\mu D_e b^2}{\alpha^2 \hbar^2} \right)}{2 \left( n + \frac{1}{2} \left( 1 + \sqrt{(1 + 2l)^2 + \frac{8\mu D_e b^2}{\alpha^2 \hbar^2}} \right) \right)} \right]^2. \quad (14)$$

### 3. Thermal properties of anharmonic Eckart potential model for some diatomic molecules

The thermal properties of the anharmonic Eckart potential model can be obtained from the ro-vibrational partition function, being the starting point to derive all thermodynamic functions of any system [52–54]. The bound state contributions to the ro-vibrational partition function of any system at a given temperature  $T$  is given as [37]

$$Z(\beta, \lambda) = \sum_{n=0}^{\lambda} e^{-\beta(E_{nl} - E_{0l})}, \quad \beta = (k_B T)^{-1}. \quad (15)$$

Here,  $k_B$  is the Boltzmann’s constant,  $\lambda$  is the upper bound vibration quantum number and  $E_{nl}$  is the ro-vibrational energy eigenvalues of the anharmonic Eckart potential model. Sequel to the fact that partition function cannot be calculated exactly in a closed form, reasonable approximation can only be obtained for high temperature where  $T \rightarrow \infty$  and for low temperature where  $T \rightarrow 0$  [55]. Hence, the ro-vibrational partition function can be evaluated using the Euler–Maclaurin formula, as defined by [42–44]

$$\sum_{n=0}^{\infty} f(x) = \frac{1}{2} f(0) + \int_0^{\infty} f(x) dx - \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} f^{(2p-1)}(0), \quad (16)$$

where  $B_{2p}$  are the Bernuoli numbers,  $f^{(2p-1)}$  is the derivative of order  $(2p - 1)$ . Taking  $p$  up to 3, we obtain

$$Z(\beta, \lambda) = \frac{1}{2} + \int_0^{\lambda} f(n) dn - \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} f^{(2p-1)}(0), \quad (17)$$

where

$$f(n) = e^{-\beta(E_{nl} - E_{0l})}, \quad B_2 = \frac{1}{6} \text{ and } B_4 = -\frac{1}{30}.$$

Employing the Mathematica software and the Euler–MacLaurin formula, the expression for the ro-vibrational partition function of the anharmonic Eckart potential model is obtained as

**Table 1.** Spectroscopic parameters for the selected diatomic molecules [44].

Molecules	$D_e$ (eV)	$r_e$ (Å)	$\alpha$ (Å <sup>-1</sup> )	$\mu$ (a.m.u.)
HCl	4.619030905	1.2746	1.8677	0.9801045
LiH	2.5152672118	1.5956	1.1280	0.8801221
H <sub>2</sub>	4.7446	0.7416	1.9426	0.50391

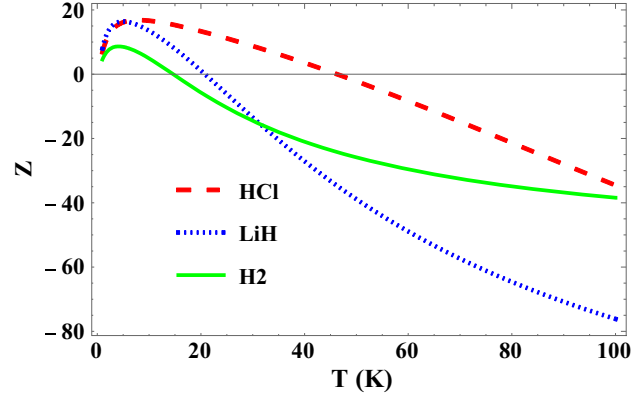
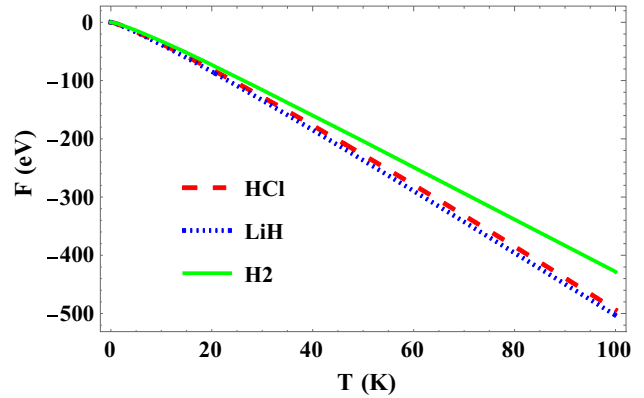
$$Z(\beta, \lambda) = \frac{1}{2} + \frac{1}{2M} \times \left\{ e^{(E_{0l} - G_2 + \frac{G_1 H}{2})\beta - \frac{LM}{2} \sqrt{\pi}} \times \left[ \begin{array}{l} \text{Erf}\left(\frac{L - M\delta^2}{2\delta}\right) + \text{Erf}\left(\frac{-L + M(\delta + \lambda)^2}{2(\delta + \lambda)}\right) \\ + e^{LM} \left( -\text{Erf}\left(\frac{L + M\delta^2}{2\delta}\right) \right. \\ \left. + \text{Erf}\left(\frac{L + M(\delta + \lambda)^2}{2(\delta + \lambda)}\right) \right) \end{array} \right] \right\} - J, \quad (18)$$

where

$$\begin{aligned} E_{0l} &= G_2 - H \left[ \frac{\delta}{2} - \frac{G_1}{2\delta} \right]^2; \\ G_1 &= - \left( \frac{4\mu D_e b}{\alpha^2 \hbar^2} + \frac{2\mu D_e b^2}{\alpha^2 \hbar^2} \right); H = \frac{\alpha^2 \hbar^2}{2\mu}; \\ G_2 &= D_e + \frac{\alpha^2 \hbar^2}{2\mu} K_0 l(l+1); \\ \lambda &= \sqrt{G_1} - \delta; L = \sqrt{G_1^2 M}; \\ M &= \sqrt{-H\beta}; J = \frac{M}{24} \left( \frac{G_1^2}{\delta^3} - \delta \right); \\ \text{Erfi}(w) &= \frac{\text{Erf}(iw)}{i} = \frac{2}{\sqrt{\pi}} \int_0^w e^{s^2} ds. \end{aligned} \quad (19)$$

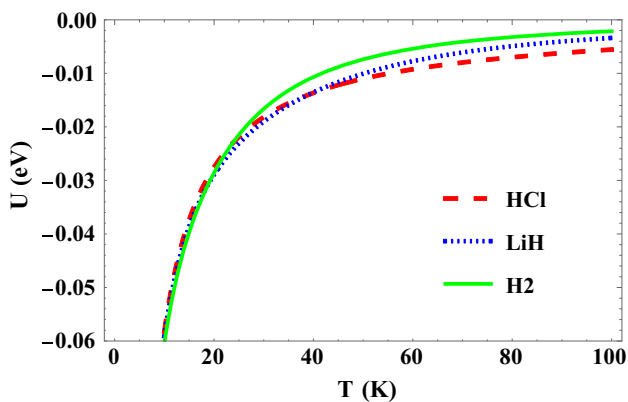
Thermodynamic properties such as Helmholtz free energy,  $F(\beta, \lambda)$ , entropy,  $S(\beta, \lambda)$ , internal energy,  $U(\beta, \lambda)$  and specific heat capacity,  $C_v(\beta, \lambda)$  can be obtained from the partition function as follows [56]:

$$\begin{aligned} F(\beta, \lambda) &= -\frac{1}{\beta} \ln Z(\beta, \lambda); \\ U(\beta, \lambda) &= -\frac{\partial(\ln Z(\beta, \lambda))}{\partial\beta}; \\ S(\beta, \lambda) &= k_B \ln Z(\beta, \lambda) - k_B \beta \frac{\partial}{\partial\beta} \ln Z(\beta, \lambda); \\ C_v(\beta, \lambda) &= k_B \beta^2 \frac{\partial^2}{\partial\beta^2} \ln Z(\beta, \lambda). \end{aligned} \quad (20)$$

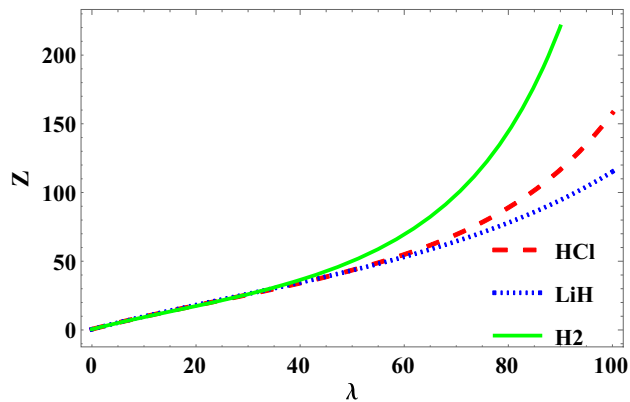
**Figure 1.** Variation of ro-vibrational partition function of anharmonic Eckart potential with temperature for selected diatomic molecules.**Figure 2.** Variation of ro-vibrational free energy of anharmonic Eckart potential with temperature for selected diatomic molecules.

#### 4. Discussion of results

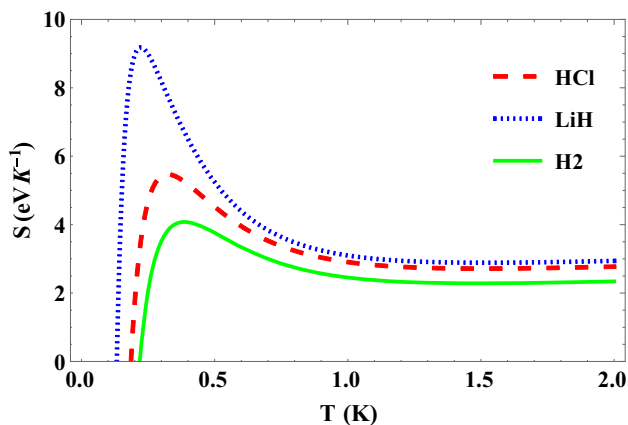
In this research, we consider some diatomic molecules with their respective spectroscopic parameters as given in table 1. The following conversions have been employed throughout our analysis [40]: 1 a.m.u. = 931.494028 MeV/c<sup>2</sup>; 1 cm<sup>-1</sup> = 1.239841875 × 10<sup>-4</sup> eV and  $\hbar c = 1973.29$  eV Å. Using these spectroscopic parameters as inputs, plots of different thermodynamic properties were generated for various temperatures and upper bound vibration quantum numbers. The effects of temperature on the thermodynamic properties



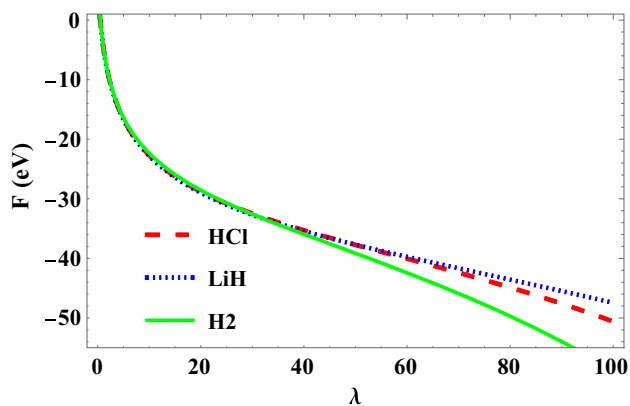
**Figure 3.** Variation of ro-vibrational internal energy of anharmonic Eckart potential with temperature for selected diatomic molecules.



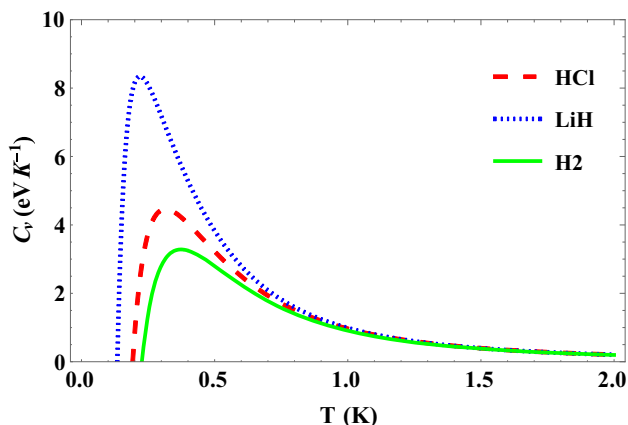
**Figure 6.** Variation of ro-vibrational partition function of anharmonic Eckart potential with upper bound vibration quantum number for selected diatomic molecules.



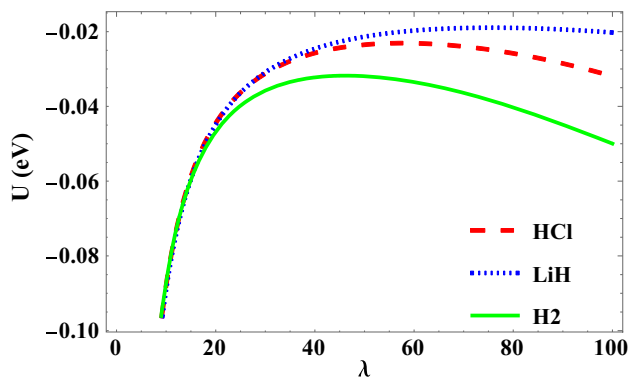
**Figure 4.** Variation of ro-vibrational entropy of anharmonic Eckart potential with temperature for selected diatomic molecules.



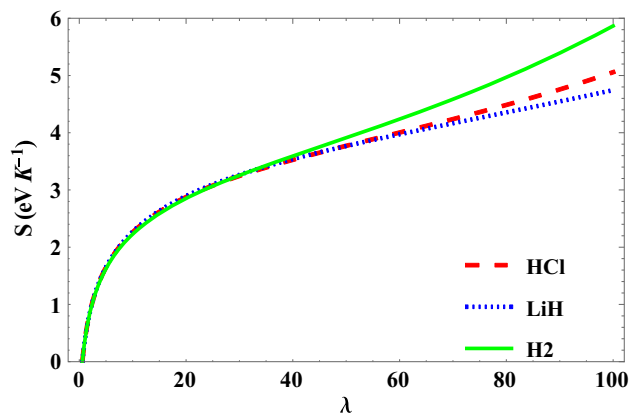
**Figure 7.** Variation of ro-vibrational free energy of anharmonic Eckart potential with upper bound vibration quantum number for selected diatomic molecules.



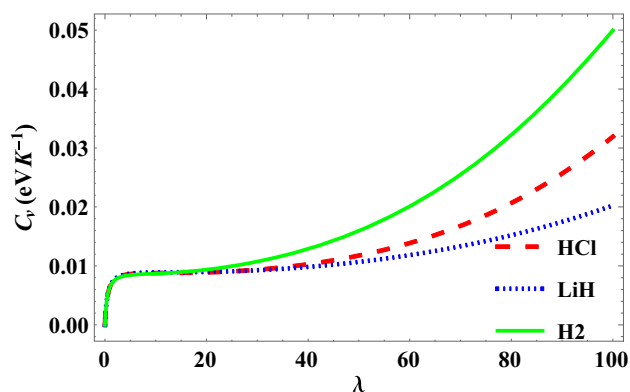
**Figure 5.** Variation of ro-vibrational specific heat capacity of anharmonic Eckart potential with temperature for selected diatomic molecules.



**Figure 8.** Variation of ro-vibrational internal energy of anharmonic Eckart potential with upper bound vibration quantum number for selected diatomic molecules.



**Figure 9.** Variation of ro-vibrational entropy of anharmonic Eckart potential with upper bound vibration quantum number for selected diatomic molecules.



**Figure 10.** Variation of ro-vibrational specific heat capacity of anharmonic Eckart potential with upper bound vibration quantum number for selected diatomic molecules.

of anharmonic Eckart potential for the selected diatomic molecules are shown in figures 1–5. Figure 1 shows a gradual decrease in the ro-vibrational partition function for the selected diatomic molecules, as temperature is enhanced. In figure 2, there exists a sharp decrease in the ro-vibrational free energy as the temperature increases. Figure 3 shows a monotonous increase in the ro-vibrational internal energy for the selected diatomic molecules as temperature is increased. In figures 4 and 5, there exists first a sharp increase in the ro-vibrational entropy and ro-vibrational specific heat capacity to a unique value called the critical temperature, corresponding to the selected diatomic molecules, respectively. Thereafter, the respective ro-vibrational entropy and ro-vibrational specific heat capacity curves begin to reduce monotonously as temperature is enhanced.

The effects of the upper bound vibration quantum number on the thermodynamic properties of anharmonic Eckart potential for the selected diatomic molecules

are shown in figures 6–10. Figure 6 shows an increase in the ro-vibrational partition function of the anharmonic Eckart potential as the upper bound vibration quantum number increases, for the selected diatomic molecules. As shown in figure 7, the ro-vibrational free energy decreases with increase in the upper bound vibration quantum number. The ro-vibrational internal energy and the ro-vibrational entropy of the anharmonic Eckart potential are seen to increase monotonously for the selected diatomic molecules, as the upper bound vibration quantum number increases, as demonstrated in figures 8 and 9, respectively. In figure 10, the ro-vibrational specific heat capacity increases sharply first at  $\lambda = 0$ ; then remains constant as  $\lambda$  increases. Further enhancement of  $\lambda$  increases the ro-vibrational specific heat capacity. Our results are seen to agree with the previously obtained results in [44,49].

## 5. Conclusion

In this paper, the thermal properties of anharmonic Eckart potential model is investigated for selected diatomic molecules including hydrogen chloride, lithium hydride and hydrogen dimer. Firstly, we determined the eigensolutions of the three-dimensional time-independent Schrödinger equation using the asymptotic iteration method. The analytical expression of the ro-vibrational energy spectra of the anharmonic Eckart potential obtained was used to derive expressions for ro-vibrational partition function and other thermal properties of the anharmonic Eckart potential model using the Euler–MacLaurin formula. With the help of the experimental data for the selected diatomic molecules taken from the literature, the thermal functions of the anharmonic Eckart potential have been evaluated graphically via the Mathematica software. The effects of temperature and upper bound vibration quantum number on the evaluated thermal functions have been discussed extensively. We specifically observed the existence of unique critical temperatures, corresponding to the selected diatomic molecules. Our results agree with that obtained in literatures. Of late, different researchers have employed different improved potential models related to the anharmonic potential model to study various thermodynamic functions for diatomic and even triatomic molecular systems [57–64]. The anharmonic Eckart potential model will be very effective in predicting thermochemical quantities for diatomic and polyatomic molecular systems.



### Appendix A

#### A.1 Review of the asymptotic iteration method (AIM)

The AIM was proposed by Ciftci *et al* [65] to solve the homogeneous linear second-order differential equation of the form [66]

$$y''(x) = \lambda_0(x)y'(x) + s_0(x)y(x), \tag{A.1}$$

where  $\lambda_0(x) \neq 0$  and the prime denotes the derivative with respect to  $x$ . The functions,  $s_0(x)$  and  $\lambda_0(x)$  must be sufficiently differentiable. Differentiating eq. (A.1) with respect to  $x$ , we obtain

$$y'''(x) = \lambda_1(x)y'(x) + s_1(x)y(x), \tag{A.2}$$

where

$$\begin{aligned} \lambda_1(x) &= \lambda_0'(x) + \lambda_0^2(x) + s_0(x), \\ s_1(x) &= s_0'(x) + s_0(x)\lambda_0(x). \end{aligned} \tag{A.3}$$

Taking the second derivative of eq. (A.1) yields

$$y''''(x) = \lambda_2(x)y'(x) + s_2(x)y(x), \tag{A.4}$$

where

$$\begin{aligned} \lambda_2(x) &= \lambda_1'(x) + \lambda_0(x)\lambda_1(x) + s_1(x), \\ s_2(x) &= s_1'(x) + s_0(x)\lambda_1(x). \end{aligned} \tag{A.5}$$

Again, by taking the  $(k + 1)$ th and  $(k + 2)$ th-order derivative of eq. (A.1) for  $k = 1, 2, 3, \dots$ , we obtain the following differential equations:

$$\begin{aligned} y^{(k+1)}(x) &= \lambda_{k-1}(x)y'(x) + s_{k-1}(x)y(x), \\ y^{(k+2)}(x) &= \lambda_k(x)y'(x) + s_k(x)y(x), \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} \lambda_{k-1}(x) &= \lambda'_{k-2}(x) + \lambda_0(x)\lambda_{k-2}(x) + s_{k-2}(x), \\ s_{k-1}(x) &= s_0(x)\lambda_{k-2}(x) + s'_{k-2}(x), \\ \lambda_k(x) &= \lambda'_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x) + s_{k-1}(x), \\ s_k(x) &= s_0(x)\lambda_{k-1}(x) + s'_{k-1}(x). \end{aligned} \tag{A.7}$$

Solving eq. (A.6), we obtain the following relation:

$$\frac{y^{(k+2)}(x)}{y^{(k+1)}(x)} = \frac{\lambda_k(x) \left[ y'(x) + \frac{s_k(x)}{\lambda_k(x)} y(x) \right]}{\lambda_{k-1}(x) \left[ y'(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} y(x) \right]}. \tag{A.8}$$

For sufficiently large values of  $k$ ,  $\alpha(x)$  values are obtained as

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x). \tag{A.9}$$

This method consists of converting the Schrödinger-like equation into the form of eq. (A.1) for a given potential

model. The corresponding energy eigenvalues are calculated by means of the quantisation condition [67]

$$\delta_k(x) = s_k(x)\lambda_{k-1}(x) - \lambda_k(x)s_{k-1}(x), \quad k=1, 2, 3, \dots \tag{A.10}$$

The general solution of eq. (A.1) is obtained from eq. (A.8) as

$$\begin{aligned} y(x) &= \exp\left(-\int^x \alpha(x_1)dx_1\right) \\ &\times \left[ C_2 + C_1 \int^x \exp\left(\int^x [\lambda_0(x_2) + 2\alpha(x_2)]dx_2\right) dx_1 \right], \end{aligned} \tag{A.11}$$

where  $C_1$  and  $C_2$  are integration constants. Also, the eigenfunction can be obtained by transforming the Schrödinger-like equation of the form

$$\begin{aligned} y''(x) &= 2\left(\frac{ax^{N+1}}{1-bx^{N+2}} - \frac{t+1}{x}\right)y'(x) \\ &\quad - \frac{Wx^N}{1-bx^{N+2}}y(x). \end{aligned} \tag{A.12}$$

The exact solutions for eq. (A.12) is given by

$$\begin{aligned} y(x) &= (-1)^2 C(N+2)(\sigma)_n {}_2F_1 \\ &\quad \left(-n, \rho + n; \sigma; bx^{N+2}\right), \end{aligned} \tag{A.13}$$

where

$$\begin{aligned} (\sigma)_n &= \frac{\Gamma(\sigma + n)}{\sigma}, \quad \sigma = \frac{2t + N + 3}{N + 2}, \\ \rho &= \frac{(2t + 1)b + 2a}{(N + 2)b}. \end{aligned} \tag{A.14}$$

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